Bi-Free Independence and the Asymptotics of Tensor Random Matrices

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Quantum Channels

Definition

A *quantum channel* is a trace-preserving, completely positive map $\Phi : M_n \to M_n$.

The Kraus decomposition implies if $\Phi : M_n \to M_n$ is a quantum channel, then there exists $\{K_j\}_{j=1}^d \subseteq M_n$ such that

$$\Phi(X) = \sum_{j=1}^{d} K_j X K_j^*.$$

Note:

- $\sum_{j=1}^d K_j^* K_j = I_n$ implies Φ is trace-preserving.
- $\sum_{j=1}^d K_j K_j^* = I_n$ implies Φ is unital.
- $\mathrm{tr}(\Phi(X)Y) = \mathrm{tr}(X\Phi(Y))$ implies we can take $\{K_j\}_{j=1}^d$ to be self-adjoint.

Random Quantum Channels

Viewing $M_n \cong \mathbb{C}^n \otimes \mathbb{C}^n$, we can examine Φ through its Kraus operator

$$K_\Phi = \sum_{j=1}^{d} K_j \otimes \overline{K_j}$$

where $\overline{K_j}$ denotes the entry-wise conjugation of K_j.

What happens when we take $\{K_j\}_{j=1}^{d}$ to be random matrices? What information about K_Φ can be obtained?

What information can we obtain about the distributions of tensor product of random matrices?

Random Tensor Matrices

Theorem (Lacien, Santos, Youssef; 2023)

Let $W_1, \ldots, W_d$ be centred, self-adjoint random $n \times n$ matrices such that

- *W_j converges weakly in probability and expectation to μ_j as $n \to \infty$,*
- *$\{W_j\}_{j=1}^d$ are in probability and expectation asymptotically free, and*
- *$\mathbb{E}\left(W_j \otimes \overline{W_j}\right)$ converges weakly to 0 as $n \to \infty$.*

Let $a_1, \ldots, a_d$ be freely independent random variables with respect to φ with distributions $\mu_1, \ldots, \mu_d$ respectively, and let

$$\Delta_{d,n} = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} \left(W_j \otimes \overline{W_j} - \mathbb{E}\left(W_j \otimes \overline{W_j}\right)\right).$$

Then in probability and expectation the distribution of $\Delta_{d,n}$ tends to the distribution of $\frac{1}{\sqrt{d}} \sum_{j=1}^{d} a_j \otimes a_j$ with respect to $\varphi \otimes \varphi$ as $n \to \infty$.

Tensors of Freely Independent Operators

Theorem (Lacien, Santos, Youssef; 2024)

Let $(\mathcal{A}, \varphi)$ *be a non-commutative probability space and let* $a \in \mathcal{A}$ *be such that* $\varphi(a) = \lambda$ *and* $\mathrm{var}(a) = \sigma^2 \neq 0$*. Let*

$$\delta^2 = \mathrm{var}(a \otimes a) \quad \textit{and} \quad q = \frac{2\lambda^2}{\sigma^2 + 2\lambda^2} \in [0, 1).$$

Let $(a_k)_{k \geq 1}$ *be a sequence of freely independent copies of* a *in* $(\mathcal{A}, \varphi)$*. Let*

$$S_n = \frac{1}{\delta\sqrt{n}} \sum_{k=1}^{n} (a_k \otimes a_k - \lambda 1 \otimes \lambda 1) \in \mathcal{A} \otimes \mathcal{A}.$$

Then the distribution of $(S_n)_{\geq 1}$ *with respect to* $\varphi \otimes \varphi$ *converges to*

$$\sqrt{q}\left(\frac{1}{\sqrt{2}}\mu_{sc} \oplus \frac{1}{\sqrt{2}}\mu_{sc}\right) \boxplus \sqrt{1-q}\,\mu_{sc}.$$

What Type of Independence?

Lacien, Santos, Youssef commented:

"The difficulty in analyzing S_n stems from the complicated dependence structure exhibited by tensors, combining classical independence (between the two legs of the tensor) and freeness (between the variables across tensors). ... It would be of interest to design a general notion of independence corresponding to the tensor case, analyze its properties, derive the corresponding limit theorems, and characterize the corresponding universal objects."

Bi-Free Independence

Definition (Voiculescu; 2014)

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Pairs of unital C*-subalgebras $(A_{\ell,1}, A_{r,1})$ and $(A_{\ell,2}, A_{r,2})$ of $\mathcal{A}$ are said to be *bi-freely independent* if there exist Hilbert spaces $\mathcal{H}_k$ and unital homomorphisms $\alpha_k : A_{\ell,k} \to \mathcal{B}(\mathcal{H}_k)$ and $\beta_k : A_{r,k} \to \mathcal{B}(\mathcal{H}_k)$ such that the following diagram commutes:

$$\begin{array}{ccccc}
A_{\ell,1} * A_{r,1} * A_{\ell,2} * A_{r,2} & \xrightarrow{\ i\ } & \mathcal{A} & \xrightarrow{\ \varphi\ } & \mathbb{C} \\
\Big\downarrow {\scriptstyle \alpha_1 * \beta_1 * \alpha_2 * \beta_2} & & & & \Big\uparrow {\scriptstyle \varphi_*} \\
\mathcal{B}(\mathcal{H}_1) * \mathcal{B}(\mathcal{H}_1) * \mathcal{B}(\mathcal{H}_2) * \mathcal{B}(\mathcal{H}_2) & & \xrightarrow{\ \lambda_1 * \rho_1 * \lambda_2 * \rho_2\ } & & \mathcal{B}(\mathcal{H}_1 * \mathcal{H}_2)
\end{array}$$

Examples of Bi-Free Independence

- Given a free product group $G = *_{k \in K} G_k$, $\{(\lambda(G_k), \rho(G_k))\}_{k \in K}$ are bi-freely independent with respect to τ_G.
- Given II_1 factors $\{(\mathfrak{M}_k, \tau_k)\}_{k \in K}$, $\{(\lambda(\mathfrak{M}_k), \rho(\mathfrak{M}'_k))\}_{k \in K}$ are bi-freely independent in $(L_2(*_{k \in K} \mathfrak{M}_k), *_{k \in K} \tau_k)$.
- Given $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \varphi)$, $(\mathcal{A}, \mathbb{C})$ and $(\mathbb{C}, \mathcal{B})$ are bi-freely independent with respect to $\tau \otimes \varphi$.

Theorem (S; 2024)

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Let $a_1, \ldots, a_n \in \mathcal{A}$ be free with respect to φ and $b_1, \ldots, b_n \in \mathcal{A}$ be free with respect to φ. Then

$$\{(a_k \otimes 1, 1 \otimes 1)\}_{k=1}^n \cup \{(1 \otimes 1, 1 \otimes b_k)\}_{k=1}^n$$

are bi-free in $(\mathcal{A} \otimes \mathcal{A}, \varphi \otimes \varphi)$.

Independence Inside Bi-Free Independence

Note if (A_1, B_1) is bi-free from (A_2, B_2), then

- A_1 is free from A_2,
- B_1 is free from B_2,
- A_1 is independent from B_2, and
- A_2 is independent from B_1.

The converse does not hold! Moreover by [S, 2016]

- Boolean independence can be modelled using bi-free independence.
- (anti-)monotone independence can be modelled using bi-free independence.

Thus all five notions of independence in non-commutative probability can be studied via bi-free independence. Furthermore:

- Free-Free-Boolean independence can modelled using bi-free independence (Pepper; 2025).

Bi-Free Central Limit

Theorem (Voiculescu; 2014), (Huang, Wang; 2016)

Let $(\mathcal{A}, \varphi)$ *be a non-commutative probability space and let* $((X_k, Y_k))_{k\geq 1}$ *be a sequence of bi-freely independent pairs in* $\mathcal{A}$ *such that* X_k *and* Y_k *are self-adjoint,* $[X_k, Y_k] = 0$, $\varphi(X_k) = \varphi(Y_k) = 0$, $\varphi(X_k^2) = \varphi(Y_k^2) = 1$, *and* $\varphi(X_k Y_k) = c$ *for some* $c \in (-1, 1)$*. Let*

$$S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k \qquad \text{and} \qquad T_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k.$$

Then the joint distribution of (S_n, T_n) *with respect to* φ *converges to*

$$d\mu_{(X,Y)}(x, y) = \frac{1 - c^2}{4\pi^2} \frac{\sqrt{4 - x^2}\sqrt{4 - y^2}}{(1 - c^2)^2 - c(1 + c^2)xy + c^2(x^2 + y^2)}\, dx\, dy.$$

Bi-Free Convolutions

Given a pair of self-adjoint operators (X, Y) in $(\mathcal{A}, \varphi)$:

- The *two-variable Green's function* is defined by

$$G_{X,Y}(z, w) = \varphi((z - X)^{-1}(w - Y)^{-1})$$

  for all $(z, w) \in (\mathbb{C} \setminus \mathbb{R})^2$.

- The *reduced bi-free partial R-transform of* (X, Y) is the power series defined by

$$\tilde{R}_{X,Y}(z, w) = \sum_{n,m \geq 1} \kappa_{n,m}(X, Y) z^n w^m.$$

- (X_1, Y_1) and (X_2, Y_2) bi-free implies

$$\tilde{R}_{X_1+X_2,Y_1+Y_2}(z, w) = \tilde{R}_{X_1,Y_1}(z, w) + \tilde{R}_{X_2,Y_2}(z, w).$$

- (Voiculescu; 2016), (S; 2016) showed

$$\tilde{R}_{X,Y}(z, w) = 1 - \frac{zw}{G_{X,Y}\left(K_X(z), K_Y(w)\right)}.$$

Bi-Free Convolutions

- (Huang, Wang; 2016) If $[X, Y] = 0$ and thus define a measure $\mu_{(X,Y)}$ on $\mathbb{R}^2$,

$$d\mu_{(X,Y)}(x, y) = \lim_{\epsilon \searrow 0} \Im \left(\frac{G_{X,Y}(x + i\epsilon, y + i\epsilon) - G_{X,Y}(x + i\epsilon, y - i\epsilon)}{2\pi^2 i} \right).$$

- (Voiculescu; 2016), (S; 2016) showed (X_1, Y_1) and (X_2, Y_2) bi-free implies

$$S_{(X_1X_2,Y_1Y_2)}(z, w) = S_{(X_1,Y_1)}(z, w)S_{(X_2,Y_2)}(z, w).$$

- (Huang, Wang; 2018), (S; 2018) showed (X_1, Y_1) and (X_2, Y_2) bi-free implies

$$S^{\mathrm{op}}_{(X_1X_2,Y_2Y_1)}(z, w) = S^{\mathrm{op}}_{(X_1,Y_1)}(z, w)S^{\mathrm{op}}_{(X_2,Y_2)}(z, w).$$

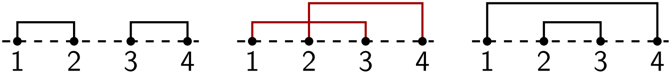
A Different Approach

How can we handle bi-free independence?

Definition

A partition π on $\{1, \ldots, n\}$ is said to be *non-crossing* if whenever $i < j < k < \ell$ are such that $i \sim_\pi k$ and $j \sim_\pi \ell$, then $i \sim_\pi j \sim_\pi k \sim_\pi \ell$.

$\{\{1,2\},\{3,4\}\}$ $\quad$ $\{\{1,3\},\{2,4\}\}$ $\quad$ $\{\{1,4\},\{2,3\}\}$



Free Cumulants

There is a connection between non-crossing partitions and free probability via the free cumulant functions:

$$\varphi(X_1 \cdots X_n) = \sum_{\pi \in NC(n)} \kappa_\pi(X_1, \ldots, X_n)$$

$$\kappa_{1_n}(X_1, \ldots, X_n) = \sum_{\pi \in NC(n)} \varphi_\pi(X_1, \ldots, X_n)\mu_{NC}(\pi, 1_n).$$

Theorem (Speicher; 1994)

Let $\{\mathcal{A}_k\}_{k \in K}$ be unital C^*-subalgebras of a non-commutative probability space $(\mathcal{A}, \varphi)$. Then $\{\mathcal{A}_k\}_{k \in K}$ are freely independent with respect to φ if and only if mixed free cumulants vanish; that is

$$\kappa_{1_n}(X_1, \ldots, X_n) = 0$$

for all $X_j \in \mathcal{A}_{k_j}$ provided $k_{j_1} \neq k_{j_2}$ for some j_1, j_2.

The Permutation

Let $\chi : \{1, \ldots, n\} \to \{\ell, r\}$ designate whether the k^{th} operator is considered a left operator ($\chi(k) = \ell$) or a right operator ($\chi(k) = r$). If

$$\chi^{-1}(\{\ell\}) = \{k_1 < k_2 < \cdots < k_m\}$$
$$\chi^{-1}(\{r\}) = \{k_{m+1} > k_{m+2} > \cdots > k_n\}$$

define the permutation s_χ of $\{1, \ldots, n\}$ via $s_\chi(t) = k_t$.

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Consider $\chi : \{1, \ldots, 7\} \to \{\ell, r\}$ with $\chi^{-1}(\{\ell\}) = \{1, 4, 6, 7\}$.
Suppose $Z_1, \ldots, Z_7$ are operators (either left or right based on χ) for which we want to consider $Z_1 \cdots Z_7 \Omega$.

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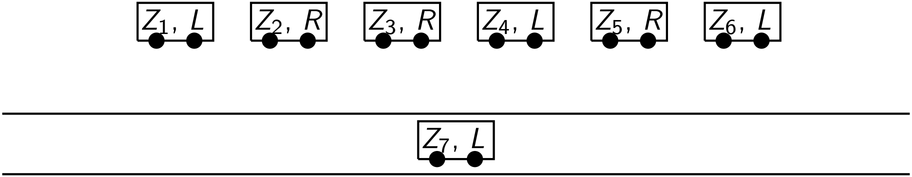

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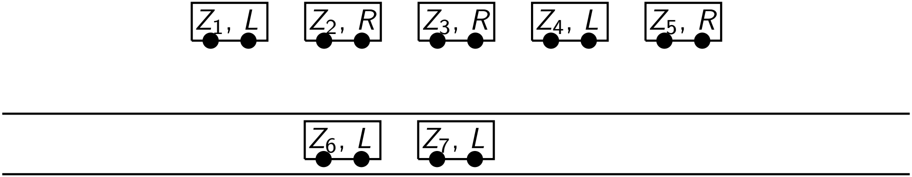

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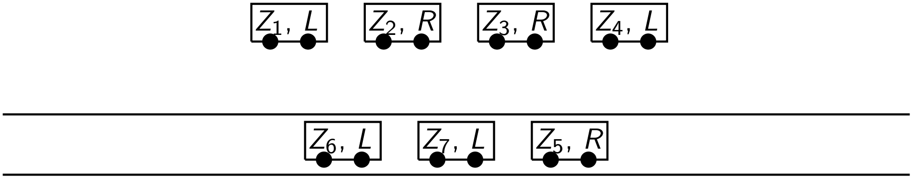

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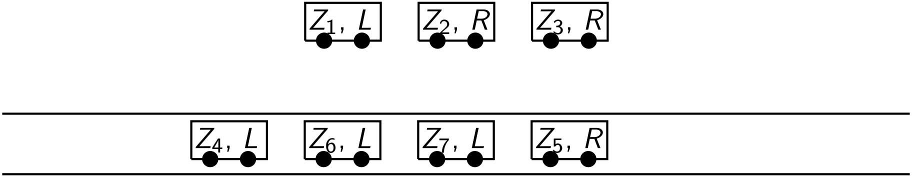

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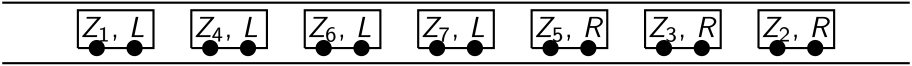

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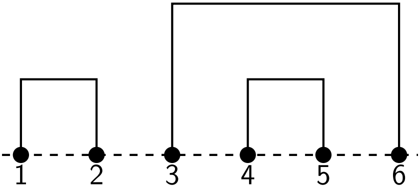
Definition (Mastnak, Nica; 2014)

Given $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, a partition π of $\{1, \ldots, n\}$ is said to be *bi-non-crossing with respect to* χ if the partition $s_\chi^{-1} \cdot \pi$ (the partition obtained by applying s_χ^{-1} to each block of π) is non-crossing.

Transferring Free Diagrams to Bi-Free



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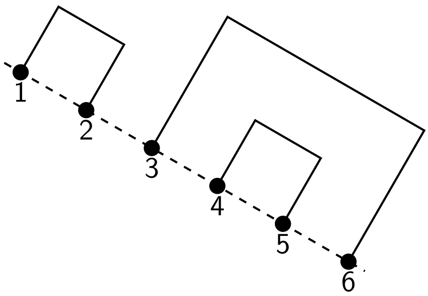

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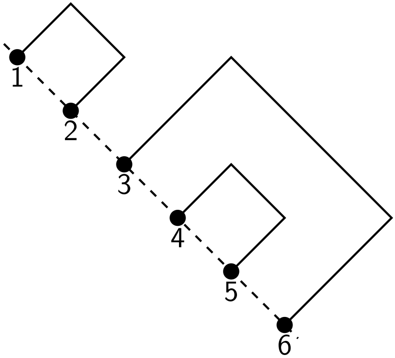

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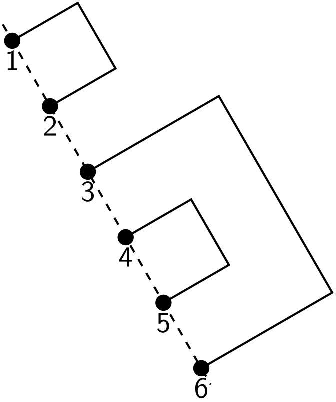

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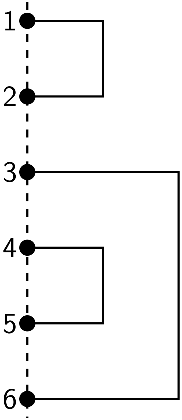

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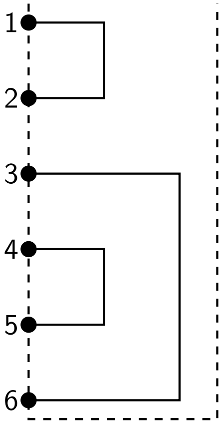

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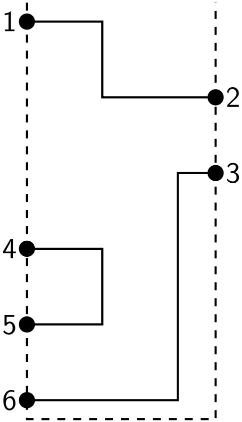


Bi-Free Cumulants

Theorem (Charlesworth, Nelson, S; 2015)

Let $\{(\mathcal{A}_{\ell,k}, \mathcal{A}_{r,k})\}_{k\in K}$ *be pairs of unital* C^**-subalgebras of a non-commutative probability space* $(\mathcal{A}, \varphi)$*. Then* $\{(\mathcal{A}_{\ell,k}, \mathcal{A}_{r,k})\}_{k\in K}$ *are bi-freely independent with respect to* φ *if and only if mixed bi-free cumulants vanish.*

That is, for all $\chi : \{1, \ldots, n\} \to \{\ell, r\}$*,* $\epsilon : \{1, \ldots, n\} \to K$ *non-constant, and* $Z_m \in A_{\chi(m),\epsilon(m)}$*,*

$$\kappa_\chi(Z_1, \ldots, Z_n) := \sum_{\pi \in BNC(n)} \varphi_\pi(Z_1, \ldots, Z_n)\mu_{BNC}(\pi, 1_n) = 0.$$

CLT for Quantum Channels

Lacien, Santos, Youssef's CLT can be obtained via bi-free computations. It begins the same:

Definition

Given a partition $\pi \in \mathcal{P}(n)$, the *intersection graph of* π is the graph whose vertices are the blocks of π where two blocks are connected via an edge if and only if they cross. Let $\mathcal{P}_2^{\mathrm{bicon}}(n)$ denote the set of all pair partitions π on $\{1, \ldots, n\}$ such that the intersection graph of π is bipartite and connected.

Lemma (Lacien, Santos, Youssef; 2024)

If κ_n denotes the n^{th} free cumulant of our target distribution, then

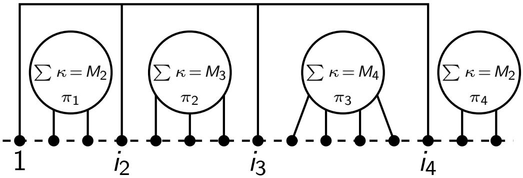
$$\kappa_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n = 2 \\ 2\left(\frac{q}{2}\right)^{\frac{n}{2}} \left|\mathcal{P}_2^{\mathrm{bicon}}(n)\right| & \text{if } n \text{ is even and } n \geq 4 \end{cases}.$$

Cumulants to Moments

Corollary

Let M_n denote the n^{th} moment of our target distribution. Then $M_n = 0$ if n is odd, $M_2 = 1$, and, if n is even with $n \geq 4$, we have that M_n is

$$\sum_{\substack{0 \leq k_1, k_2 \leq n-2 \\ k_1 + k_2 = n-2}} M_{k_1} M_{k_2} + \sum_{j \geq 2} \sum_{\substack{0 \leq k_1, \ldots, k_{2j} \leq n-2j \\ k_1 + \cdots + k_{2j} = n-2j}} 2 \left(\frac{q}{2}\right)^j |\mathcal{P}_2^{\text{bicon}}(2j)| M_{k_1} \cdots M_{k_{2j}}.$$



The Bi-Free Computations

With

$$S_N = \frac{1}{\delta\sqrt{N}} \sum_{k=1}^{N} (a_k \otimes a_k - \lambda 1 \otimes \lambda 1),$$

we need show $\lim_{N\to\infty}(\varphi \otimes \varphi)(S_N^n) = M_n$.

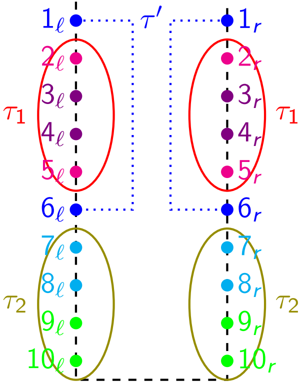
Expanding $(\varphi \otimes \varphi)(S_N^n)$ yields moments of the form

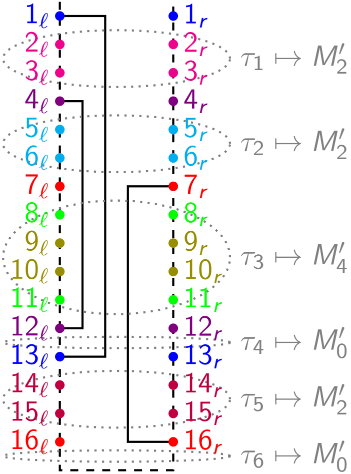
$$(\varphi \otimes \varphi)\left(\prod_{k=1}^{n} L_{\theta(k)} R_{\theta(k)}\right)$$

for maps $\theta : [n] \to [N]$.

Asymptotics, bi-freeness, and equal distributions allows us to pair up the operators that occur. We then expand moments via bi-free cumulants, and add over bi-non-crossing partitions with the same block containing the first elements.

The Bi-Free Diagrams





Operator-Valued Free Independence

Replace $(\mathcal{A}, \varphi)$ with $(\mathcal{A}, E)$ where $E : \mathcal{A} \to \mathcal{B}$ is a conditional expectation of $\mathcal{A}$ onto a unital C*-subalgebra $\mathcal{B}$.

Proposition

Let $\mathcal{B}$ be a unital C*-algebra and let $\{\mathcal{A}_k\}_{k \in K}$ be freely independent unital C*-algebras in a non-commutative probability space $(\mathcal{A}, \varphi)$. Then $\{\mathcal{A}_k \otimes \mathcal{B}\}_{k \in K}$ are freely independent over $\mathcal{B}$ with respect to $\Phi = \varphi \otimes I_{\mathcal{B}} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$.

When $\mathcal{A} = \mathcal{L}_{\infty}(\mu)$ and $\mathcal{B} = M_n$, one obtains the picture for random matrices where $\varphi = \mathbb{E}$ and $\mathrm{tr} \circ \Phi$ are used.

Operator-Valued Bi-Free Independence

Theorem (S; 2016)

Let $\mathcal{B}$ be a unital C^*-algebra and let $\{(\mathcal{A}_{\ell,k}, \mathcal{A}_{r,k})\}_{k\in K}$ be bi-freely independent pairs of unital C^*-algebras in a non-commutative probability space $(\mathcal{A}, \varphi)$. Consider

$$\mathfrak{A} = \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{B}^{\mathrm{op}}$$

and $E : \mathfrak{A} \to \mathcal{B}$ by

$$E(T \otimes x \otimes y) = \varphi(T)xy.$$

Then $\{(\mathcal{A}_{\ell,k} \otimes \mathcal{B} \otimes 1, \mathcal{A}_{r,k} \otimes 1 \otimes \mathcal{B}^{\mathrm{op}})\}_{k\in K}$ are bi-freely independent over $\mathcal{B}$ with respect to E.

For $\mathfrak{A} = \mathcal{L}_{\overline{\infty}}(\mu) \otimes M_n \otimes M_n^{\mathrm{op}}$, let $L, R : \mathcal{L}_{\overline{\infty}}(\mu) \otimes M_n \to \mathcal{A}$ be defined by

$$L(f \otimes T) = f \otimes T \otimes I_n \qquad \text{and} \qquad R(g \otimes T) = g \otimes I_n \otimes T.$$

Bi-Free Matrix Models

Theorem (S; 2017)

If $\{W_j\}_{j=1}^{2d}$ are asymptotically freely independent with respect to $\mathrm{tr} \circ \mathbb{E}$, then $\{(L(W_k), R(W_k))\}_{j=1}^{2d}$ are asymptotically bi-free independent with respect to $\mathrm{tr} \circ E$.

Thus

$$\Gamma = \{(L(W_j), R(1)\}_{j=1}^{d} \cup \{(L(1), R(W_j))\}_{j=d+1}^{2d}$$

are asymptotically bi-freely independent with respect to $\mathrm{tr} \circ E$. Hence if W_j converges weakly in probability and expectation to μ_j and $a_1, \ldots, a_d$ be freely independent random variables with distributions $\mu_1, \ldots, \mu_d$, then the joint distribution of Γ tends to the joint distribution of

$$\{(a_j \otimes 1, 1 \otimes 1)\}_{j=1}^{d} \cup \{(1 \otimes 1, 1 \otimes a_j)\}_{j=d+1}^{2d}.$$

A form of Lacien, Santos, and Youssef's result immediately follows.

Significance

Bi-free probability can solve problems involving mixing of lefts and rights.

Thanks for Listening!