How Certain Multi-Algebra Independences Can Arise in Bi-Free Probability

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Introduction

In a few papers, Liu noticed that by tweaking the definition of bi-free independence, one could define other types of multi-algebra independences. He developed the several common tools, like central limit theorems and cumulants for these new independences.

But we do not discard bi-free independence. In fact, bi-free probability remains robust in its ability to model these independences. We will show that one can embed many of these multi-algebra independences into bi-free families.

Outline

1. Liu's multi-algebra independences
2. Upgrading Skoufranis' bi-free representation for Boolean independence to free-free-Boolean independence
3. Bi-free combinatorics of free-free-Boolean independence
4. A sketch of the main argument

Notations and Setup

- B will always denote a unital algebra over $\mathbb{C}$.
- Triples of the form $(\mathcal{A}, E, \varepsilon)$ will denote B-B-probability spaces, where $\varepsilon : B \otimes B^{\mathrm{op}} \to \mathcal{A}$ is a unital homomorphism, and E is the corresponding expectation.
- Triples of the form $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ will denote B-B-bimodules with specified B-projections

$$\mathcal{X} = B \oplus \mathring{\mathcal{X}}$$

- $\mathcal{L}(\mathcal{X})$ will denote the $\mathbb{C}$-linear operators on $\mathcal{X}$, while $\mathcal{L}_\ell$ and $\mathcal{L}_r$ denote the left and right algebras of $\mathcal{L}(\mathcal{X})$. $E_{\mathcal{L}(\mathcal{X})}$ is the expectation given by

$$E_{\mathcal{L}(\mathcal{X})}(T) = pT1_B$$

Product Spaces

Given $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$, form their reduced free product $(\mathcal{X}, \mathring{\mathcal{X}}, p)$, where

$$\mathcal{X} = B \oplus \bigoplus_{n\geq 1} \bigoplus_{\substack{k_1,\ldots,k_n\in K \\ k_1\neq\cdots\neq k_n}} \mathring{\mathcal{X}}_{k_1} \otimes_B \cdots \otimes_B \mathring{\mathcal{X}}_{k_n}.$$

Then we have the following projections, for each $k \in K$

- the k^{th} Boolean projection, $P_{\uplus,k}$ which projects onto the subspace

$$\mathcal{X}_{\uplus}(k) = B \oplus \mathring{\mathcal{X}}_k \subseteq \mathcal{X}$$

- the k^{th} monotone projection, $P_{\lhd,k}$, if K is ordered, which projects onto the subspace

$$\mathcal{X}(\lhd, k) = B \oplus \bigoplus_{n\geq 1} \bigoplus_{k_1<\cdots<k_n=k} \mathring{\mathcal{X}}_{k_1} \otimes_B \cdots \otimes_B \mathring{\mathcal{X}}_{k_n}$$

Bi-Free Independence (when $B = \mathbb{C}$)

Definition

A family of pairs $\Gamma = \{(C_k, D_k)\}_{k \in K}$ of unital subalgebras of $\mathcal{A}$ is *bi-free* if for each $k \in K$, there are $(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)$ and unital homomorphisms

$$l_k : C_k \to \mathcal{L}(\mathcal{X}_k) \quad \text{and} \quad r_k : D_k \to \mathcal{L}(\mathcal{X}_k)$$

and the joint distribution of Γ is equal to the joint distribution of the family

$$\{\lambda_k(l_k(C_k)), \rho_k(r_k(D_k))\}_{k \in K}$$

in $(\mathcal{L}(\mathcal{X}), E_{\mathcal{L}(\mathcal{X})})$.

Liu's Multi-Algebra Independences ($B = \mathbb{C}$)

Definition (Liu, 2019)

In $(\mathcal{A}, \varphi)$, let $\Gamma = \{(C_k, D_k)\}_{k \in K}$ be a family of pairs of subalgebras of $\mathcal{A}$. Suppose for each $k \in K$ there are $(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)$ and maps $\ell_i : C_k \to \mathcal{L}(\mathcal{X}_k)$, $r_i : D_k \to \mathcal{L}(\mathcal{X}_k)$, and projections P_k and Q_k such that the joint distribution of Γ equals the joint distribution of the family

$$\{(P_k \lambda_k(\ell_k(C_k)) P_k, Q_k \rho_k(r_k(D_k)) Q_k)\}_{k \in K}$$

in the the space $(\mathcal{L}(\mathcal{X}), E_{\mathcal{L}(\mathcal{X})})$. Say Γ is

- *free-Boolean independent* if $P_k = I_{\mathcal{X}}$ and $Q_k = P_{\uplus,k}$ for all $k \in K$,
- *free-monotone independent* if $P_k = I_{\mathcal{X}}$ and $Q_k = P_{\lhd,k}$ for all $k \in K$.

Free-Free-Boolean Independence ($B = \mathbb{C}$)

Definition

$(\mathcal{A}, \varphi)$ a non-commutative probability space. A family $\Gamma = \{(A_k^\ell, A_k^r, A_k^b)\}_{k \in K}$ of triples of subalgebras of $\mathcal{A}$ is *free-free-Boolean independent* if for each $k \in K$

- A_k^ℓ, A_k^r, are unital, A_k^b is not necessarily.
- there are $(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)$ and homomorphisms

$$l_k : A_k^\ell \to \mathcal{L}(\mathcal{X}_k), \quad r_k : A_k^r \to \mathcal{L}(\mathcal{X}_k), \text{ and } m_k : A_k^b \to \mathcal{L}(\mathcal{X}_k),$$

  where ℓ_k and r_k are unital,

the joint distribution of Γ equals the joint distribution of the family

$$\{(\lambda_k(l_k(A_k^\ell)), \rho_k(r_k(A_k^r)), P_{\uplus,k}\lambda_k(m_k(A_k^b))P_{\uplus,k})\}_{k \in K}$$

in $(\mathcal{L}(\mathcal{X}), E_{\mathcal{L}(\mathcal{X})})$.

Free-Free-Boolean Independence with Amalgamation

For general B, we work in $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$. The definition remains the same, but we require in addition for each $k \in K$,

$$\varepsilon(B \otimes 1_B)\, A_k^\ell\, \varepsilon(B \otimes 1_B) \subseteq A_k^\ell \text{ and } l_k : A_k^\ell \to \mathcal{L}_\ell(\mathcal{X}_k),$$

$$\varepsilon(B \otimes 1_B)\, A_k^b\, \varepsilon(B \otimes 1_B) \subseteq A_k^b \text{ and } m_k : A_k^b \to \mathcal{L}_\ell(\mathcal{X}_k),$$

and

$$\varepsilon(1_B \otimes B^{\mathrm{op}})\, A_k^r\, \varepsilon(1_B \otimes B^{\mathrm{op}}) \subseteq A_k^r \text{ and } r_k : A_k^r \to \mathcal{L}_r(\mathcal{X}_k).$$

Lingering Independences

Note that if $\{(A_k^\ell, A_k^r, A_k^b)\}_{k \in K}$ is free-free-Boolean over B, then

- $\{(A_k^\ell, A_k^r)\}_{k \in K}$ is bi-free over B,
- $\{A_k^b\}_{k \in K}$ is Boolean independent over B, and
- $\{(A_k^\ell, A_k^b)\}_{k \in K}$ is free-Boolean over B, in the sense of Liu + Zhong.

Bi-Free Representation of Boolean Independence

Skoufranis (2015) had the following construction to represent Boolean independence with bi-free operators. Starting with a B-valued probability space $(\mathcal{A}, \Phi)$, consider $(\mathcal{A} \oplus \mathcal{A}, \mathring{\mathcal{A}} \oplus \mathcal{A}, \Phi \oplus 0)$. For $Z \in \mathcal{A}$ consider

$$T_Z(Z_1 \oplus Z_2) = ZZ_2 \oplus 0,$$

and

$$S_{1_B}(Z_1 \oplus Z_2) = 0 \oplus Z_1.$$

Then if $\{A_k^b\}_{k \in K}$ is Boolean over B, for each $Z \in A_k^b$, send

$$Z \longmapsto \lambda_k(T_Z)\rho_k(S_{1_B}).$$

Bi-Free Boolean B-Systems

Definition (Skoufranis, 2015)

Let $\{(C_k, D_k)\}_{k\in K}$ be bi-free in $(\mathcal{A}, E, \varepsilon)$. For each $k \in K$, let

$$C_k' \subseteq C_k \quad \text{and} \quad D_k' \subseteq D_k \cap \mathcal{A}_\ell$$

be subsets such that $L_b C_k', C_k' L_b \subseteq C_k'$ for all $b \in B$. Then $\{C_k', D_k'\}_{k\in K}$ is a *bi-free Boolean B-system* if for all $k \in K$ and $n \geq 0$,

1. $(C_k')^2 = \{0\} = (D_k')^2$,
2. $E(C_k'(D_k'C_k')^n) = \{0\}$, and
3. $E(D_k'(C_k'D_k')^n) = \{0\}$.

Skoufranis showed that for such a structure, the family $\{\mathrm{alg}(C_k'D_k')\}_{k\in K}$ is Boolean independent over B.

Embedding Result for Boolean Independence

Theorem (Skoufranis, 2015)

Given a Boolean independent family $\{A_k\}_{k \in K}$ in $(\mathcal{A}, \Phi)$, there is a bi-free Boolean B-system $\{C'_k, D'_k\}_{k \in K}$ in some $(\mathcal{A}_0, E, \varepsilon)$ and injective B-linear maps $\beta_k \to C'_k D'_k$ such that

$$E(\beta_{k_1}(Z_1) \cdots \beta_{k_n}(Z_n)) = \Phi(Z_1 \cdots Z_n)$$

for all $Z_m \in A_{k_m}$, $k_m \in K$.

We build a close analog of this process, and aim for an analogous result for free-free-Boolean independence.

Adjusting for free-free-Boolean Independence

The construction works almost the same, except we need to keep track of lefts and rights simultaneously. So we need a proper faithful representation of $\mathcal{A}$ as operators. We use the following construction from [Nelson, Charlesworth, Skoufranis, 2015]:

Given $(\mathcal{A}, E, \varepsilon)$, let

$$\tilde{\mathcal{A}} = B \oplus (\ker(E)/\mathrm{span}\{TL_b - TR_b | T \in \mathcal{A}, b \in B\})$$

Now let $\theta : \mathcal{A} \to \mathcal{L}(\tilde{\mathcal{A}})$ be the homomorphism such that

$$\theta(T)(b) = E(TL_b) \oplus q(TL_b - L_{E(TL_b)})$$

and

$$\theta(T)(q(A)) = E(TA) \oplus q(TA - L_{E(TA)})$$

Operator Representation

Now consider the space $(\tilde{\mathcal{A}} \oplus \tilde{\mathcal{A}}, \mathring{\tilde{\mathcal{A}}} \oplus \tilde{\mathcal{A}}, \tilde{E} \oplus 0)$, and as before, define for each $Z \in \mathcal{A}$, let

$$T_Z(a_1 \oplus a_2) = \theta(Z)a_2 \oplus 0,$$

and

$$S_{1_B}(a_1 \oplus a_2) = 0 \oplus a_1.$$

But now we also consider operators D_Z defined by

$$D_Z(a_1 \oplus a_2) = \theta(Z)a_1 \oplus \theta(Z)a_2.$$

Given a free-free-Boolean independent family $\{(A_k^\ell, A_k^r, A_k^b)\}_{k \in K}$, for each $Z_1 \in A_k^\ell, Z_2 \in A_k^r$, and $Z_3 \in A_k^b$ send

$$Z_1 \mapsto \lambda_k(D_{Z_1}), \quad Z_2 \mapsto \rho_k(D_{Z_2}), \quad \text{and } Z_3 \mapsto \lambda_k(T_{Z_3})\rho_k(S_{1_B}).$$

Bi-Free ffb B-systems

The abstract structures we get then look like

Definition

Consider a family $\{(A_k^\ell, A_k^r, C_k, D_k)\}_{k\in K}$ of quadruples of subalgebras of $\mathcal{A}$ such that (A_k^ℓ, A_k^r) is a pair of B-faces for each $k \in K$, and the family $\{(\mathrm{alg}(A_k^\ell, C_k), \mathrm{alg}(A_k^r, D_k)\}_{k\in K}$ is bi-free. For each $k \in K$, let $C_k' \subseteq C_k$ and $D_k' \subseteq D_k \cap \mathcal{A}_\ell$ be subsets such that $L_b C_k', C_k' L_b \subseteq C_k'$ for all $b \in B$. Then $\{A_k^\ell, A_k^r, C_k', D_k'\}_{k\in K}$ is a *bi-free ffb B-system* if for all $k \in K$ and $n \geq 0$,

1. $C_k' \mathbb{A}_k C_k' = \{0\} = D_k' \mathbb{A}_k D_k'$,
2. $E(\mathbb{A}_k C_k' \mathbb{A}_k (D_k' \mathbb{A}_k C_k')^n \mathbb{A}_k) = \{0\}$, and
3. $E(\mathbb{A}_k D_k' \mathbb{A}_k (C_k' \mathbb{A}_k D_k')^n \mathbb{A}_k) = \{0\}$

where $\mathbb{A}_k = \mathrm{alg}(A_k^\ell, A_k^r)$.

Towards a Complete Embedding Result

Our construction above gives an embedding of a free-free-Boolean independent family $\{(A_k^\ell, A_k^r, A_k^b)\}_{k\in K}$ into a bi-free ffb B-system that preserves the individual joint distributions (for fixed $k \in K$).

To guarantee total joint distribution preservation, we need to know that for a bi-free ffb B system $\{(A_k^\ell, A_k^r, C_k', D_k')\}_{k\in K}$, that the family $\{(A_k^\ell, A_k^r, \mathrm{alg}(C_k' D_k'))\}_{k\in K}$ is free-free-Boolean independent.

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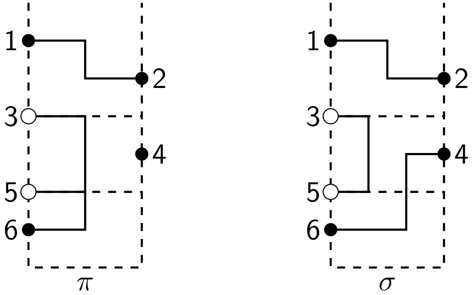
We turn to some combinatorics for help.

Interval Bi-Non-Crossing Partitions

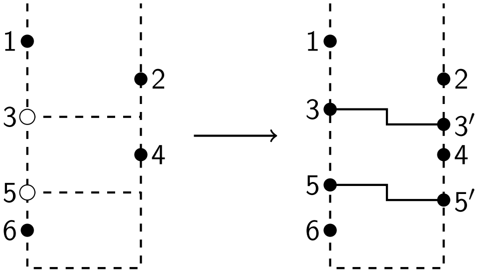
Liu observed that the appropriate partitions to use for the free-free-Boolean cumulants are the *interval bi-non-crossing partitions*. They arise from labelings $\chi : \{1, \cdots, n\} \to \{\ell, r, b\}$. For example, $n = 6$ and $\chi^{-1}(\{\ell\}) = \{1, 6\}$ and $\chi^{-1}(\{b\}) = \{3, 5\}$, then consider



π is an interval bi-non-crossing partition, while σ is not.

Comparison with Bi-Non-Crossing Diagrams

Replacing each Boolean-labeled number a connected left and right should give us the same combinatorial behaviour.



$BNC_{ffb}(\hat{\chi})$

Given $n \in \mathbb{N}$ and $\hat{\chi} : \{1, \ldots, n\} \to \{\ell, r, b\}$, let $\chi : \{1, \ldots, n + |\hat{\chi}^{-1}(\{b\})|\} \to \{\ell, r\}$ be the labeling obtained by replacing all instances of b in $\hat{\chi}$ with ℓr in that order. Let $f(i) = i + |\hat{\chi}^{-1}(\{b\}) \cap [1, i-1]|$ for each $i \in \{1, \ldots, n\}$.

Let $BNC_{\mathrm{ffb}}(\hat{\chi}) \subseteq BNC(\chi)$ such that whenever $\pi \in BNC_{\mathrm{ffb}}(\hat{\chi})$ and $i \in \hat{\chi}^{-1}(\{b\})$, then $f(i)$ and $f(i+1)$ share a block of π.

It follows that $BNC_{\mathrm{ffb}}(\hat{\chi})$ is an interval of $BNC(\chi)$ with maximum element 1_χ. So we can restrict μ_{BNC} to $BNC_{\mathrm{ffb}}(\hat{\chi})$ to get the Möbius inversion on $BNC_{\mathrm{ffb}}(\hat{\chi})$.

Operator-Valued Bi-Multiplicative Functions

Definition

In $(\mathcal{A}, E, \varepsilon)$, fix $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, $\pi \in BNC(\chi)$, and $Z_1, \ldots, Z_n \in \mathcal{A}$. Define $E_\pi(Z_1, \ldots, Z_n) \in B$ recursively. Let V be the block of π with largest minimum element.

- If $\pi = 1_\chi$, then $E_\pi(Z_1, \ldots, Z_n) = E(Z_1 \cdots Z_n)$.
- If there is some k such that $V = \{k+1, \ldots, n\}$, then

$$E_\pi(Z_1, \ldots, Z_n) = E_{\pi|_{V^c}}(Z_1, \ldots, Z_k L_{E(Z_{k+1}\cdots Z_n)})$$

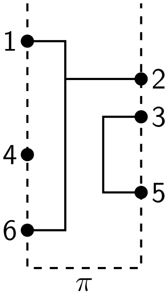
- Otherwise $\min(V)$ is adjacent to a spine, say of the block W. Let k be the smallest element of W such that $k > \min V$, define

$$E_\pi(Z_1, \ldots, Z_n) = E_{\pi|_{V^c}}(Z_1, \ldots, c_V Z_k, \ldots, Z_n),$$

where $c_V = L_{E_{\pi|_V}(Z_1, \ldots, Z_n)}$ if $\chi(\min(V)) = \ell$ and $c_V = R_{E_{\pi|_V}(Z_1, \ldots, Z_n)}$ otherwise.

A Simple Example

Consider the partition given by the diagram



Then

$$E_\pi(Z_1, \ldots, Z_n) = E(Z_1 Z_2 R_{E(Z_3 Z_5)} L_{E(Z_4)} Z_6).$$

Vanishing of non-$BNC_{\text{ffb}}(\hat{\chi})$ moments

Theorem

Let $\{(A_k^\ell, A_k^r, C_k', D_k')\}_{k\in K}$ be a bi-free ffb B-system in $(\mathcal{A}, E, \varepsilon)$. With $n \in \mathbb{N}$, $\hat{\chi}$, χ, f, as before, $n' = n + |\hat{\chi}^{-1}(\{b\})|$, and let $\epsilon : \{1, \ldots, n'\} \to K$ such that $\epsilon(i) = \epsilon(i+1)$ whenever $i \in f(\hat{\chi}^{-1}(\{b\}))$. Let

$$Z_i \in \begin{cases} A_{\epsilon(i)}^{\chi(i)} & \text{if } i \in f(\{1, \ldots, n\} \setminus \hat{\chi}^{-1}(\{b\}) \\ C_{\epsilon(i)}' & \text{if } i \in f(\hat{\chi}^{-1}(\{b\}) \\ D_{\epsilon(i)}' & \text{otherwise} \end{cases}.$$

Then

$$E_\pi(Z_1, \ldots, Z_{n'}) = 0$$

Unless $\pi \in BNC_{ffb}(\hat{\chi})$.

Sketch of Proof - I

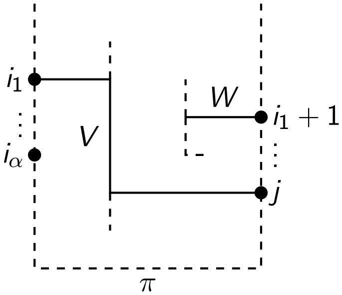
We simply show that if $i \in \hat{\chi}^{-1}(\{b\})$ with $f(i)$ and $f(i)+1$ not sharing a block of π, then

$$E_\pi(Z_1, \cdots, Z_{n'}) = 0.$$

The argument is inductive, but largely illustrated by a base case. Let $f(\hat{\chi}^{-1}(\{b\}) = \{i_1 < \cdots < i_m\}$. Let $i_1 \in V \in \pi$ and suppose $i_1 + 1 \in W \neq V$.
Consider two cases corresponding to V. Either there is some $j > i_1$ such that $\chi(j) = r$ and $j \in V$, or there is no such j.

Sketch of Proof - Case 1

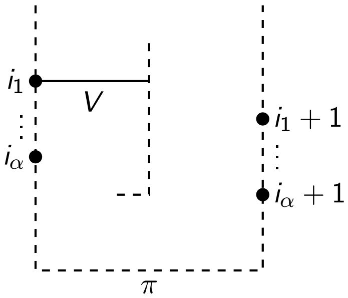
In the first case, we have a diagram that looks like



In which case, $E_{\pi|_W}(Z_1, \ldots, Z_n) = 0$ by definition of bi-free ffb B-system. This implies that $E_\pi(Z_1, \ldots, Z_n) = 0$.

Sketch of Proof - Case 2

Otherwise we get a diagram like this



In which case $E_{\pi|_V}(Z_1, \ldots, Z_n) = 0$, so our moment

$$E_\pi(Z_1, \ldots, Z_n) = 0.$$

The Final Stretch

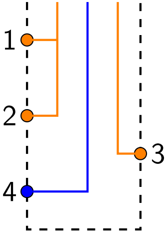
Our goal is to show that given a bi-free ffb B-system $\{(A_k^\ell, A_k^r, C_k', D_k')\}_{k\in K}$, it should follow that the family $\{(A_k^\ell, A_k^r, \mathrm{alg}(C_k' D_k'))\}_{k\in K}$ is free-free-Boolean independent over B.

To do so, we need to be able to associate our representation with the definition. That is, if $Z = S_C S_D \in \mathrm{alg}(C_k' D_k')$ then how do $\lambda_k(S_C)\rho_k(S_D)$ and $P_{\uplus,k}\lambda_k(m_k(Z))P_{\uplus,k}$ compare in expectation, when mixed in with other operators?

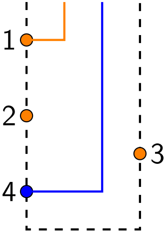
The most convenient tool for the job appears to be the LR diagrams.

LR diagrams

LR-diagrams are combinatorial tools, similar to the bi-non-crossing partitions, but where spines are allowed to reach the top gap. Consider for example



D_1



D_2

The LR Diagram Calculus - I

The most useful property for our purposes is that these diagrams used to keep track of the vector components of the left and right actions of operators on free product. Suppose we have (C', D') and (C'', D'') which are bi-free. Let

$$\mu_i(Z_i) = \begin{cases} \lambda_{\epsilon(i)}(\ell_{\epsilon(i)}(Z_i)) & \text{if } Z_i \in C^{\epsilon(i)} \\ \rho_{\epsilon(i)}(r_{\epsilon(i)}(Z_i)) & \text{otherwise} \end{cases}$$

Then

$$\mu_1(Z_1)\cdots\mu_n(Z_n)1_B = \sum_{D \in LR(\chi,\epsilon)} c_D E_D(\mu_1(Z_1), \ldots, \mu_n(Z_n)),$$

for some constants c_D.

The LR Diagram Calculus - II

Here the E_D correspond to the vector components as follows. Take for example the diagram D_1 before, and let $n = 4$, $\epsilon(4) ='' $ and $\epsilon(1) = \epsilon(2) = \epsilon(3) =' $ Then

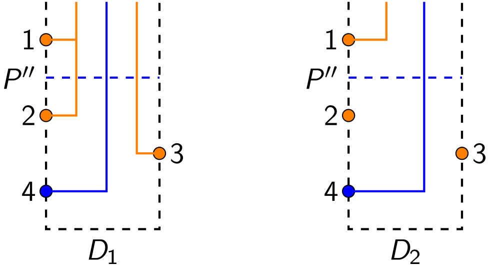
$$E_{D_1}(\mu_1(Z_1), \mu_2(Z_2), \mu_3(Z_3), \mu_4(Z_4)) = \\ (1 - p')Z_1 Z_2 1_B \otimes (1 - p'')Z_4 1_B \otimes (1 - p')Z_3 1_B$$

Boolean Projections with LR diagrams

Boolean projections annihilate many diagrams when they are first applied. Consider for example the product

$$\mu_1(Z_1)P''\mu_2(Z_2)\mu_3(Z_3)\mu_4(Z_4)1_B,$$

then looking at diagrams that appear therein,



Here E_{D_1} would get annihilated by the projection P'', while E_{D_2} would not.

Showing Free-Free-Boolean Independence - I

We need an arbitrary moment of elements of $\{(A_k^\ell, A_k^r, \mathrm{alg}(C_k' D_k')\}_{k\in K}$. So pick $Z_1, \ldots, Z_n$ such that

$$Z_i \in \begin{cases} A_{\hat{\epsilon}(i)}^{\hat{\chi}(i)} & \text{if } \hat{\chi}(i) \neq b \\ \mathrm{alg}(C'_{\hat{\epsilon}(i)} D'_{\hat{\epsilon}(i)}) & \text{otherwise} \end{cases}.$$

If $\hat{\chi}(i) = b$ then let $Z_i = S_{C,i} S_{D_i}$ where $S_{C_i} \in C'_{\hat{\epsilon}(i)}$ and $S_{D,i} \in D'_{\hat{\epsilon}(i)}$.

For each $k \in K$ let $(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)$ be a copy of $(\tilde{\mathcal{A}}, \mathring{\tilde{\mathcal{A}}}, \tilde{E})$ and let ℓ_k, r_k, m_k be restrictions of the map θ. Let $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ be their reduced free product. Let

$$\tilde{\mu}_i(Z_i) = \begin{cases} \lambda_{\hat{\epsilon}(i)}(\ell_{\hat{\epsilon}(i)}(Z_i)) & \text{if } \hat{\chi}(i) = \ell \\ \rho_{\hat{\epsilon}(i)}(r_{\hat{\epsilon}(i)}(Z_i)) & \text{if } \hat{\chi}(i) = r \\ P_{\uplus,\hat{\epsilon}(i)} \lambda_{\hat{\epsilon}(i)}(m_{\hat{\epsilon}(i)}(Z_i)) P_{\uplus,\hat{\epsilon}(i)} & \text{if } \hat{\chi}(i) = b \end{cases}$$

Showing Free-Free-Boolean Independence - II

We only need to see that

$$E_{\mathcal{A}}(Z_1 \cdots Z_n) = E_{\mathcal{L}(\mathcal{X})}(\tilde{\mu}_1(Z_1) \cdots \tilde{\mu}_n(Z_n)).$$

Let $T_{f(i)} = Z_i$ if $i \notin \hat{\chi}^{-1}(\{b\})$, $T_{f(i)} = S_{C,i}$ if $i \in \hat{\chi}^{-1}(\{b\})$, and $T_{f(i)+1} = S_{D,i}$ for $i \in \hat{\chi}^{-1}(\{b\})$. Then

$$T_1 \cdots T_{n'} = Z_1 \cdots Z_n.$$

So we get

$$E_{\mathcal{A}}(Z_1 \cdots Z_n) = E_{\mathcal{A}}(T_1 \cdots T_{n'}) = E_{\mathcal{L}(\mathcal{X})}(\mu_1(T_1) \cdots \mu_{n'}(T_{n'})),$$

which equals

$$p\mu_1(T_1) \cdots \mu_{n'}(T_{n'})1_B = p \sum_{D \in LR} c_D E_D(\mu_1(T_1), \ldots, \mu_{n'}(T_{n'}))$$

Idea is then to add in the Boolean projections and separate the diagrams into those which respect the Boolean projections and those which get annihilated by them.

Showing Free-Free-Boolean Independence - III

Note that if $i \in \hat{\chi}^{-1}(\{b\})$ then

$$P_{\uplus,\epsilon(f(i))}\mu_{f(i)}(T_f(i))\mu_{f(i)+1}(T_{f(i)+1}) = \tilde{\mu}_i(Z_i)$$

Let $A \subseteq LR(\chi, \epsilon)$ be the collection of diagrams which get annihilated by any of the Boolean projections when added in the correct location.
So that

$$\mu_1(T_1)\cdots\mu_{n'}(T_{n'})1_B = \tilde{\mu}_1(Z_1)\cdots\tilde{\mu}_n(Z_n)1_B + \eta$$

where

$$\eta = \sum_{D\in A} c_D E_D(\mu_1(T_1), \ldots, \mu_{n'}(T_{n'})).$$

But it is easy to see that $p\eta 1_B = 0$ since we get only those diagrams corresponding to bi-non-crossing partitions which aren't in $BNC_{ffb}(\hat{\chi})$, which we know have expectation 0.

Concluding Remarks

As noted earlier, representing free-free-Boolean independence gives us a representation of free-Boolean independence.

Skoufranis also had a representation of monotone independence between two algebras. Though the finer details haven't yet been checked, it seems likely that a similar process yields an embedding of free-monontone for two pairs of algebras.

Thank you for listening!