Graph Constructions as Free Random Variables

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Joint works with

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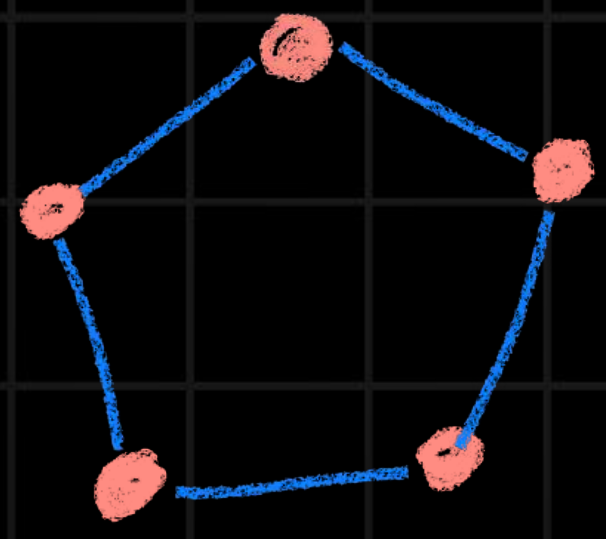
Spectral Expanders 101



G

$$\Longrightarrow \begin{pmatrix} \cdot & 1 & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 \\ 1 & \cdot & \cdot & 1 & \cdot \end{pmatrix}$$

A_G

Eigenvalues of A_G:

$$d = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq -d$$

$$\lambda(G) = \max_{i \geq 2} \{ |\lambda_i| \} \in [0, d]$$

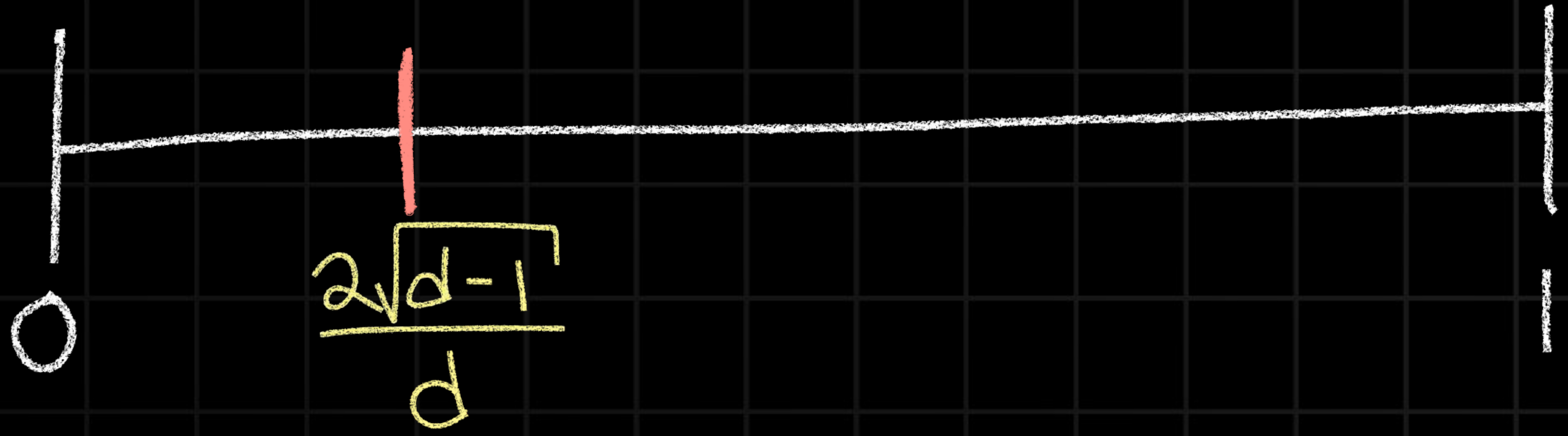
$$\omega_G = \frac{\lambda(G)}{d} \in [0,1]$$

Alon-Boppana: $\lambda(G) \geq 2\sqrt{d-1} - o_n(1)$

$$\frac{1}{d} A_G = J + E$$

perfect randomness

$\|E\| = \omega_G$



Spectral Expanders 101

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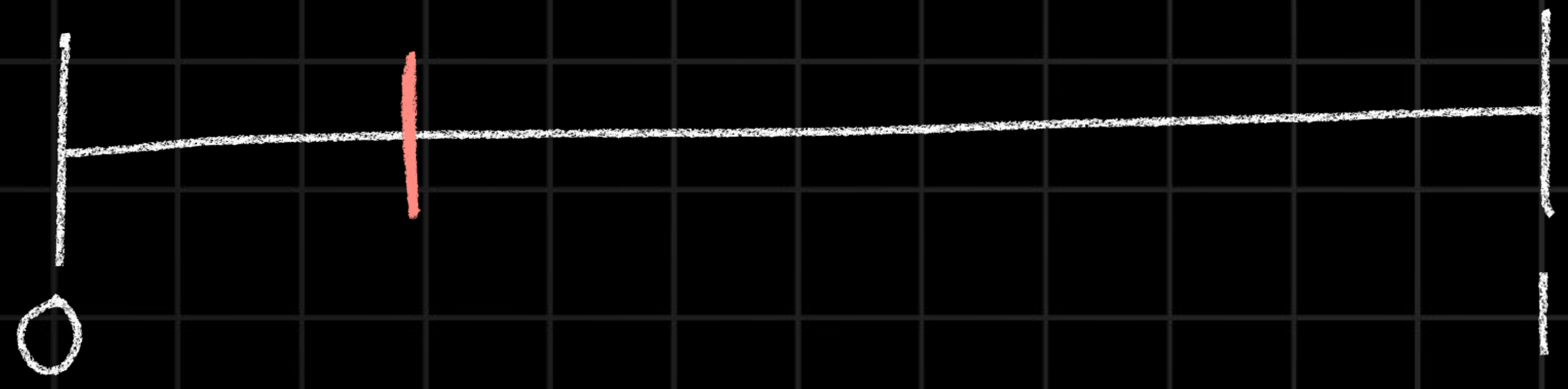

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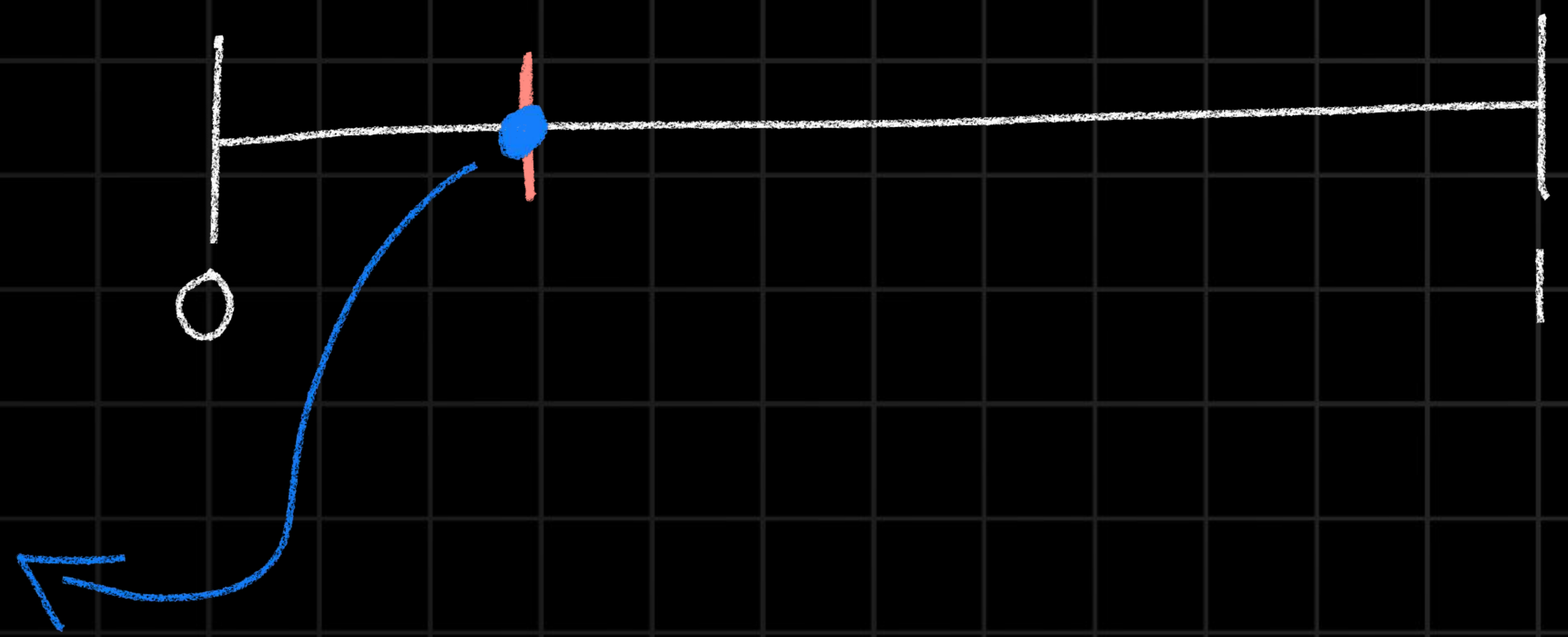

Ramanujan
[LPS'88]
[MSS'15]

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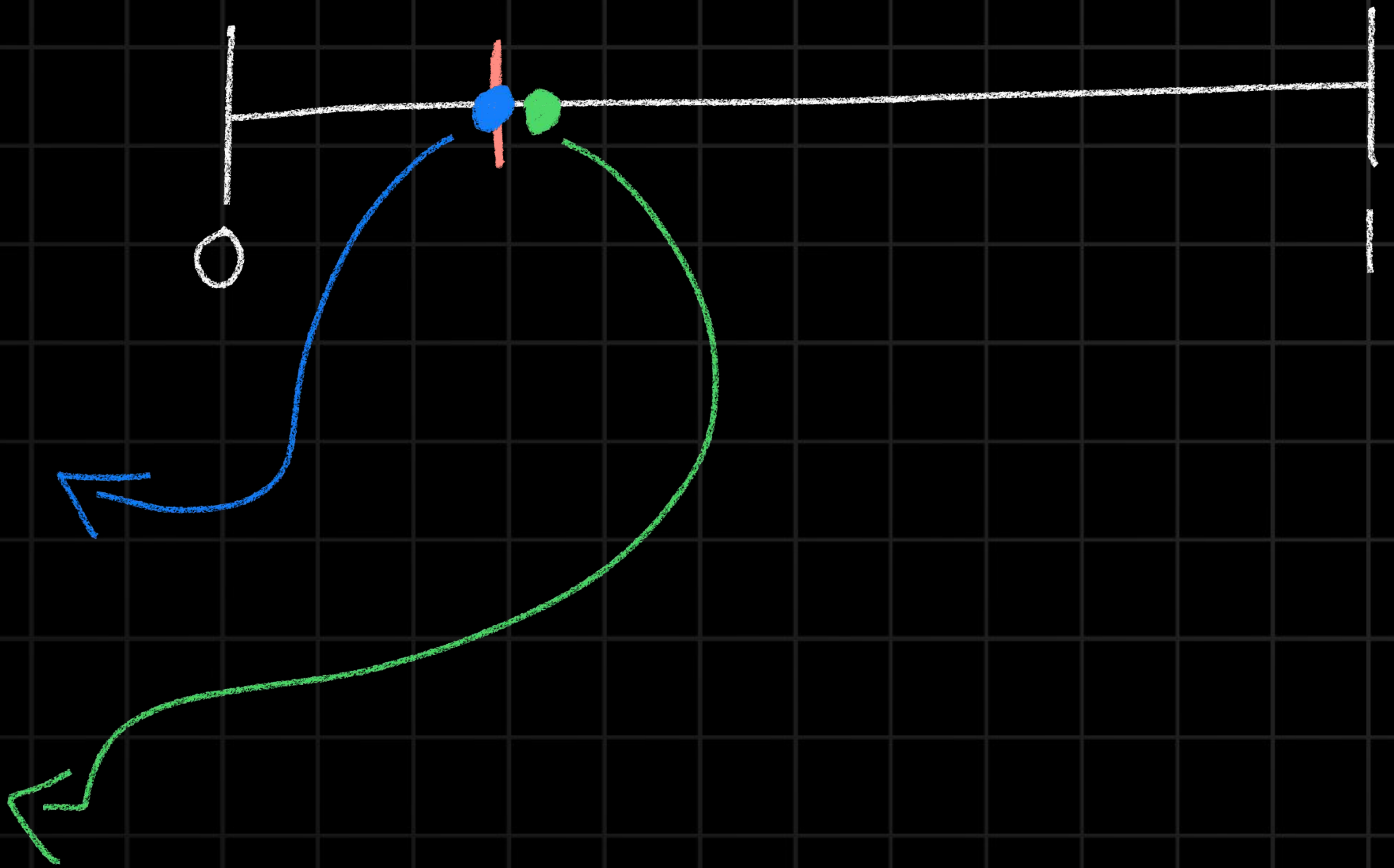

Ramanujan
[LPS'88]
[MSS'15]

Random
[Fri'06]
[Bor'15]

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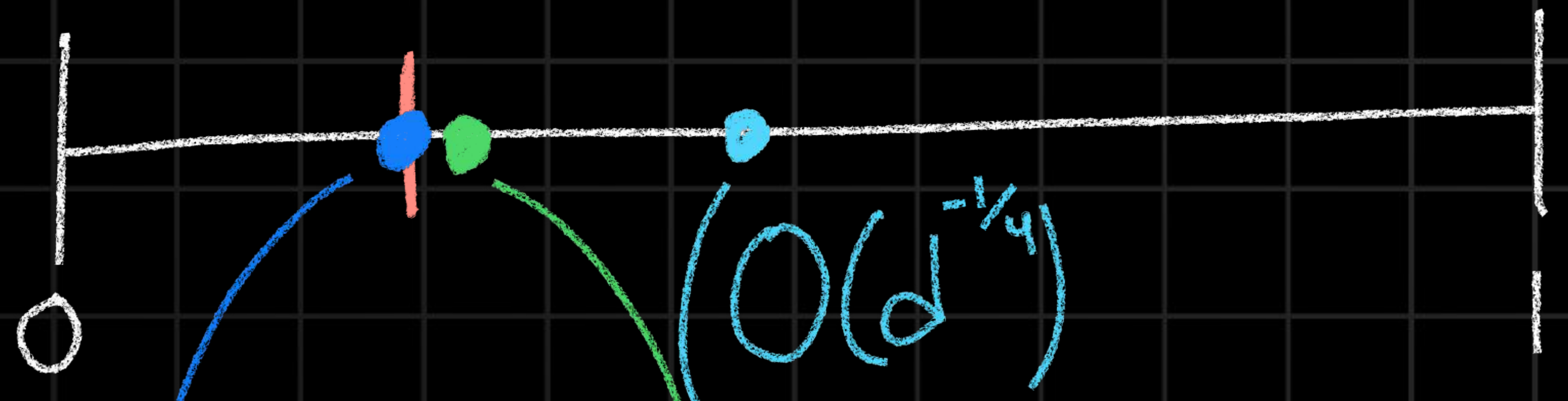

Ramanujan
[LPS'88]
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Random
[Fri'06]
[Bor'15]

Zig-Zag construction
[RVW'00]

Purely combinatorial!

Can it be improved?

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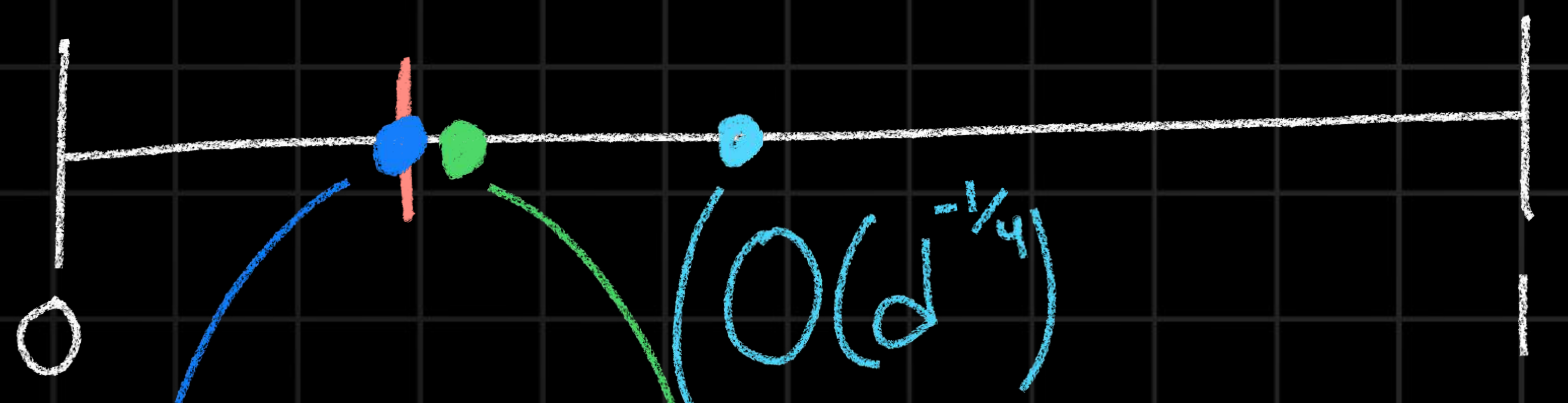


$O(d^{-1/4})$

Ramanujan
[LPS'88]
[MSS'15]

Random
[Fri'06]
[Bor'15]

Zig-Zag construction
[RVW'00]

Purely combinatorial!

Can it be improved?

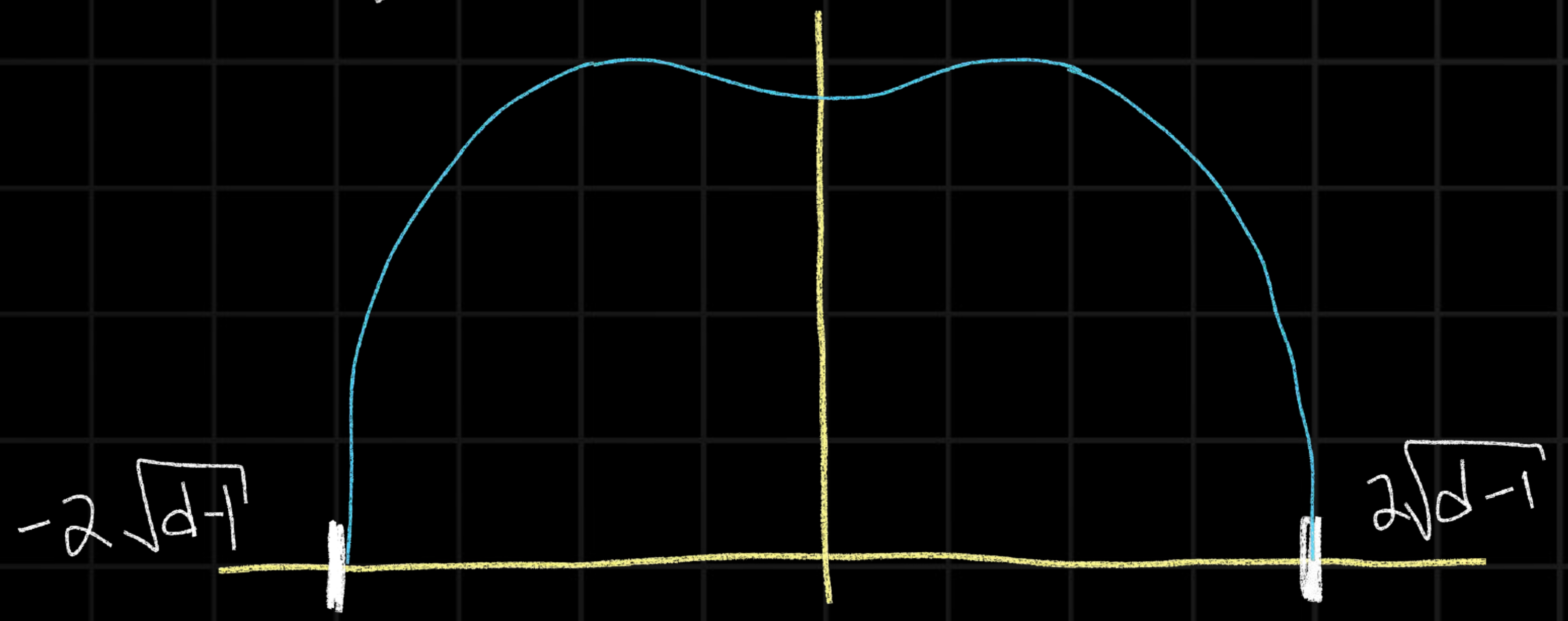
MSS Overview

Alon-Boppana: $\lambda(G) \geq 2\sqrt{d-1} - o_n(1)$

What is $2\sqrt{d-1}$?

- Spectral radius of T_d (infinite d-ary tree)
- Support of KM_d (Kesten-McKay with param d)

$$KM_d = \underbrace{\mu \boxplus \mu \boxplus \cdots \boxplus \mu}_{d \text{ times}}, \quad \mu = \frac{1}{2}(\delta_1 + \delta_{-1})$$



MSS Overview

$$KM_d = \underbrace{\mu \boxplus \mu \boxplus \cdots \boxplus \mu}_{d \text{ times}}, \qquad \mu = \frac{1}{2}(\delta_1 + \delta_{-1})$$

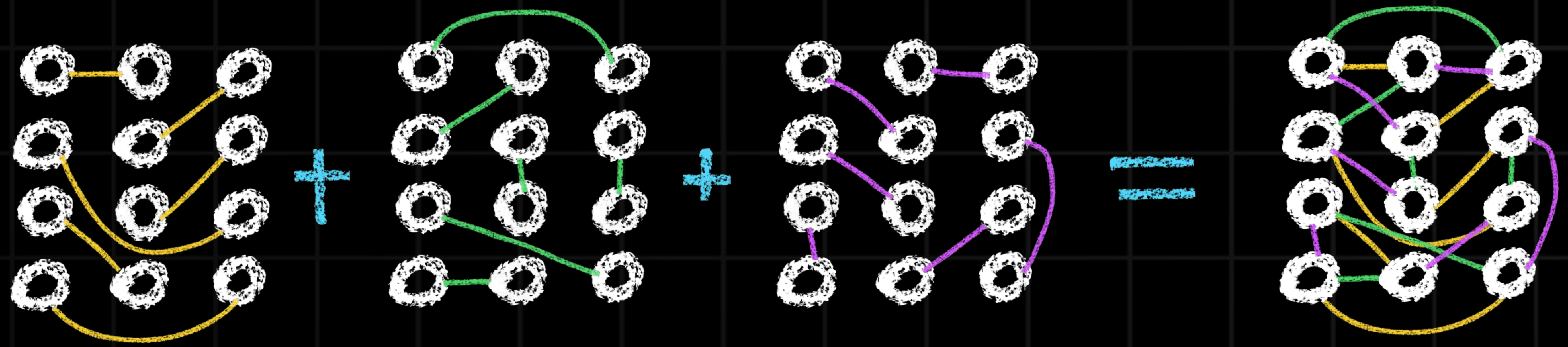
What is the graph analog?

MSS Overview

$$KM_d = \mu \boxplus \mu \boxplus \cdots \boxplus \mu, \qquad \mu = \tfrac{1}{2}(\delta_1 + \delta_{-1})$$

Graph:

$$G_P = M_1 + M_2 + \cdots + M_d, \qquad M_i = P_i^T M P_i$$



Thm: $\lambda_2(G_P) \leq \text{max-support}(KM_d)$

($\exists P$ s.t.)

(Interlacing & Quadrature)

MSS Inspired Constructions

- $G_p = M_1 + M_2 + \cdots + M_d$ $\qquad M_i = P_i^T M P_i$

[MSS'15] $\lambda_2(G_p) \le \text{max-support}(\mu_M^{\boxplus d}) = 2\sqrt{d-1}$

- $G_p = G_1 + G_2 + \cdots + G_t$ $\qquad G_i = P_i^T G P_i$

$$\lambda_2(G_p) \le \text{max-support}(\mu_G^{\boxplus t})$$

- $G_p = G_1 G_2 \cdots G_t^2 \cdots G_2 G_1$ $\qquad$ degree $D = d^{2t}$

$$\lambda_2(G_p) \underset{?}{\le} \text{max-support}(\mu_{G^2}^{\boxtimes t}) = ?$$

MSS Inspired Constructions

- $G_P = M_1 + M_2 + \cdots + M_d \qquad M_i = P_i^T M P_i$

[MSS'15] $\lambda_2(G_P) \leq \text{max-support}\left(\mu_M^{\boxplus d}\right) = 2\sqrt{d-1}$

- $G_P = G_1 + G_2 + \cdots + G_t \qquad G_i = P_i^T G P_i$

$$\lambda_2(G_P) \leq \text{max-support}\left(\mu_G^{\boxplus t}\right)$$

- $G_P = G_1 G_2 \cdots G_t^2 \cdots G_2 G_1$

$$\lambda_2(G_P) \leq \text{max-support}\left(\mu_{G^2}^{\boxtimes t}\right) \approx \, !$$

Proving I&Q for $\boxtimes$

Reminder: the convolution recipe

$$
\begin{tikzcd}
\mu \arrow[r] & G_\mu(x) \arrow[d] \\
 & R_\mu(y) \arrow[d] \\
 & R_{\mu^{\boxplus d}} = d \cdot R_\mu \arrow[d] \\
\mu^{\boxplus d} & G_{\mu^{\boxplus d}}(x) \arrow[l]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mu \arrow[r] & G_\mu(x) \arrow[d] \\
 & S_\mu(y) \arrow[d] \\
 & S_{\mu^{\boxtimes d}} = S_\mu^d \arrow[d] \\
\mu^{\boxtimes d} & G_{\mu^{\boxtimes d}}(x) \arrow[l]
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\mu^{\boxplus d} & G_{\mu^{\boxplus d}}(x) \arrow[l]
\end{tikzcd}
\qquad
\text{Inverting}
\qquad
\begin{tikzcd}
\mu \arrow[r] & G_\mu(x) \arrow[d] \\
& S_\mu(y) \arrow[d] \\
& S_{\mu^{\boxtimes d}} = S_\mu^d \arrow[d] \\
\mu^{\boxtimes d} & G_{\mu^{\boxtimes d}}(x) \arrow[l]
\end{tikzcd}
$$

Reminder: the convolution recipe

$$
\begin{tikzcd}
\mu \arrow[r] & G_\mu(x) \arrow[d] & & \mu \arrow[r] & G_\mu(x) \arrow[d] \\
& R_\mu(y) \arrow[d] & & & S_\mu(y) \arrow[d] \\
& R_{\mu^{\boxplus d}} = d \cdot R_\mu \arrow[d, "\text{[MSS'15]} \quad \text{Bounding } \mu^{\boxplus d}"] & & & S_{\mu^{\boxtimes d}} = S_\mu^d \arrow[d] \\
\mu^{\boxplus d} & G_{\mu^{\boxplus d}}(x) \arrow[l] & & \mu^{\boxtimes d} & G_{\mu^{\boxtimes d}}(x) \arrow[l]
\end{tikzcd}
$$

Rotating Expanders

$$G_p = G_1 G_2 \cdots G_t^2 \cdots G_2 G_1$$

Idea (the Adapter)

- Use KM_d instead of μ.
- Prove $S_\mu \approx S_{KM_d}$

$$\begin{tikzcd}
\cancel{\mu} \arrow[r] & G_\mu(x) \arrow[d] \\
& S_\mu(y) \arrow[d] \\
& S_{\mu^{\boxtimes t}} = S_\mu^t \arrow[d] \\
\mu^{\boxtimes t} & G_{\mu^{\boxtimes t}}(x) \arrow[l]
\end{tikzcd}$$

The Adapter

Ideally, $o_n(1)$

Goal: $G_{\mu}(x) - G_{KM}(x)$ is small.

Recall: $G(x) = \sum_{r=0}^{\infty} \frac{m_r}{x^{r+1}}$

For G with eigenvalues μ,

$m_r(\mu) = \frac{1}{n} \cdot \#\{\text{closed walks of length } r\}$

The Adapter: Proof Sketch

Recall: $G(x) = \sum_{r=0}^{\infty} \frac{m_r}{x^{r+1}}$

$$|G_\mu(x) - G_{KM}(x)| = \left| \sum_{r=0}^{\infty} \frac{m_r(\mu) - m_r(KM)}{x^{r+1}} \right|$$

$$\leq \underbrace{\left| \sum_{r=0}^{h} \frac{m_r(\mu) - m_r(KM)}{x^{r+1}} \right|}_{0} + \underbrace{\left| \sum_{r=h+1}^{\infty} \frac{m_r(\mu) - m_r(KM)}{x^{r+1}} \right|}_{\frac{2}{x-\lambda} \cdot \left(\frac{\lambda}{x}\right)^h}$$

taking h = girth

$m_r(KM) = \#$ in T_d

[McKay '81]

The Adapter: Proof Sketch

Recall: $G(x) = \sum_{r=0}^{\infty} \frac{m_r}{x^{r+1}}$

$$G_\mu(x) - G_{KM}(x) = \sum_{r=0}^{\infty} \frac{m_r(\mu) - m_r(KM)}{x^{r+1}}$$

$$\leq \underbrace{\left| \sum_{r=0}^{h} \frac{m_r(\mu) - m_r(KM)}{x^{r+1}} \right|}_{0} + \underbrace{\left| \sum_{r=h+1}^{\infty} \frac{m_r(\mu) - m_r(KM)}{x^{r+1}} \right|}_{\frac{2}{x-\lambda} \cdot \left(\frac{\lambda}{x}\right)^{h}}$$

taking h=girth

$m_r(KM) = \#$ in T_d

[McKay '81]

depends on G!

The Adapter: Proof Sketch

Recall: $G(x) = \sum_{r=0}^{\infty} \frac{m_r}{x^{r+1}}$

$$G_\mu(x) - G_{KM}(x) = \sum_{r=0}^{\infty} \frac{m_r(\mu) - m_r(KM)}{x^{r+1}}$$

$$\leq \underbrace{\left| \sum_{r=0}^{h} \frac{m_r(\mu) - m_r(KM)}{x^{r+1}} \right|}_{0} + \underbrace{\left| \sum_{r=h+1}^{\infty} \frac{m_r(\mu) - m_r(KM)}{x^{r+1}} \right|}_{\frac{2}{x-\lambda} \cdot \left(\frac{\lambda}{x}\right)^h}$$

taking h = girth

$m_r(KM) = \#$ in T_d

[McKay'81]

(getting S_μ is calculus)

MSS Inspired Construction I Rotating Expanders

- $G_P = G_1 G_2 \cdots G_t^2 \cdots G_2 G_1$, $\quad G_i = P_i^T G P_i$, $\quad D = d^{2t}$

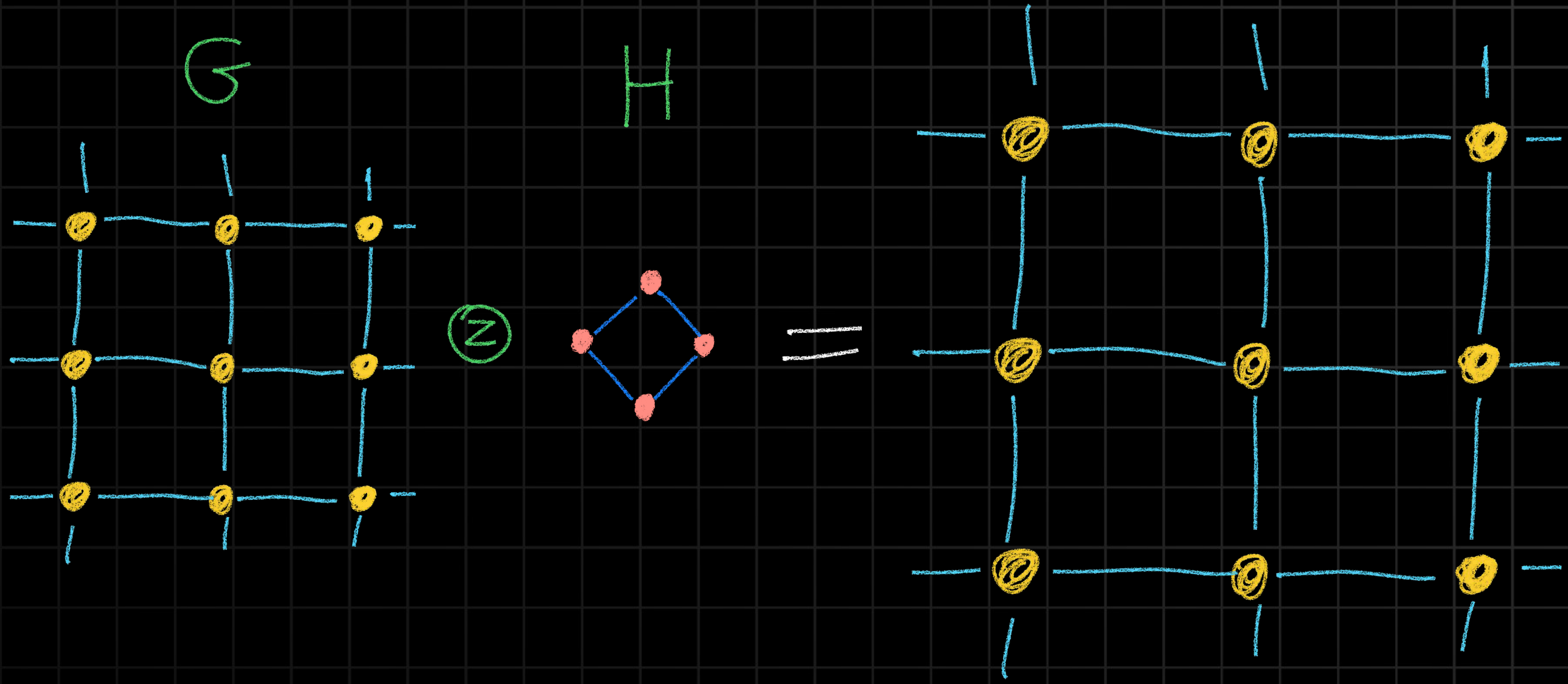
Thm [CM'23]: for G Ramanujan,

$$\lambda(G_P) \leq \underbrace{\frac{(t+1)^{t+1}}{t^t}}_{\text{Fuss-Catalan growth rate}} d \cdot (d-1)^{t-1} + O(t \cdot 2^{-h}) < \underbrace{e \cdot (t+1)}_{\ll 2^t} \sqrt{D}$$

($\exists P$ s.t.)

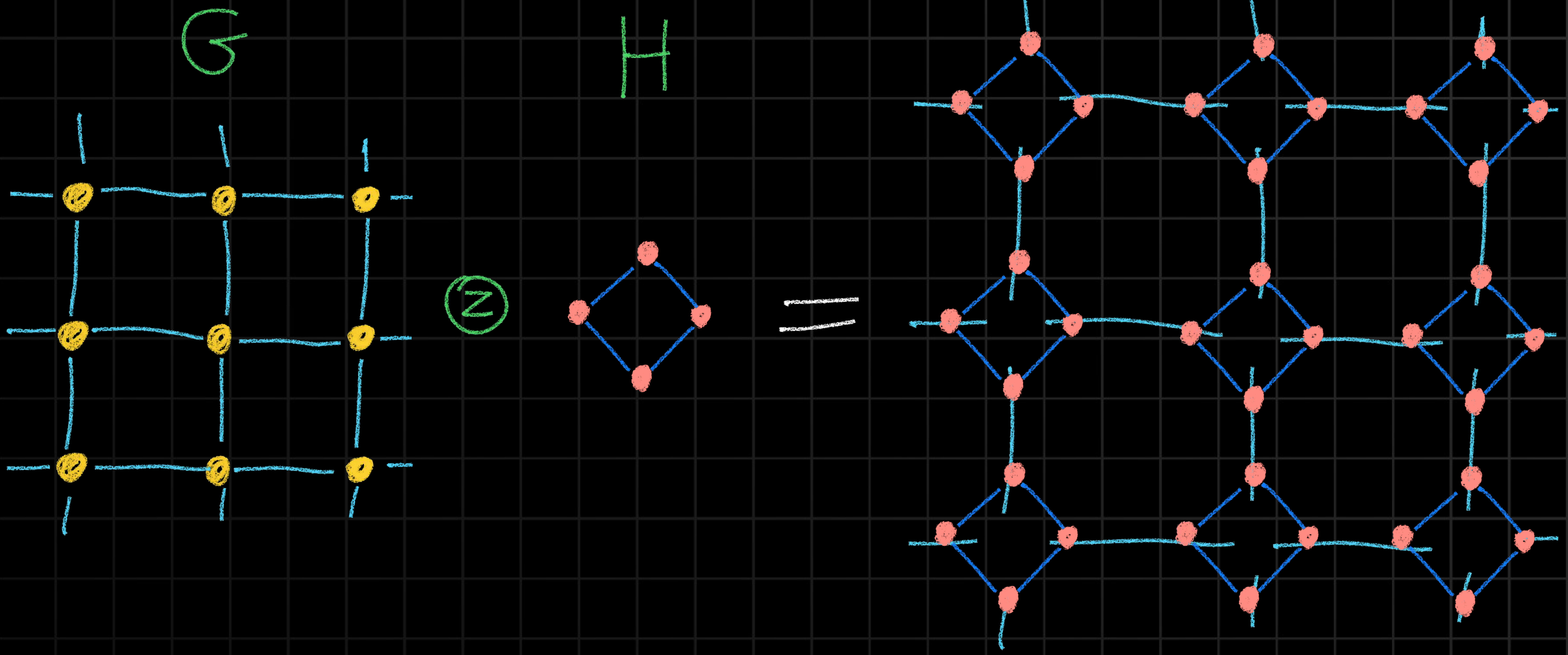
Tight! Lower bound proven in [CCMP'24]

The Zig-Zag product $G \circledZ H$

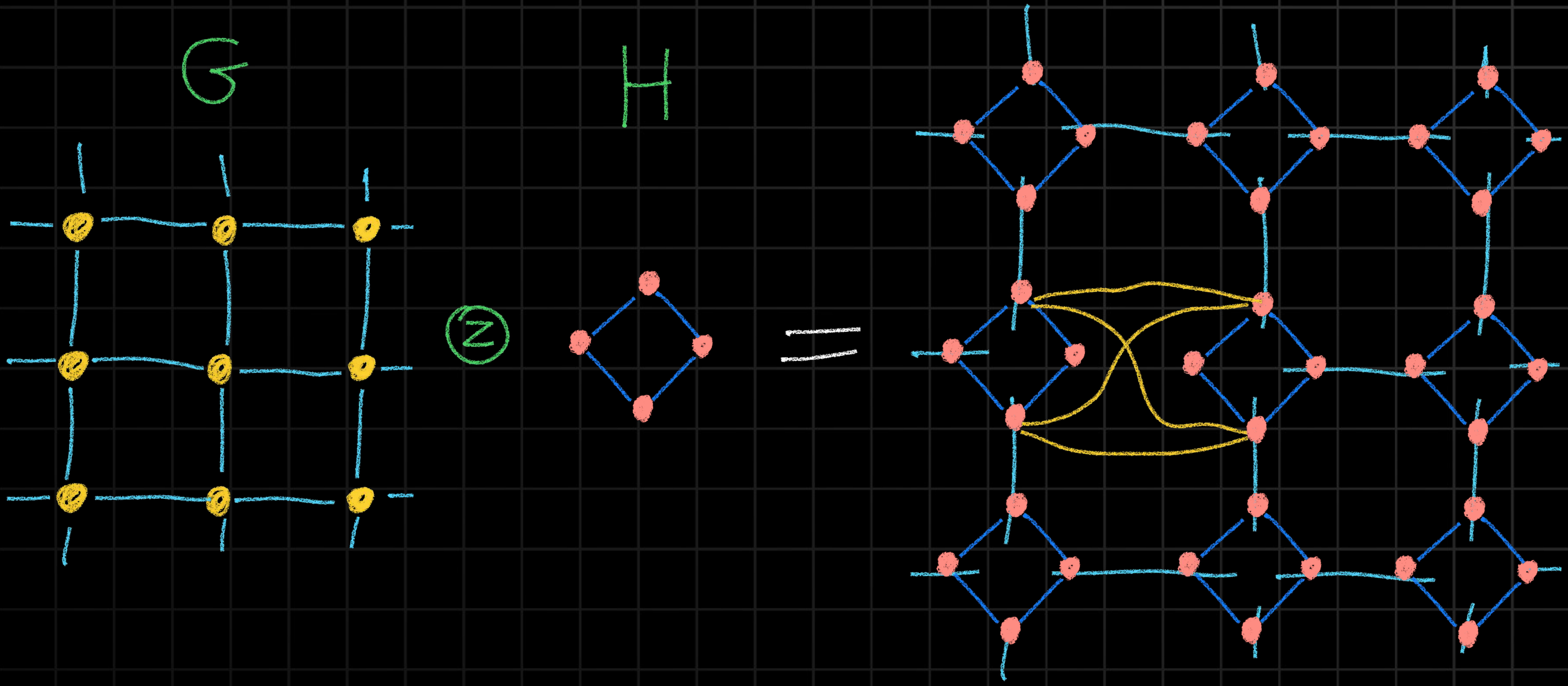


The Zig-Zag product $G \textcircled{z} H$

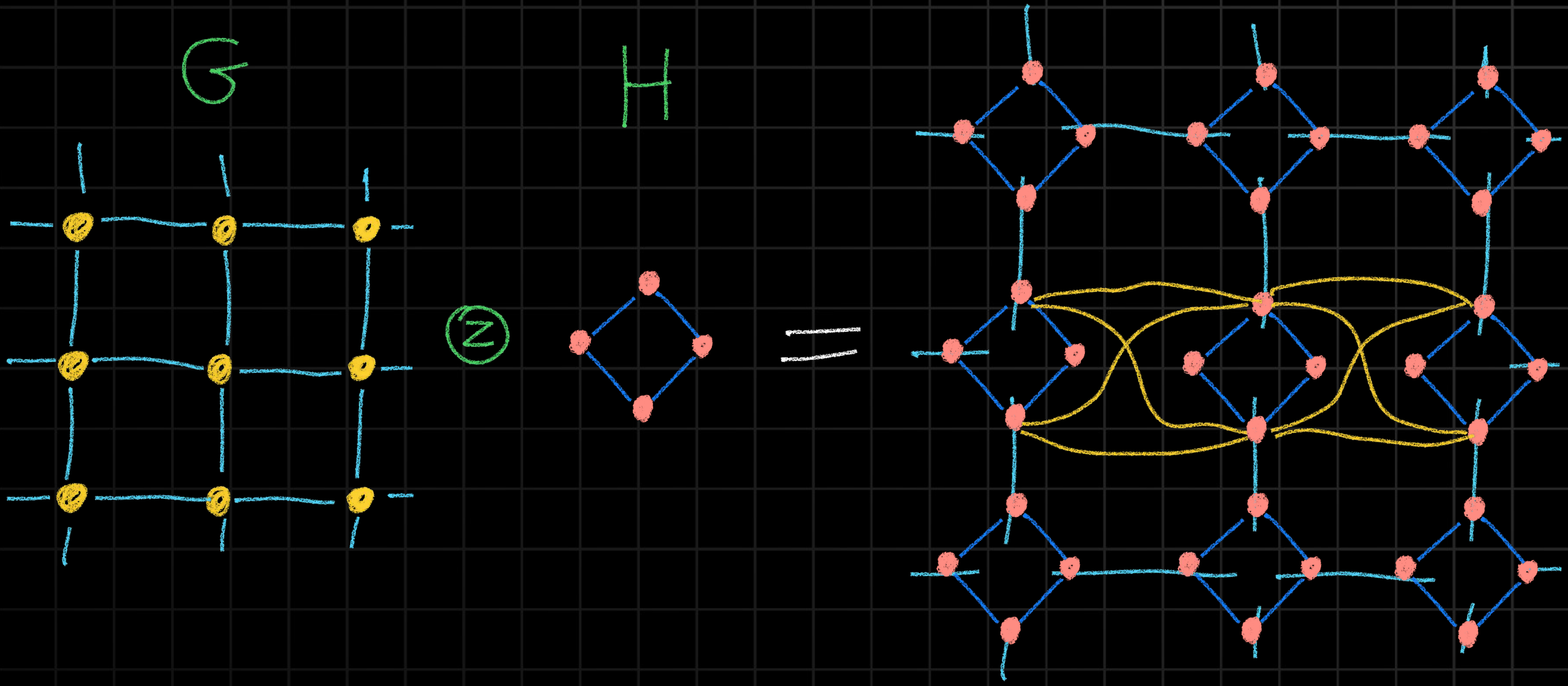
$(G \textcircled{r} H)$



The Zig-Zag product $G ⓩ H$



The Zig-Zag product G ⓩ H

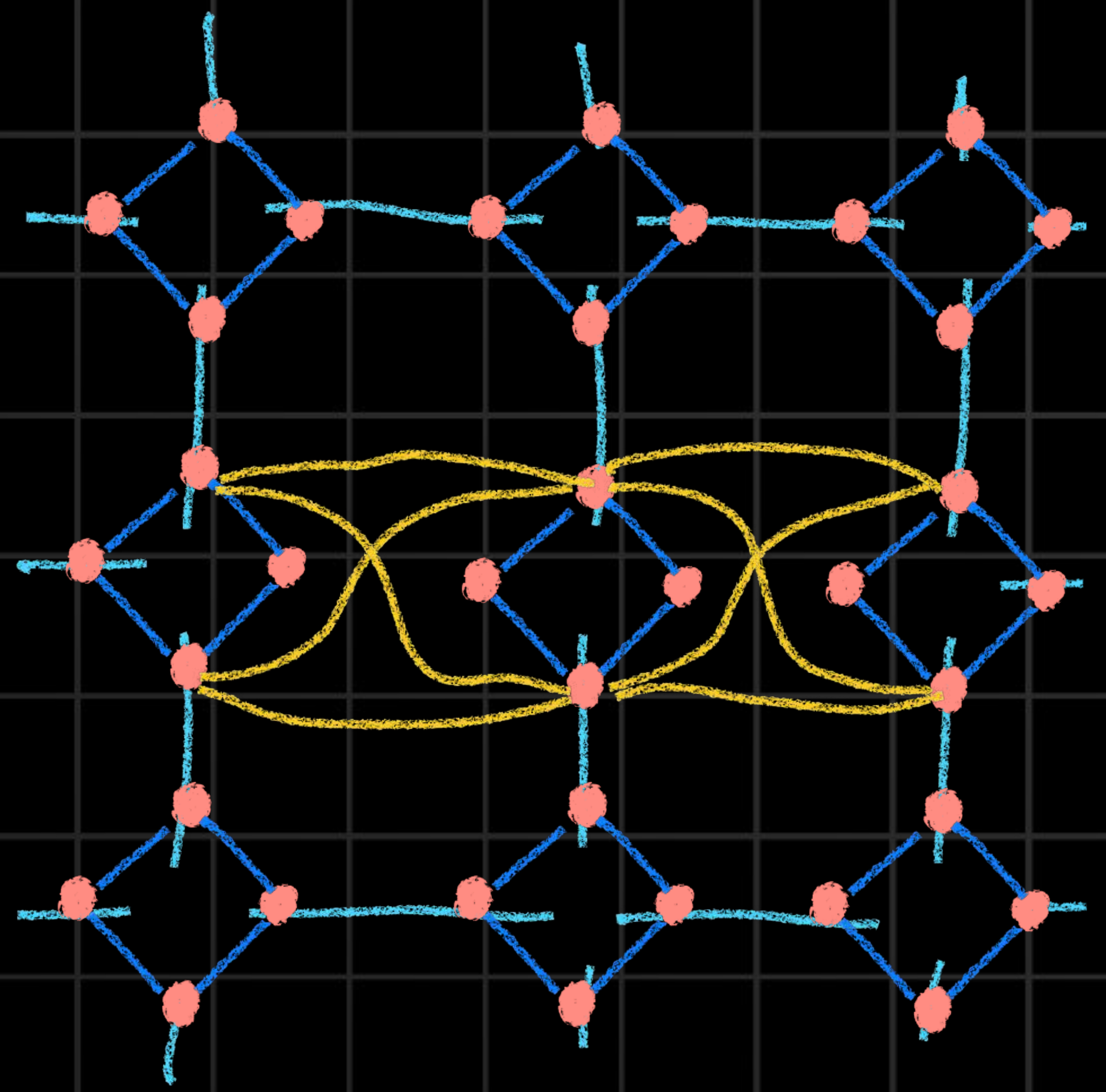


Properties of ⓩ

	Size	D	ω
G	n	d	ω_G
H	d	c	ω_H
G ⓩ H	nd	c^2	$\lesssim \omega_G + \omega_H$ [RVW]

$$= O\left(D^{-1/4}\right)$$

$G \circledz H$



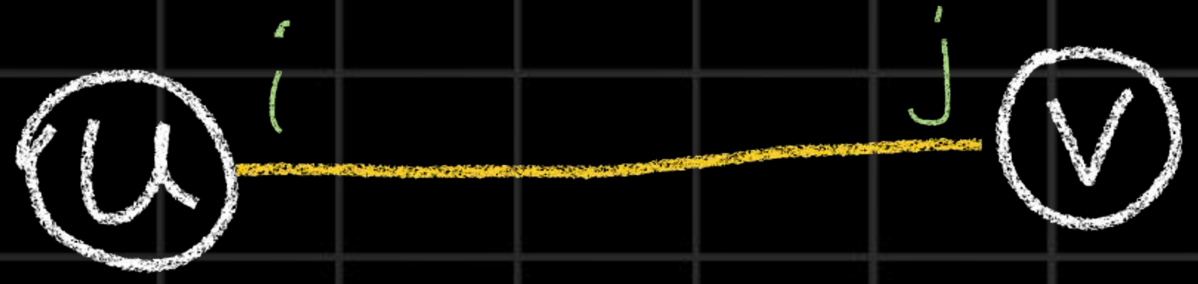
Analyzing $G \textcircled{z} H$ in Free Probability

$G \textcircled{z} H = \tilde{H}\dot{G}\tilde{H}$

$\tilde{H} = I_n \otimes H = \begin{pmatrix} H & & & \\ & H & & \\ & & \ddots & \\ & & & H \end{pmatrix}$

Random G: $\tilde{H} P^{T} M P \tilde{H}$

arbitrary matching, ± 1



$\dot{G}_{(u,i),(v,j)} = 1$

Thm [CCM'24]:

$\lambda_2(G \textcircled{z} H) \leq \text{max-support}(\mu_{H^2} \boxtimes \mu_M)$

($\exists G$ s.t.)

Analyzing $G \circledZ H$ in Free Probability

Thm [CCM'24]: $\lambda_2(G \circledZ H) \leq \underbrace{\text{max-support}(\mu_{H^2} \boxtimes \mu_M)}_{Z_H}$
($\exists G$ s.t.)

Moreover, this is a lower bound:

$$\lambda(G \circledZ H) \geq Z_H - O_n(1)$$

Proven using analytic combinatorics.

Analyzing G ⓩ H in Free Probability

Thm [CCM'24]: $\lambda_2(G ⓩ H) \le \underbrace{\text{max-support}(\mu_{H^2} \boxtimes \mu_M)}_{Z_H}$

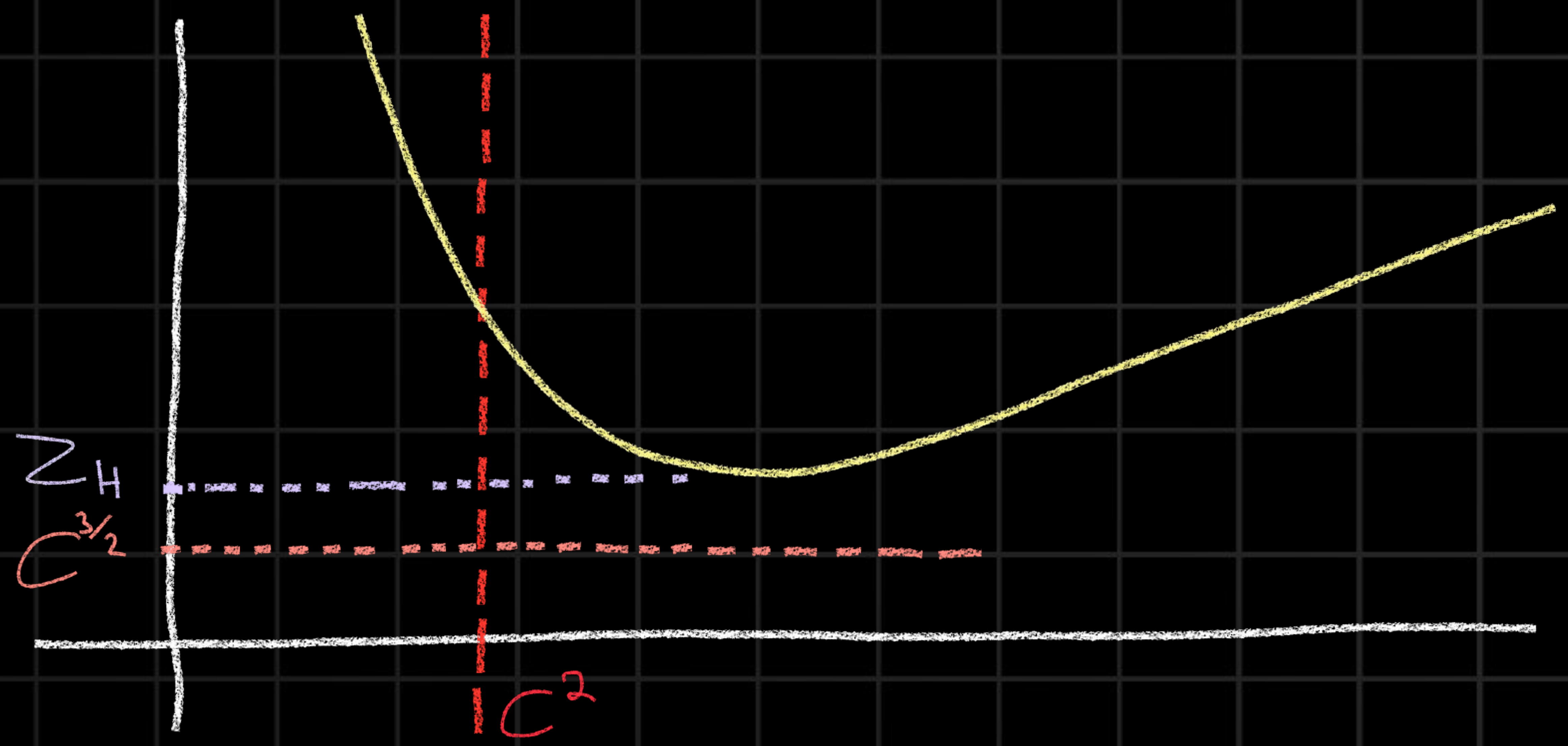
(∃G s.t.)

Moreover, this is a lower bound:

$$\lambda(G ⓩ H) \geq Z_H - O_n(1)$$

$$\omega_{G ⓩ H} \geq \frac{C^{3/2}}{C^2} = C^{-1/2} = D^{-1/4}$$

$$Z_H = \min_{x > C^2} x\sqrt{1 - \frac{1}{x \cdot G_{H^2}(x)}}$$



Analyzing $G \textcircled{z} H$ in Free Probability

Thm [CCM'24]: $\lambda_2(G \textcircled{z} H) \leq \underbrace{\text{max-support}(\mu_{H^2} \boxtimes \mu_M)}_{Z_H}$

$(\exists G \text{ s.t.})$

Moreover, this is a lower bound:

$$\lambda(G \textcircled{z} H) \geq Z_H - O_n(1)$$

Corollary:

$$\omega(G \textcircled{z} H) \geq D^{-1/4}$$

(and not $D^{-1/2}$)

The [RVW] bound is asimptotically tight

Recap

- Using FP, proved tight bounds for the ZigZag product ⓩ and others (derand. square ⓢ, replacement product ⓡ).
- "Free Graphs" yield better expansion properties (rotating expanders)

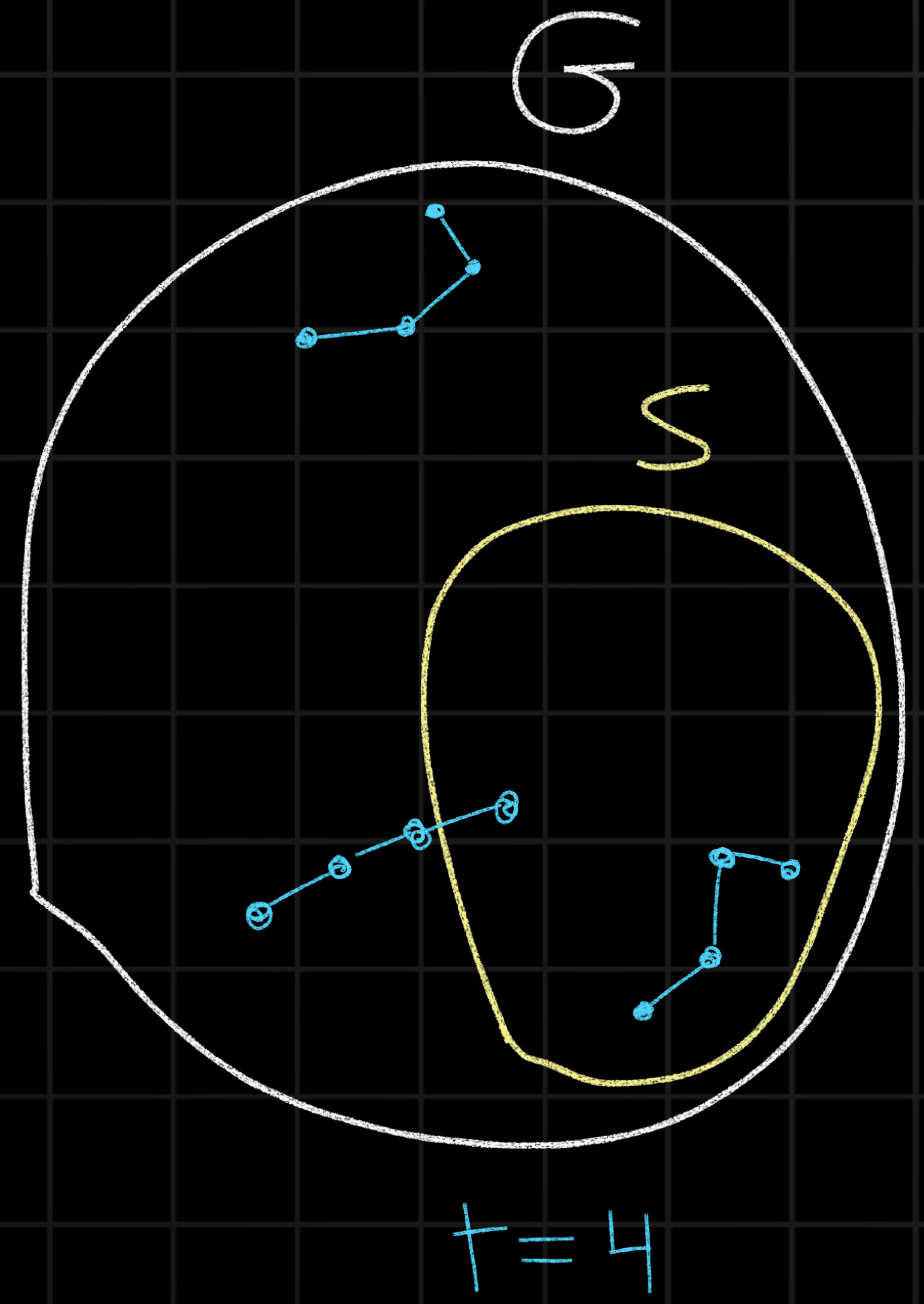
Questions?

Open Problem I

Trapped Random Walks
(or: the Expander Hitting Property)

G is an ω-expander. $|S| = \mu \cdot |V|$

$$\Pr[\ \subseteq S] = ?$$



Open Problem I

Trapped Random Walks
(or: the Expander Hitting Property)

G is an ω-expander. $|S| = \mu \cdot |V|$

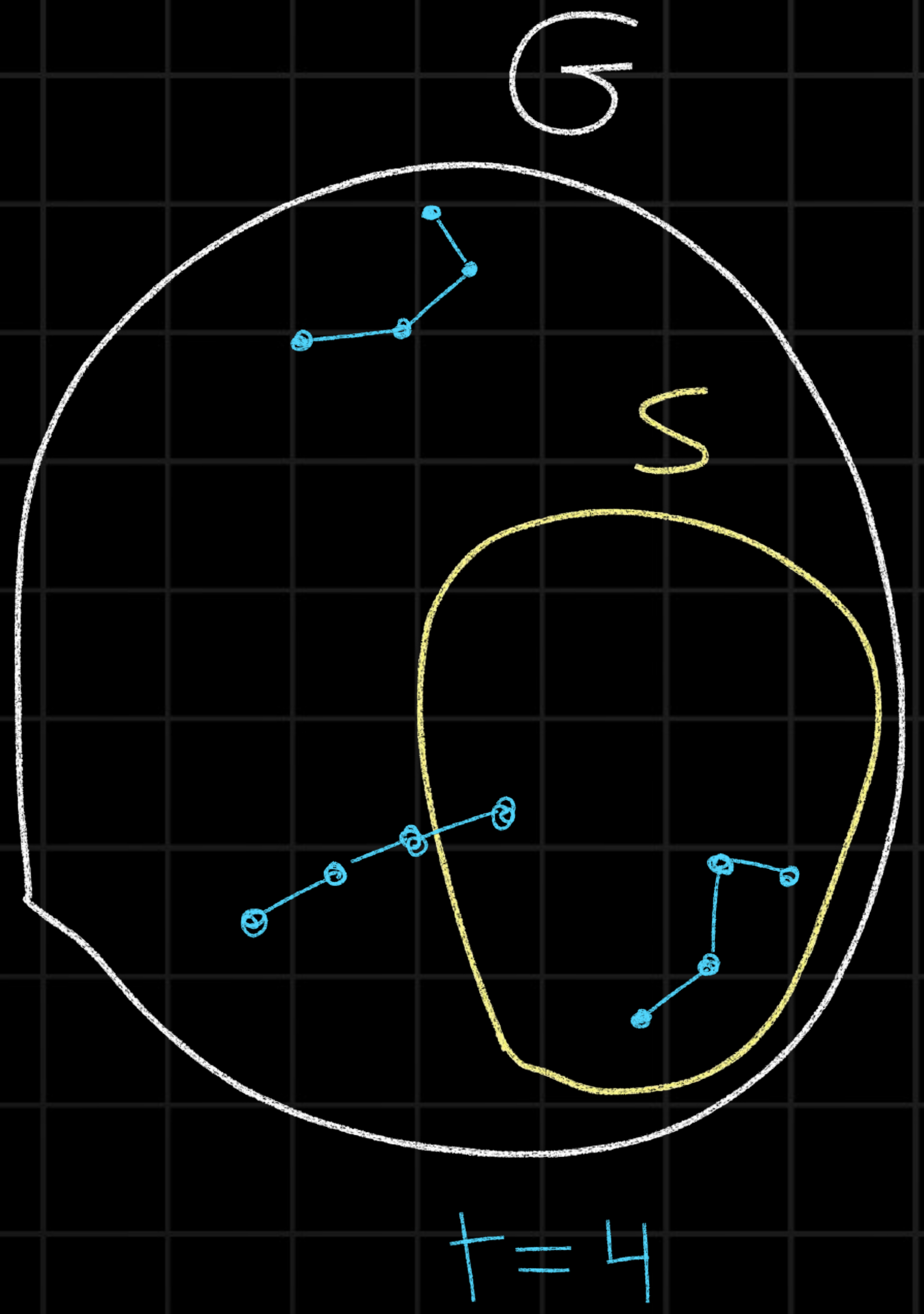
$$\Pr[\;\; \subseteq S] \le \mu \cdot (\mu + \omega)^{t-1}$$

(Ideal: μ^t)

- The ideal requires $t \cdot \log n$ bits.
- Random Walk: $\log n + (t-1) \cdot \log d$.

Taking $\omega = \mu$: $\Pr[\;\; \subseteq S] \approx 2^t \mu^t$

$(d \approx 4/\mu^2)$



Open Problem I

Trapped Random Walks

(or: the Expander Hitting Property)

$$\Pr[\ \subseteq S\ ] \le \mu\cdot(\mu+\omega)^{t-1}$$

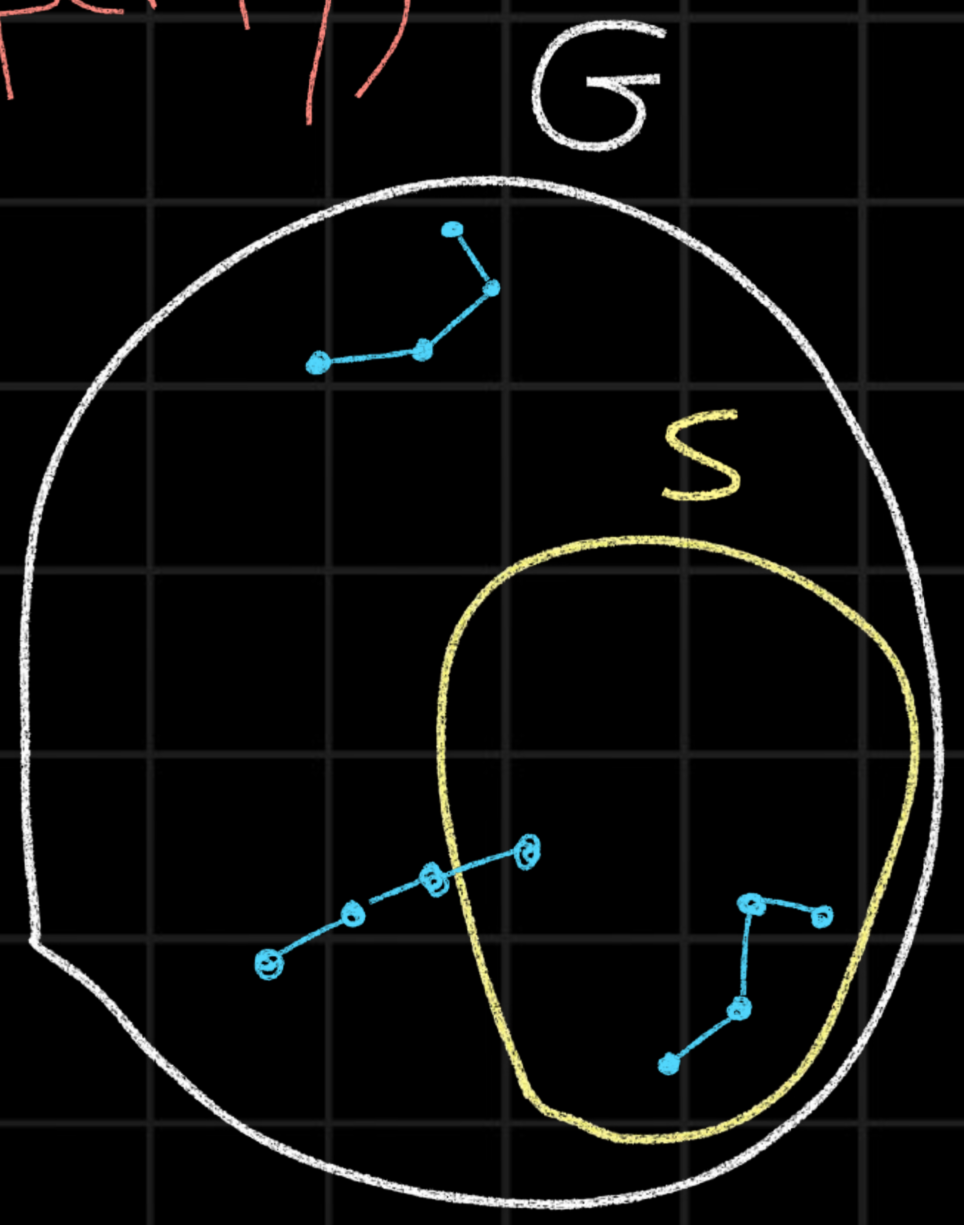
Proof: $\frac{1}{n}\mathbb{1}^T SGSGSGS\,\mathbb{1}$ $\quad (S_{ii}=1 \text{ for } i\in S)$

$$= \frac{1}{n}\mathbb{1}^T S^2 G S^2 G S^2 G S^2\,\mathbb{1}$$

$$\le \mu\cdot\|SGS\|\cdot\|SGS\|\cdot\|SGS\| \quad \left(\frac{1}{n}\mathbb{1}^T S^2\mathbb{1}=\mu\right)$$

$$\|SGS\| \le \underbrace{\|SJS\|}_{\mu} + \underbrace{\|SES\|}_{\le\,\omega}$$

$\square$



Open Problem I — Hitting Property for Rotating Expanders

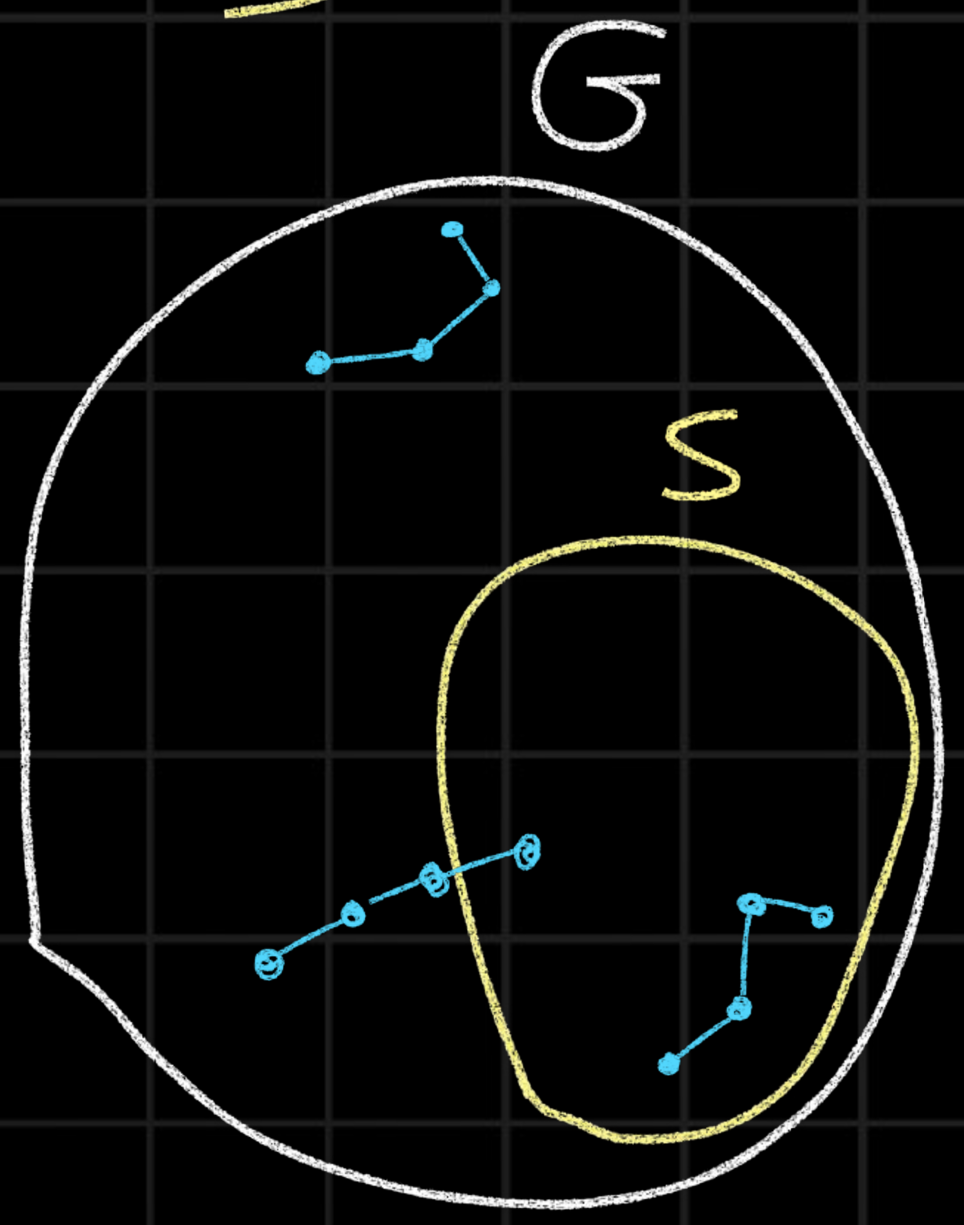
$$\Pr[\ \ \subseteq S] \le \mu\cdot(\mu+\omega)^{t-1}$$

Proof: $= \frac{1}{n}\mathbf{1}^T S G_3 S G_2 S G_1 S \mathbf{1}$

$$= \frac{1}{n}\mathbf{1}^T S^2 G_3 S^2 G_2 S^2 G_1 S^2 \mathbf{1}$$

$$\le \mu\cdot\|SG_3S\|\cdot\|SG_2S\|\cdot\|SG_1S\|$$

$$\|SG_iS\| \le \underbrace{\|SJS\|}_{\mu} + \underbrace{\|SE_iS\|}_{\le ?}$$



Open Problem I

Hitting Property for Rotating Expanders

$$\Pr\left[\quad \subseteq S\right] \le \mu\cdot(\mu+\omega)^{t-1}$$

$$\|$$

Proof: $\frac{1}{n}\mathbf{1}^T S G_3 S G_2 S G_1 S \mathbf{1}$

$$= \frac{1}{n}\mathbf{1}^T S^2 G_3 S^2 G_2 S^2 G_1 S^2 \mathbf{1}$$

or, analyze this term.

$$\le \mu\cdot\|SG_3S\|\cdot\|SG_2S\|\cdot\|SG_1S\|$$

for $t=3$: $\mu\left(\mu+\frac{\omega}{\sqrt{2}}\right)^2$

$$\|SG_iS\| \le \|SJS\| + \|SE_iS\|$$

We suspect $\mu\left(\mu+\frac{\omega}{\sqrt{t-1}}\right)^{t-1}$

Open Problem II   Small-Bias sets

Goal: find a small set $S \subseteq \{\pm 1\}^n$ that ε-fools all linear tests.

$J \subseteq [n]$ $f_J(x) = \prod_{j \in J} x_j$ $x \in \{\pm 1\}^n$

$\forall J$ $\mathop{\mathbb{E}}_{x \in \{\pm 1\}^n} f_J(x) = 0$

$$\left| \mathop{\mathbb{E}}_{x \in S} f_J(x) \right| \leq \varepsilon$$

$|S| = ?$

Open Problem II — Small-Bias sets

Goal: find a small set $S \subseteq \{\pm 1\}^n$ that ε-fools all linear tests.

$J \subseteq [n]$ $\qquad f_J(x) = \prod_{j \in J} x_j$ $\qquad |S| = \Omega\left(\frac{n}{\varepsilon^2 \cdot \log \frac{1}{\varepsilon}}\right)$

$\forall J$ $\qquad \mathbb{E}_{x \in \{\pm 1\}^n} f_J(x) = 0$ $\qquad$ non-explicit:

$$\left| \mathbb{E}_{x \in S} f_J(x) \right| \leq \varepsilon \qquad |S| = O\left(\frac{n}{\varepsilon^2}\right)$$

Open Problem II   Small-Bias sets

Amplification using expanders: Let $|S| = m$, ε_0-biased.

G a d-regular ω-expander. New set S_t:

$$x_1 - x_2 - \cdots - x_t \in G \qquad |S_t| = m \cdot d^{t-1}$$

$$y \in S_t : \quad y_i = x_{1_i} \cdot x_{2_i} \cdots x_{t_i}$$

$J \subseteq [n]$, P diag. matrix, $P_{xx} = f_J(x)$.

$$\varepsilon_t = \frac{1}{n} 1^T \underbrace{PGPGPGP}_{t} 1 \leq O\left(\left(\varepsilon_0^2 + \omega \right)^{t/2} \right)$$

Open Problem II Small-Bias sets

$|S| = m$, ε_0-biased, $|S_t| = m \cdot d^{t-1}$, $\varepsilon_t \leq O\left(\left(\varepsilon_0^2 + \omega\right)^{t/2}\right)$

Enables constructing ε-biased T of size $O\left(\frac{n}{\varepsilon^4}\right)$

What if we analyzed

$$\varepsilon_t = \frac{1}{n} 1^T P G_3 P G_2 P G_1 P 1 \quad ?$$

Thank You

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