Tensor free independence and central limit theorem

Sang-Jun Park [1]
Joint work with Ion Nechita [1]
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[1]CNRS, LPT (Laboratoire de Physique Théorique) Toulouse

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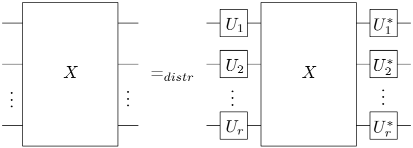
Overview

We consider random matrices $X = X_N \in M_N(\mathbb{C})^{\otimes r} \cong M_{N^r}(\mathbb{C})$ satisfying the local unitary invariance (LUI):

$$X =_{\text{prob.distr}} UXU^*, \quad \forall\, U = \bigotimes_{s=1}^{r} U_s \in \mathcal{U}_N^{\otimes r}.$$



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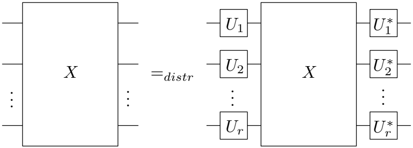

Main question

What can be expected about the convergences of LUI random matrices in the limit $N \to \infty$? For example,

$$X_N, Y_N\text{: indep, LUI, each having a limit} \implies \exists \text{ limit of } X_N + Y_N?$$

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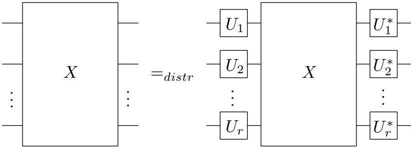

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$\longrightarrow$ Answered via "tensor free probability"!

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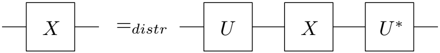
Unitary invariant random matrices

A family $\mathcal{W} \subset M_N\big(L^{\infty-}(\mathbb{P})\big)$ of $N \times N$ random matrices is called unitary invariant if

$$\mathcal{W} =_{\mathrm{prob.distr}} U\mathcal{W}U^* := \left\{ UXU^* \,\middle|\, X \in \mathcal{W} \right\}, \quad \forall\, U \in \mathcal{U}_N.$$



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EXAMPLE:

- (Independent) GUE matrices,
- (Independent) Haar random unitary matrices,
- $U\mathcal{W}U^*$ where U is a Haar random unitary matrix and $\mathcal{W}$ is deterministic (or possibly random independent from U).
- $\{U, U^\top\}$ is not UI even if both U and $U^\top$ are UI.

Unitary invariance and asymptotic freeness

Let $\mathrm{tr} := \frac{1}{N}\mathrm{Tr}$.

Theorem (Voiculescu 1991, 1998; Collins 2003)

$\mathcal{W}_N^{(1)}, \ldots, \mathcal{W}_N^{(L)}$: *independent $N \times N$ random matrices such that*

- *each $\mathcal{W}_N^{(i)}$ is unitary invariant (except possibly one),*
- *each $\mathcal{W}_N^{(i)}$ converges in distribution i.e.,*

$$\lim_{N\to\infty} \mathbb{E}\left[\mathrm{tr}\big(P(\mathcal{W}_N^{(i)})\big)\right] \text{ exists for all polynomials } P,$$

- *factorization property: for $i \in [L]$ and polynomials P_j,*

$$\mathbb{E}\big[\mathrm{tr}(P_1(\mathcal{W}_N^{(i)}))\cdots\mathrm{tr}(P_r(\mathcal{W}_N^{(i)}))\big] = \prod_{j=1}^r \mathbb{E}\big[\mathrm{tr}(P_j(\mathcal{W}_N^{(i)}))\big] + o(1).$$

Then $\mathcal{W}_N^{(1)}, \ldots \mathcal{W}_N^{(L)}$ are asymptotically freely independent as $N \to \infty$.

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REMARK:

- If X_N is Hermitian and $X_N \xrightarrow{\mathrm{distr}} x$ (in expectation), then

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- Variants: a.s. / higher-order / strong convergence, etc.

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Then $\mathcal{W}_N^{(1)}, \ldots \mathcal{W}_N^{(L)}$ *are asymptotically freely independent as* $N \to \infty$.

- The asymptotic freeness remains valid for
  - (Dykema 1993) Wigner matrices,
  - (Collins and Sniady 2006) Orthogonal invariant matrices.
- (Male 2020; Cebron, Dahlqvist, and Male 2024) Permutation invariant matrices and traffic independence.

Partial order between permutations

Let $|\cdot|: S_p \to \mathbb{Z}_{\geq 0}$ be the length function:

$$|\alpha| := \min\left\{k \geq 0 \,\middle|\, \alpha = \tau_1 \cdots \tau_k \text{ for some transpositions } \tau_1, \ldots, \tau_k\right\}.$$

- $|\alpha| = p - \#\alpha$ for $\alpha \in S_p$, where $\#\alpha$ is the number of cycles in α.
- (Triangle inequality) $|\alpha\beta| \leq |\alpha| + |\beta|, \quad \alpha, \beta \in S_p$.
- "$\beta \leq \alpha$" if $|\beta| + |\beta^{-1}\alpha| = |\alpha|$, i.e., $\mathrm{id}_p \to \beta \to \alpha$ is a geodesic.

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Proposition (cf. Textbook of Nica and Speicher)

- $\leq$ *defines a partial order on* S_p.
- $\left(\left\{\beta \in S_p \,\middle|\, \beta \leq \gamma_p\right\}, \leq\right) \cong (NC(p), \leq)$ *where* $\gamma_p := (1\,2\,\cdots\,p)$ *and* $NC(p)$ *is the set of non-crossing partitions of* $[p] := \{1, 2, \ldots, p\}$.

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EXAMPLE: For $\gamma_4 = (1\,2\,3\,4)$,

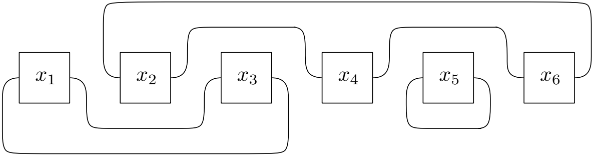
$$(1\,4)(2\,3) \leq \gamma_4 \text{ and } (1\,3\,4) \leq \gamma_4 \text{ while } (1\,3)(2\,4) \not\leq \gamma_4 \text{ and } (1\,4\,3) \not\leq \gamma_4$$

Moments associated to permutations

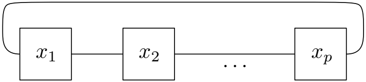
Let $(\mathcal{A}, \varphi)$ be a NC probability space with a tracial state φ. Let us define

$$\varphi_\alpha(x_1, \dots, x_p) = \prod_{\substack{c \in \mathrm{Cycle}(\alpha) \\ c=(i_1\, i_2 \cdots i_n)}} \varphi(x_{i_1} \cdots x_{i_n}), \quad \alpha \in S_p.$$

- $\varphi_{(13)(624)(5)}(x_1, \dots, x_6) = \varphi(x_1 x_3)\varphi(x_6 x_2 x_4)\varphi(x_5)$.



- $\varphi_{\gamma_p}(x_1, \dots, x_p) = \varphi(x_1 \cdots x_p)$.



Free cumulants from permutations

Free cumulant $\kappa_\alpha : \mathcal{A}^p \to \mathbb{C}$ is defined from the moment-cumulant relation

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- By Möbius inversion, $\kappa_\alpha(x_1, \dots, x_p) = \sum_{\beta \leq \alpha} \varphi_\beta(x_1, \dots, x_p) \,\text{Möb}(\beta^{-1}\alpha)$ where $\text{Möb}(\sigma) := \prod_{c \in \mathrm{Cycle}(\sigma)} (-1)^{|c|} \,\text{Cat}_{|c|}$ is the Möbius function.
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Theorem (Speicher 1994)

$\mathcal{W}_1, \dots, \mathcal{W}_L \subset (\mathcal{A}, \varphi)$ are free if and only if every mixed free cumulant vanishes: $\kappa_p(x_{i_1}, \dots, x_{i_p}) = 0$ whenever $x_j \in \mathcal{W}_{i_j}$ and $i_l \neq i_k$ for some l, k.

Free cumulants from permutations

RECALL: $\varphi(x_1 \cdots x_p) = \sum_{\pi \in NC(p)} \kappa_\pi(x_1, \dots, x_p)$.

For a function $f : [p] \to [L]$, define the partition

$$\operatorname{Ker} f := \left\{ f^{-1}(i) \;\middle|\; i \in f([p]) \right\} \in \mathcal{P}(p).$$

Corollary (Joint distribution from free cumulants)

$\mathcal{W}_1, \dots, \mathcal{W}_L \subseteq (\mathcal{A}, \varphi)$ are freely independent if only if for every function $f : [p] \to [L]$ and $x_j \in \mathcal{W}_{f(j)}$,

$$\varphi(x_1 \cdots x_p) = \sum_{\substack{\pi \in NC(p), \\ \pi \leq \operatorname{Ker} f}} \prod_{i=1}^{L} \kappa_{\pi|_{f^{-1}(i)}}((x_j)_{j \in f^{-1}(i)}).$$

Outline

Tensor distribution of matrices

Definition (Collins, Gurau, and Lionni 2023)

For $X^{(1)}, \dots, X^{(p)} \in M_N(\mathbb{C})^{\otimes r}$*, define the* *trace invariant* *associated to* $\underline{\alpha} = (\alpha_1, \dots, \alpha_r) \in (S_p)^r$ *by*

$$\mathrm{Tr}_{\underline{\alpha}}(X^{(1)}, \dots, X^{(p)}) := \sum_{\text{all indices}} \left(\prod_{k=1}^{p} X^{(k)}_{i_1^{(k)} \dots i_r^{(k)}, j_1^{(k)} \dots j_r^{(k)}} \right) \prod_{s=1}^{r} \left(\prod_{k=1}^{p} \delta_{i_s^{(\alpha_s(k))}, j_s^{(k)}} \right).$$

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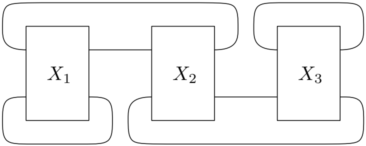
- $\mathrm{Tr}_{(12)(3),(1)(23)}(X^{(1)}, X^{(2)}, X^{(3)}) = \sum_{a_{1,2,3}, b_{1,2,3} \in [N]} X^{(1)}_{a_1 b_1, a_2 b_1} X^{(2)}_{a_2 b_2, a_1 b_3} X^{(3)}_{a_3 b_3, a_3 b_2}$



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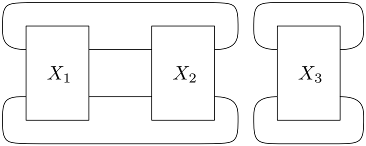
- $\mathrm{Tr}_{(12)(3),(12)(3)}(X_1, X_2, X_3) = \mathrm{Tr}(X_1 X_2) \cdot \mathrm{Tr}(X_3) = \mathrm{Tr}_{(12)(3)}(X_1, X_2, X_3).$



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- $\mathrm{Tr}_{\underline{\alpha}}$ is invariant under local unitaries:
  $\mathrm{Tr}_{\underline{\alpha}}(UX^{(1)}U^*, \dots, UX^{(p)}U^*) = \mathrm{Tr}_{\underline{\alpha}}(X^{(1)}, \dots, X^{(p)}), \quad U = \bigotimes_{s=1}^{r} U_s.$
- Tensor distribution = data of $\left(\mathbb{E} \circ \mathrm{tr}_{\underline{\alpha}} := \frac{1}{N^{\#\alpha_1 + \dots + \#\alpha_r}} \mathbb{E} \circ \mathrm{Tr}_{\underline{\alpha}}\right)_{\underline{\alpha} \in \bigcup_{p=1}^{\infty} (S_p)^r}$ for a family of random matrices $\mathcal{W}_N \subset M_N^{\otimes r}(L^{\infty-}(\mathbb{P}))$.

Algebraic tensor probability space

Definition (Nechita and P., 2025+)

An r-partite (algebraic) tensor probability space is a triple $(\mathcal{A}, \varphi, (\varphi_{\underline{\alpha}}))$:

- $(\mathcal{A}, \varphi)$ *is a NC probability space with a tracial state* φ*,*
- *for each* $p \geq 1$ *and* $\underline{\alpha} \in (S_p)^r$, $\varphi_{\underline{\alpha}} : \mathcal{A}^p \to \mathbb{C}$ *is a multilinear functional satisfying "reasonable conditions", e.g.* $\varphi_{\underline{\alpha}} = \varphi_\sigma$ *whenever* $\alpha_s \equiv \sigma$*.*

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EXAMPLE:

1. $(M_N(\mathbb{C})^{\otimes r}, \mathrm{tr}, (\mathrm{tr}_{\underline{\alpha}}))$.
2. $(\mathcal{A}^{\otimes r}, \varphi^{\otimes r}, (\bigotimes_{s=1}^r \varphi_{\alpha_s}))$ where $(\mathcal{A}, \varphi)$ is a NC probability space.

Tensor free cumulants and tensor free independence

Definition (Nechita and P., 2025+)

1. *The tensor free cumulants* $\kappa_{\underline{\alpha}} : \mathcal{A}^p \to \mathbb{C}$ *for* $\underline{\alpha} \in (S_p)^r$ *is defined by*

$$\varphi_{\underline{\alpha}}(x_1, \dots, x_p) = \sum_{\underline{\beta}:\, \beta_s \le \alpha_s \ \forall s} \kappa_{\underline{\beta}}(x_1, \dots, x_p).$$

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2. *Unital subalgebras* $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(L)}$ *of* $\mathcal{A}$ *are called* *tensor freely independent* *(or* *tensor free**) if* *every mixed tensor free cumulant vanishes**, i.e.,*

$$\kappa_{\underline{\alpha}}(x_1, \ldots, x_p) = 0 \text{ whenever } x_j \in \mathcal{A}_{f(j)},\ f(k) \neq f(l) \text{ for some } k, l \in [p], \text{ and } \Pi(\alpha_1) \vee_{\mathcal{P}} \cdots \vee_{\mathcal{P}} \Pi(\alpha_r) = 1_p$$

where $\Pi : S_p \to \mathcal{P}(p)$ *is the natural projection.*

EXAMPLE: if $x, y \in \mathcal{A}$ are tensor free, then

$$\kappa_{(12)(3),(1)(23)}(x, y, x) = \kappa_{(12)(3),(1)(23)}(x, x, y) = 0.$$

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2. $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(L)}$ *are tensor free if* $\kappa_{\underline{\alpha}}(x_1, \ldots, x_p) = 0$ *for every mixed tensor free cumulant.*

REMARK:

- $$\kappa_{\underline{\alpha}}(x_1, \ldots, x_p) := \sum_{\underline{\beta}\colon \beta_s \le \alpha_s \ \forall s} \varphi_{\underline{\beta}}(x_1, \ldots, x_p) \left(\prod_{s=1}^{r} \mathrm{Möb}(\beta_s^{-1}\alpha_s) \right).$$
- Tensor freeness $\longrightarrow$ the joint tensor distribution depends only on the marginal tensor distributions : for every $\underline{\alpha} \in (S_p)^r$ and $x_j \in \mathcal{A}^{(f(j))}$,

$$\varphi_{\underline{\alpha}}(x_1, \ldots, x_p) = \sum_{\substack{\forall s\colon \beta_s \le \alpha_s, \\ \bigvee_{\mathcal{P}} \Pi(\beta_s) \le \mathrm{Ker} f}} \prod_{i \in [L]} \kappa_{\underline{\beta}|_{f^{-1}(i)}}((x_j)_{j \in f^{-1}(i)}).$$

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Definition (Nechita and P., 2025+)

1. $$\varphi_{\underline{\alpha}}(x_1,\dots,x_p) = \sum_{\underline{\beta}\colon \beta_s \le \alpha_s\ \forall\, s} \kappa_{\underline{\beta}}(x_1,\dots,x_p).$$
2. *$\mathcal{A}^{(1)},\dots,\mathcal{A}^{(L)}$ are tensor free if $\kappa_{\underline{\alpha}}(x_1,\dots,x_p)=0$ for every mixed tensor free cumulant.*

REMARK:

- $$\kappa_{\underline{\alpha}}(x_1,\dots,x_p) := \sum_{\underline{\beta}\colon \beta_s \le \alpha_s\ \forall\, s} \varphi_{\underline{\beta}}(x_1,\dots,x_p)\left(\prod_{s=1}^{r} \mathrm{M\ddot{o}b}(\beta_s^{-1}\alpha_s)\right).$$
- Tensor freeness $\longrightarrow$ the joint tensor distribution depends only on the marginal tensor distributions.
- If $r=1$, $\kappa_{\underline{\alpha}} = \kappa_{\alpha}$ and free $=$ tensor free.
- If s_1, s_2 are free semicircular elements, then $s_1 \otimes s_1$ and $s_2 \otimes s_2$ are tensor free while they are not free.

Result 1: Local unitary invariance and tensor freeness

RECALL: $\mathrm{tr}_{\underline{\alpha}} = \frac{1}{N^{\#\alpha_1+\cdots+\#\alpha_r}}\mathrm{Tr}_{\underline{\alpha}}$.

Theorem (Nechita and P., 2025+)

$\mathcal{W}_N^{(1)}, \ldots, \mathcal{W}_N^{(L)}$: *independent families of $N^r \times N^r$ random matrices s.t.*

- *each $\mathcal{W}_N^{(i)}$ is local unitary invariant (except possibly one),*
- *each $\mathcal{W}_N^{(i)}$ converges in tensor distribution:*

$$\lim_{N\to\infty} \mathbb{E}\left[\mathrm{tr}_{\underline{\alpha}}(X_1, \ldots, X_p)\right] \text{ exists for all } X_j \in \mathcal{W}_N^{(i)} \text{ and } \underline{\alpha} \in (S_p)^r,$$

- *"tensor" factorization property: for $\underline{\alpha} \in (S_p)^r$, $\underline{\beta} \in (S_q)^r$, and $X_1, \ldots, X_p, Y_1, \ldots, Y_q \in \mathcal{W}_N^{(i)}$,*

$$\mathbb{E}\left[\mathrm{tr}_{\underline{\alpha}\sqcup\underline{\beta}}(\underline{X}, \underline{Y})\right] = \mathbb{E}[\mathrm{tr}_{\underline{\alpha}}(\underline{X})]\mathbb{E}[\mathrm{tr}_{\underline{\beta}}(\underline{Y})] + o(1).$$

Then $\mathcal{W}_N^{(1)}, \ldots \mathcal{W}_N^{(L)}$ are asymptotically tensor free as $N \to \infty$.

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- *+ "tensor" factorization property.*

Then $\mathcal{W}_N^{(1)}, \ldots \mathcal{W}_N^{(L)}$ are asymptotically tensor free as $N \to \infty$.

Idea of proof: apply (graphical) Weingarten calculus to show that

$$\lim_{N\to\infty} \mathbb{E}\left[\mathrm{tr}_{\underline{\alpha}}(X_{N,1}^{(f(1))}, \ldots, X_{N,p}^{(f(p))})\right] = \sum_{\substack{\forall s:\beta_s \le \alpha_s,\\ \Pi(\beta_s) \le \mathrm{Ker} f}} \prod_{i=1}^{L} \kappa_{\underline{\beta}|_{f^{-1}(i)}}((x_j)_{j\in f^{-1}(i)}).$$

Partial transpose and free independence

QUESTION: Can we find examples of non-independent random matrices which are asymptotically tensor free?

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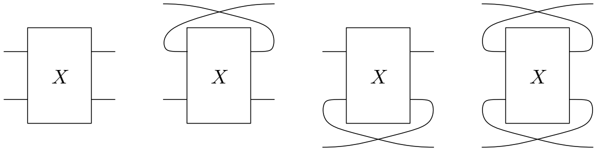
QUESTION: Can we find examples of non-independent random matrices which are asymptotically tensor free?

Examples for free independence:

- (Mingo and Popa 2016) If X is UI, then X and $X^\top$ are asymp. free.
- (Mingo and Popa 2019) If $X = GG^*$ is an $N^2 \times N^2$ Wishart matrix, then all the partial transposes

$$X, \ (\top \otimes \mathrm{id}_N)(X), \ (\mathrm{id}_N \otimes \top)(X), \ X^\top = (\top \otimes \top)(X)$$

  of X are asymptotically free.
- (Mingo and Popa 2024) The same holds if $X \in M_N^{\otimes 2}(L^{\infty-}(\mathbb{P}))$ is UI.



Result 1': Partial transpose and tensor freeness

Theorem (Nechita and P., 2025+)

Let X_N be an $N^r \times N^r$ random matrix such that

- *X_N is (globally) unitary invariant,*
- *X_N converges in distribution and satisfies the factorization property.*

Then, the family of 2^r partial transposes of X_N

$$\left\{(\Phi_1 \otimes \cdots \otimes \Phi_r)(X_N) \,\middle|\, \Phi_s \in \{\mathrm{id}_N, \top\} \;\forall\, s\right\},$$

becomes both asymp. free and asymp. tensor free as $N \to \infty$.

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In particular, for a UI matrix X_N having (usual) distribution limit,

- X_N converges in tensor distribution,
- All partial transposes $(\Phi_1 \otimes \cdots \otimes \Phi_r)(X_N)$ converges in tensor distribution.

Similar statement can be shown for orthogonal invariant random matrix.

Outline

Free CLT

Theorem (Voiculescu 1983; 1985)

Let $\{x_i\}_{i=1}^{\infty} \subset (\mathcal{A}, \varphi)$ *be a sequence of centered, identically distributed, and free elements. Then* $\frac{1}{\sqrt{N}}(x_1 + \cdots + x_N)$ *converges in distribution to a semicircular element of variance* $\sigma^2 = \varphi(x_1^2)$*:*

$$\lim_{N\to\infty} \varphi\left(\left(\frac{1}{\sqrt{N}}(x_1 + \cdots x_N)\right)^p\right) = \sigma^p \int x^p d\mu_{SC}(x), \quad p \geq 1.$$

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Simple argument using free cumulants:

$$\kappa_p\left(\frac{x_1 + \cdots + x_N}{\sqrt{N}}\right) = N^{-p/2} \cdot N\kappa_p(x_1) \to \begin{cases} \kappa_2(x_1) = \varphi(x_1^2) & \text{if } p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

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RECALL: $\Pi : S_p \to \mathcal{P}(p)$, the natural projection.

Proposition

If $x_1, \ldots, x_k$ are tensor free and $\underline{\alpha} \in (S_p)^r$ satisfies

$$\Pi(\alpha_1) \vee_{\mathcal{P}} \cdots \vee_{\mathcal{P}} \Pi(\alpha_r) = 1_p, \text{ then } \kappa_{\underline{\alpha}}(x_1 + \cdots + x_k) = \sum_{i=1}^{k} \kappa_{\underline{\alpha}}(x_i).$$

Result 2: Tensor free CLT

Theorem (Nechita and P., 2025+)

Let $\{x_i\}_{i=1}^{\infty} \subset (\mathcal{A}, \varphi, (\varphi_{\underline{\alpha}}))$ be a sequence of centered, identically tensor distributed, and tensor free elements. Then

$$\frac{1}{\sqrt{N}}(x_1 + \cdots + x_N) \to \sum_{\underline{\alpha} \in (S_2)^r \setminus \{\underline{\mathrm{id}_2}\}} \sqrt{\kappa_{\underline{\alpha}}(x_1)}\, s_{\underline{\alpha}}$$

in tensor distribution as $N \to \infty$, where $(s_{\underline{\alpha}})$ are tensor free family of semicircular elements with

$$\kappa_{\underline{\beta}}(s_{\underline{\alpha}}) = \begin{cases} \delta_{\underline{\alpha},\underline{\beta}} & \text{if } p = 2, \\ 0 & \text{otherwise,} \end{cases}, \quad \underline{\beta} \in (S_p)^r.$$

In particular, the corresponding limit is universally governed by $2^r - 1$ semicircular elements.

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- $r = 1$: Free CLT.
- $r = 2$: governed by three semicircular elements

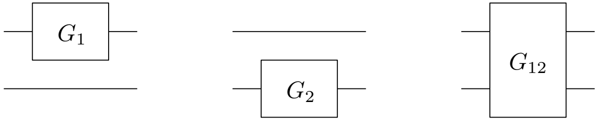
$$s_1 := s_{\gamma_2, \mathrm{id}_2}, \quad s_2 := s_{\mathrm{id}_2, \gamma_2}, \quad s_{12} := s_{\gamma_2, \gamma_2},$$

$$\text{where} \begin{cases} s_1 \text{ and } s_2 \text{ are (classically) independent,} \\ s_{12} \text{ and } \{s_1, s_2\} \text{ are free.} \end{cases}$$

Result 2: Tensor free CLT

- Indeed, $s_1 := s_{\gamma_2, \mathrm{id}_2}$, $\quad s_2 := s_{\mathrm{id}_2, \gamma_2}$, $\quad s_{12} := s_{\gamma_2, \gamma_2}$ can be understood as a tensor distribution limit of tensor GUE models:

$$G_1 \otimes I_N,\ I_N \otimes G_2,\ G_{12} \xrightarrow{\otimes\text{-distr}} s_1,\ s_2,\ s_{12}$$



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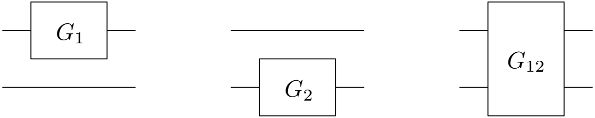
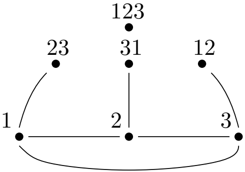


- For general $r \geq 2$, the family $(s_{\underline{\alpha}})_{\underline{\alpha} \in (S_2)^r \setminus \{\underline{\mathrm{id}_2}\}}$ are ε-free wrt the graph $\varepsilon = (V, E)$: $V = \{I \subseteq [r] \mid I \neq \varnothing\}$; $(I, J) \in E$ iff $I \cap J = \varnothing$.



Application: CLT for tensor products of free variables

Suppose that $\{a_i\}_{i=1}^{\infty} \subset (\mathcal{A}, \varphi)$ is a family of self-adjoint, identically distributed free elements with mean $\varphi(a_i) = \lambda$ and variance $\mathrm{var}(a_i) = \sigma^2$. Consider the identically tensor distributed and tensor free elements

$$x_i := a_i \otimes a_i \in (\mathcal{A}^{\otimes 2}, \varphi^{\otimes 2}, (\varphi_{\alpha_1} \otimes \varphi_{\alpha_2})).$$

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Corollary (Lancien, Santos, and Youssef 2024; Skoufranis 2024+)

The normalized sum $\frac{1}{\sqrt{N}}(x_1 + \cdots + x_N - N\lambda^2)$ *converges in distribution to*

$$\sigma\lambda(s_1 + s_2) + \sigma^2 s_{12} \sim_{\mathrm{distr}} (D_{\sigma\lambda}[\mu_{SC}] * D_{\sigma\lambda}[\mu_{SC}]) \boxplus D_{\sigma^2}[\mu_{SC}]$$

where $\begin{cases} * = & \text{(classical) convolution,} \\ \boxplus = & \text{free convolution,} \\ D_c[\mu_{SC}] = & \text{push-forward measure of } \mu_{SC} \text{ wrt the map } t \mapsto ct. \end{cases}$

Conclusions and outlook

In this work, we provided

- Convergence of LUI random matrices in the framework of tensor free probability.
- Asymptotic behavior of local orthogonal invariant random matrices.
- CLT for tensor free elements and its applications.

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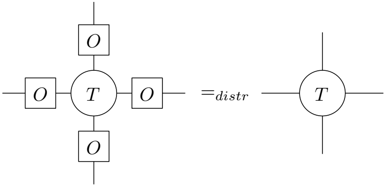
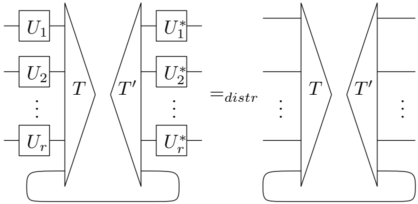
Recent works on random tensors + free probability:

Kunisky, Moore, and Wein: [arXiv:2404.18735],
Bonnin and Bordenave: [arXiv:2407.18881]



Collins, Gurau, and Lionni: [arXiv:2410.00908]



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Future works:

- Any possible integrated theory of random tensors:
  - Kunisky, Moore, and Wein: arXiv:2404.18735,
  - Bonnin and Bordenave: arXiv:2407.18881,
  - Collins, Gurau, and Lionni: arXiv:2410.00908.
- Higher-order / strong behavior of random tensors.
- Application to Quantum Information Theory.

Thank you for your attention!