Trace Simplex of free products

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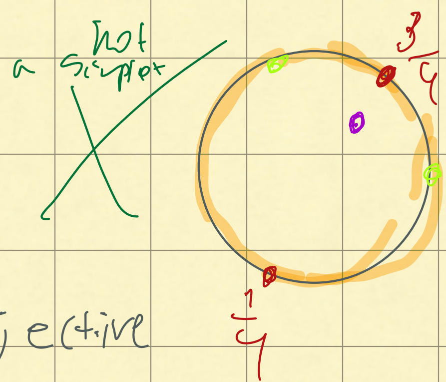
I. Choquet Theory

Let V be a metrizable TVS loc convex

$C \subseteq V$ compact convex

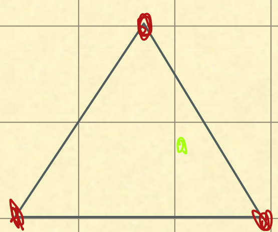
$$\mathrm{Prob}(\partial_e C) \xrightarrow{\text{barycenter}} C$$

extreme pts



By Choquet's thm bary is surjective

Def: C is called a simplex if bary is bijective



is a simplex

Warning: $\partial_e C$ is not nec compact.

$\emptyset, \{0\} \neq \mathcal{P}$

In fact (Poulsen 61) there exist a metrizable simplex $\mathcal{P}$ s.t. $\overline{\partial_e \mathcal{P}} = \mathcal{P}$.

Def: a simplex w/ this prop is called a Poulsen simplex

<u>Fact:</u> (LOS 78')

1) Any two (non-trivial, metrizable) Poulsen simplices affinely homeomorphic (<u>The</u> Poulsen simplex)

2) $\mathcal{P}$ is characterized by the following two prop.

a) <u>universality</u>: any met. simplex is a face of $\mathcal{P}$

b) <u>homogeneity</u>: $\forall F_1, F_2 \subseteq_f \mathcal{P}$ faces, if $F_1 \cong F_2$ then $\exists \varphi \in Aut(\mathcal{P})$ $\varphi(F_1) = F_2$

3) $\partial_e \mathcal{P}$ is homeomorphic to $\ell^2(\mathbb{Z})$

II. Trace simplices.

Let A be a unital and separable C^*-alg.

$T(A)$ – the space of traces.

$\varphi \in T(A) \xleftrightarrow{\quad GNS \quad}$
$$\begin{tikzcd} A \arrow[r, hook, "\pi"] \arrow[dr, dashed, "\varphi"'] & M \subseteq B(\mathcal{H}) \arrow[d, "\tau"] \\ & \mathbb{C} \end{tikzcd} \qquad M = \pi(A)''$$

$\varphi = \tau \circ \pi$ $\quad \varphi$ is extreme (metrizable) $\longleftrightarrow$ M is a factor

<u>Fact:</u> $T(A)$ is a Γ simplex.

In fact any met. simplex arise as such.

III. Free Products.

$A = A_1 * A_2$ (coproduct in the catagory of unital C^*-algs)

$$\begin{tikzcd}
A_1 \arrow[dr] \arrow[drr, bend left] & & \\
& A \arrow[r, dashed] & B \\
A_2 \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

$$\overline{\mathrm{span}_{\mathbb{C}}\, x_1 y_1 \cdots x_n y_n}^{\|\cdot\|_{max}}$$

<u>General problem</u>: Describe $T(A_1 * A_2)$ in terms of $T(A_1), T(A_2)$

<u>Our result</u>: $T(A_1 * A_2)$ is "typically" the Poulsen simplex.

Ex: $C^*(\mathbb{F}_n)$, $M_n(\mathbb{C}) * M_n(\mathbb{C})$, $n \geq 4$

Orovitz-Slutsky-V aui, ...

<u>Cor</u>: A_1, A_2 are f.d then $T(A_1 * A_2)$ is Poulsen unless at least one of A_1, A_2 have a $\mathbb{C}$-direct summand.

<u>Obstruction</u> (2 types

I. triviality:

- $T(A_1) = \emptyset \Rightarrow T(A_1 * A_2) = \emptyset$
- $T(A_1) = \{\varphi\} \Rightarrow T(A_1 * A_2) = T(A_2)$

φ is 1-dim character

II. Isolated characters.

$\varphi_i \in \partial_e T(A_i)$ 1-dim (character) and if isolated

$\Rightarrow \varphi = \varphi_1 \varphi_2 \in \partial_e T(A)$ 1-dim, isolated.

Pf $\Rightarrow T(A)$ is not Poulsen because $T_e \mathcal{P} \cong \ell^2(\mathbb{C})$

connected.

Suppose $\psi \in \partial_e T(A)$ $\psi \approx \varphi = \tau \circ \pi$ $\pi : A \to (M, \tau)$

$\psi|_{A_i} \approx \varphi_i$ isolated $\psi|_{A_i} = 0.9\, \varphi_i + 0.1\, \xi$

$p_i \in \pi(A_i)'' \subseteq M$

$\tau(p_i) = 0.9$

$p = p_1 \wedge p_2 \in Z(M)$

$\tau(p) = \tau(p_1) + \tau(p_2) - \tau(p_1 \vee p_2) \geq 0.9 + 0.9 - 1 > 0$

$p \neq 0$. Contradiction.

– $\varphi_1 \in \partial_e T(A_1)$ 1-dim and isolated

$\varphi_2 \in \partial_e T(A_2)$ fin-dim and isolated

$\varphi = \varphi_1 * \varphi_2 \in \partial_e T(A)$ isolated $\Rightarrow T(A)$ not Poulsen.

<u>Thm</u>: Assume $T(A_i)$ is not empty, and not $= \{\varphi\}$ for φ 1-dim.

TFAE:

1) $T(A_1 * A_2)$ is Poulsen simplex

2) $T(A_1 * A_2)$ does not consist of of an isolated extremal pt.

3) If $T(A_1)$ has an extreme ~~fin~~ 1-dim trace then $T(A_2)$ does not have an extreme 1-dim character (and vice versa)

$$\overbrace{\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}} * \overbrace{\mathbb{Z}/2\mathbb{Z}}$$

$\downarrow$

isolated 1-dim character

$$D_\infty = \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}$$

1, sgn $\theta \in S^1$

$\mathrm{Ind}_{\mathbb{Z}}^{D_\infty} z_\theta$ zero outside $\mathbb{Z}$.

<u>Pf Ideas</u> $A = A_1 * A_2$

Let $\varphi \in T(A)$. Want to find $\psi \in \partial_e T(A)$ $\psi \approx \varphi$.

$\pi : A \longrightarrow M$

$\pi|_{A_i} = \pi_i : A \to M$ Goal: Find $u \in \mathcal{U}(M)$ $\|u - 1\|_2 < \varepsilon$

$\hat{\pi} : A \to M$ $\hat{\pi}|_{A_1} = \pi|_{A_1}$ clearly $\hat{\pi} \approx \pi$

$\hat{\pi}|_{A_2} = \mathrm{Ad}_u \circ \pi|_{A_2}$ $\hat{\varphi} \approx \varphi$

Is $\hat{\pi}(A)''$ a factor (equiv is $\hat{\varphi}$ extreme)

Rmk: This strategy shows densitity in the Wassserstein topology.

$\pi : A \longrightarrow M$

$\pi(A_1)'' = M_1$ $\qquad$ $\pi(A_2)'' = M_2$ $\qquad$ (M, M_1, M_2)

$M_1 \vee M_2 = M$

Def: A triple (M, M_1, M_2) is <u>approximately factorial</u> if $\forall \varepsilon > 0$ $\exists \hat{M} \supseteq M$ $\exists u \in \mathcal{U}(\hat{M})$ $\|u - 1\|_2 < \varepsilon$ s.t $\underbrace{M_1 \vee u M_2 u^*}_{\cap \hat{M}}$ is II_1 factor.

Obstructions: $-$ $M_1 = \mathbb{C}$ or $M_2 = \mathbb{C}$

$-$ $M_1 = M_2 = \mathbb{C}^2$

Define (for a tracial vN alg (M, τ))

$$e(M) = \max \left\{ \tau(p) \,\middle|\, p \text{ minimal proj} \right\}$$

Note: If $e(M_1) + e(M_2) > 1$

$\underset{\text{minimal}}{p_1 \in M_1}$ $\qquad$ $\underset{\text{minimal}}{p_2 \in M_2}$ $\qquad$ $\tau(p_1) + \tau(p_2) > 1$

then $p = p_1 \wedge p_2 \in Z(M)$ $\qquad$ $\tau(p) > 0$

Thm B: Let $(M, \underset{\cup\!\!|}{M_1}, \underset{\cup\!\!|}{M_2})$, $M_1 \vee M_2 = M$. Assume $M_1, M_2 \neq \mathbb{C}$ that $\dim M_1 + \dim M_2 \geq 5$, and assume
$$e(M_1) + e(M_2) \leq 1.$$
Then (M, M_1, M_2) is app. factorial.

Can you always chose $\hat{M} = M * L(\mathbb{Z})$?

or even $\tilde{M} = M$?

Ideas from the pf of Thm B.

- Proof is easy if M_1 (or M_2) is diffuse.

$$\hat{M} = M * L(\underset{\underset{u_t}{\cup}}{\mathbb{Z}}) \qquad M = M_1 \vee M_2$$

u_t haar unitary

$u_t \xrightarrow[\|\cdot\|_2]{} 1$

$M_1 \vee u_t M_2 u_t^*$.

- We were very influence by the work of Dykema.

$$(\underset{\alpha_1}{\mathbb{C}p_1} \oplus \dots \oplus \underset{\alpha_n}{\mathbb{C}p_n} \oplus D) * (\underset{\beta_1}{\mathbb{C}q_1} \oplus \dots \oplus \underset{\beta_n}{\mathbb{C}q_n} \oplus D')$$

$$\mathbb{C}p_i \wedge q_j$$

Assume $\underset{\underset{A_1}{\cup\!\!|}}{M_1}, \underset{\underset{A_2}{\cup\!\!|}}{M_2}$ are not diffuse.

MASA

$$\tilde{M} = M *_{A_2} (A_2 \otimes L(\underset{\underset{u_t}{\cup}}{\mathbb{Z}})) \qquad \text{IPP05}$$

Can reduce to the case M_1 and M_2 are abelian and finite dimensional.

$M_1 = \underset{\frac{1}{2}}{\mathbb{C}} \oplus \underset{\frac{1}{2}}{\mathbb{C}}$ $\qquad$ $M_2 = \underset{\frac{1}{2}}{\mathbb{C}} \oplus \underset{\frac{1}{2}}{\mathbb{C}}$ $\qquad$ Collins - Kemp

Diffuse.

diffuse abelian

<u>Thm C:</u> $M_1 = \mathbb{C}p_1 \oplus \dots \oplus \mathbb{C}p_m \oplus M_1 p$

$M_2 = \mathbb{C}q_1 \oplus \dots \oplus \mathbb{C}q_n \oplus M_2 q$ $\qquad$ $M = M_1 * M_2$

Assume $M_1, M_2 \neq \mathbb{C}$ $\qquad$ $\dim M_1 + \dim M_2 \geq 5$

$$[e(M_1) + e(M_2) \leq 1]$$

Then $\forall \epsilon$ $\exists \hat{M} \supseteq M$ and $u \in \hat{M}$ $\|u - 1\|_2 < \epsilon$ s.t. p_i and uq_ju^* are in general position $\forall i,j$

$\tau(p_i \vee q_j) = \min\{\tau(p_i) + \tau(q_j), 1\}$

and $M_1 \vee uM_2u^* \cong \bigoplus_{i,j} \mathbb{C}\cancel{(p_i \wedge (uq_ju^*))} \oplus B$

where B is a II_1 factor

Cor if $e(M_1) + e(M_2) < 1$