The Quasicentral Modulus for a New Class of Operators

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History I

- Weyl-von Neumann: T self-adjoint $\implies$ there exists diagonal D such that $T - D \in K(\mathcal{H})$ (vN: $\mathfrak{S}_2$)
- Kuroda: Can be improved to every ideal larger than $\mathfrak{S}_1$
- Berg: T can be normal so that $T - D \in K(\mathcal{H})$
- Voiculescu: $\tau = (T_1, T_2, \ldots, T_n)$, $(n \geq 2)$ commuting, self-adjoint $\implies$ there exists diagonal n-tuple Δ so that $\tau - \Delta \in \mathfrak{S}_n$

Question

How "good" can we make our ideal?

History II

- Can self-adjoint T (with ac spectral measure) be diagonal modulo $\mathfrak{S}_1$?
  Kato-Rosenblum: No.
- What's the largest ideal $\mathcal{J}$ s.t. commuting, self-adjoint n-tuple τ with ac spectral measure is *not* diagonal modulo $\mathcal{J}$?
  Bercovici-Voiculescu: $\mathfrak{S}_n^-$ ($\subset \mathfrak{S}_n$)
- What if the spectral measure of the n-tuple is *not* ac with respect to Lesbegue measure?

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Preliminary notation

- $\mathcal{H}$ a separable, infinite dimensional, complex Hilbert space
- c_{00} the vector space of real sequences with finitely many nonzero terms
- Denote by $\mathcal{SN}$ the set of norms Φ on c_{00} that are permutation invariant and invariant under $|\cdot|$
  - We consider the norms that satisfy $\Phi((1,0,0,\dots)) = 1$
  - These norms are called *symmetric norming functions*
- F a positive finite rank operator, $\{s_j\}_{j=1}^N$ the singular values of F:

$$|F|_\Phi := \Phi((s_1, s_2, \dots, s_N, 0, 0, \dots))$$

The ideals $\mathfrak{S}_\Phi$

- T an arbitrary bounded linear operator, $\mathcal{P}$ the directed set of orthogonal projections:

$$|T|_\Phi := \sup_{P \in \mathcal{P}} |PT|_\Phi$$

- Let $\mathfrak{S}_\Phi := \{T \in B(\mathcal{H}) : |T|_\Phi < +\infty\}$.
  - $\mathfrak{S}_\Phi$ is an ideal
  - $(\mathfrak{S}_\Phi, |\cdot|_\Phi)$ is a Banach space

Examples of $\mathfrak{S}_\Phi$

Ex. $\mathfrak{S}_p$, $p \geq 1$

Ex. $\{s_k^*\}_{k=1}^\infty :=$ non-increasing rearrangement of $\{s_k\}_{k=1}^\infty$
$|F|_p^- := \sum_{k=1}^\infty k^{-(1-1/p)} s_k^*$
$\mathfrak{S}_p^- := \{T \in B(\mathcal{H}) : |T|_p^- < +\infty\}, p \geq 1$

Ex. $\pi = \{\pi_k\}_{k=1}^\infty \subset \mathbb{R}$ so that:

- $\pi_1 \geq \pi_2 \geq \dots$
- $\pi_k \to 0$
- $\sum_{k=1}^\infty \pi_k = +\infty$
- $\sum_{k=1}^m \pi_k \leq \alpha m \pi_m$ for every $m \in \mathbb{N}$

$|F|_\pi := \sum_{k=1}^\infty \pi_k s_k^*$
$\mathfrak{S}_\pi := \{T \in B(\mathcal{H}) : |T|_\pi < +\infty\}$

Operations on n-tuples of operators

Let $\tau^{(j)} = (T_1^{(j)}, \ldots, T_n^{(j)})$, $j = 0, 1, 2$ with $T_k^{(j)} \in B(\mathcal{H})$

- $A, B \in B(\mathcal{H})$, $A\tau^{(0)}B := (AT_1^{(0)}B, \ldots, AT_n^{(0)}B)$
- $\tau^{(1)} + \tau^{(2)} := (T_1^{(1)} + T_1^{(2)}, \ldots, T_n^{(1)} + T_n^{(2)})$
- $||\tau^{(0)}|| := \max_k ||T_k^{(0)}||$, $|\tau^{(0)}|_\Phi := \max_k |T_k^{(0)}|_\Phi$

The quasicentral modulus

Denote by $\mathcal{R}_1^+$ the directed set of positive, finite rank contractions

Definition

The *quasicentral modulus* is defined as

$$k_\Phi(\tau) := \liminf_{A \in \mathcal{R}_1^+} |[A, \tau]|_\Phi$$

Note (Voiculescu): τ an n-tuple of commuting, SA operators, $k_\Phi(\tau) = 0$ iff there exists diagonal Δ s.t. $\tau - \Delta \in \mathfrak{S}_\Phi^{(0)}$

Obstruction ideals

Definition

If $k_\Phi(\tau) > 0$, $\mathfrak{S}_\Phi$ is said to be an *obstruction ideal* for τ.

Goal 1

We seek to find the maximum obstruction ideals for a class of tuples.

Fact (Bercovici, Voiculescu): Any obstruction ideal of τ is contained in another obstruction ideal of the form $\mathfrak{S}_\pi$.

Hausdorff measures

Fix $f : [0,\infty) \to [0,\infty)$ so that

- f is increasing
- $f((0,+\infty)) \subset (0,+\infty)$
- $\lim_{t\to 0} f(t) = 0$

Definition

The *outer Hausdorff measure* corresponding to *gauge function f* is given by

$$H_f^*(A) := \liminf_{r\to 0} \left\{ \sum_{j=1}^{\infty} f(r_j) : A \subset \bigcup_{j=1}^{\infty} B(x_j, r_j) \text{ and } r_j < r \right\}$$

Denote by H_f the restriction of H_f^* to the Borel σ-algebra.

Relevant gauge functions

Definition

Given $s \geq 1$, we say that a gauge function $f : [0, \infty) \to [0, \infty)$ has property (R_s) if

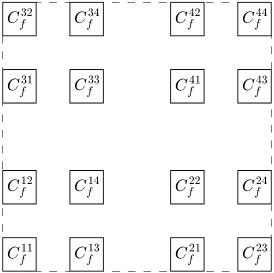
- $f \in C^2$
- $f'(0) = 0$
- for every $a \in \mathbb{R}$, $\lim_{x\to 0} f(ax)/f(x) = a^s$

Fact: If f is a gauge function with property (R_s), then

$$\lim_{x\to 0} \frac{f^{-1}(x)}{f^{-1}(ax)} = \frac{1}{a^{1/s}}$$

Construction of relevant fractals

Given a function f with property (R_s), we construct the following fractal:



Properites of SGCS

1. $C_f = \bigcup_{|w|=L} C_f^w$
2. $0 < H_f(C_f) < +\infty$
3. $\xi \cdot 2^{-n|w|} < H_f(C_f^w) < 2^{-n|w|}$
4. $Cf(r) \leq H_f(C_f \cap B(x,r)) \leq Df(r)$ for every $r > 0$

Preliminary lemmas I

Lemma

Let $\tau \in (B(\mathcal{H}))^n$ and let $(A_j)_{j=1}^{\infty} \subset \mathcal{R}_1^+$ be a sequence that converges to the identity operator in WOT. Then

$$k_\Phi(\tau) \leq \liminf_j |[A_j, \tau]|_\Phi$$

Lemma

Let $\tau \in (B(\mathcal{H}))^n$. Then there exists an increasing sequence $(A_j)_{j=1}^{\infty} \subset \mathcal{R}_1^+$ such that converges to the identity operator in WOT, and

$$k_\Phi(\tau) = \lim_{j\to\infty} |[A_j, \tau]|_\Phi$$

Preliminary lemmas II

Lemma (Voiculescu, 1979)

If $\{\tau^{(j)}\}_{j=1}^{\infty} \subset B(\mathcal{H})^n$, $\lambda^{(j)} \in \mathbb{C}^n$, then:

1. $\max_{j=1,2}\{k_\Phi(\tau^{(j)})\} \leq k_\Phi(\tau^{(1)} \oplus \tau^{(2)}) \leq k_\Phi(\tau^{(1)}) + k_\Phi(\tau^{(2)})$
2. $k_\Phi(\bigoplus_{j=1}^{\infty} \tau_j) = \lim_{N\to\infty} k_\Phi(\bigoplus_{j=1}^{N} \tau_j)$
3. $k_\Phi(\tau^{(1)} \oplus \cdots \oplus \tau^{(N)}) =$
   $k_\Phi((\tau^{(1)} - \lambda^{(1)} \otimes I) \oplus \cdots \oplus (\tau^{(N)} - \lambda^{(N)} \otimes I))$.

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τ_{Ω,H_f}

The spectral theorem allows a reduction to finding the maximum obstruction ideals for the following:

Notation

Let f have property (R_s) and $\Omega \subset \mathbb{R}^n$ such that $Cf(r) \leq H_f(B(x,r) \cap \Omega) \leq Df(r)$.

$$\tau_{\Omega,\mu} := (T_{1,\mu}, T_{2,\mu}, \dots, T_{n,\mu})$$

$$(T_{j,\mu}g)(x) := x_j g(x), g \in L^2(\Omega, \mu)$$

Goal 1 (refined)

We seek to find the maximum obstruction ideals for τ_{Ω,H_f}.

When is $0 < k_\pi(\tau_{\Omega,H_f}) < +\infty$? I

Lemma (Voiculescu, 2021)

If $0 < H_p(\Omega) < +\infty$ for appropriate Ω (ex. Cantor dust), $0 < k_p^-(\tau_{\Omega,H_p}) < +\infty$.

Lemma

Let $f : [0,+\infty) \to [0,+\infty)$ be a non-decreasing function with property (R), and let $\pi = (\pi_k)_{k=1}^\infty$ be a regular non-increasing sequence such that $\pi_k \to 0$ and $\sum_{k=1}^\infty \pi_k = +\infty$. Suppose that

$$0 < \inf_m m\pi_m f^{-1}(1/m) \le \sup_m m\pi_m f^{-1}(1/m) < +\infty$$

Then $0 < k_\pi(\tau_{\Omega,H_f}) < +\infty$.

When is $0 < k_\pi(\tau_{\Omega,H_f}) < +\infty$? II

Sketch of proof: $0 < k_\pi(\tau_{\Omega,H_f})$ follows directly from a result of David and Voiculescu (1990). We show $k_\pi(\tau_{\Omega,H_f}) < +\infty$. We find a sequence of projections P_j so that $\lim_{j\to\infty} ||[P_j, \tau_{\Omega,H_f}]||_\pi < +\infty$.

- Cover Ω by disjoint Borel sets $\{\omega_{\ell,m}\}_{\ell=1}^{m^n}$ such that $H_f(\omega_{\ell,m}) \leq D/m^n$
- Consider $P_m^{(\ell)} v = \langle v, \chi_{\omega_{\ell,m}}\rangle \frac{\chi_{\omega_{\ell,m}}}{||\chi_{\omega_{\ell,m}}||_2^2}, v \in L^2(\Omega, H_f)$ and $P_m = \sum_{\ell=1}^{m^n} P_m^{(\ell)}$
- Note: $[P_m^{(\ell)}, T_j]$ has at most rank 2, so $|||[P_m^{(\ell)}, T_j]||| = |||[P_m^{(\ell)} - \lambda I, T_j]||| \leq 2\text{diam}(\omega_{\ell,m}) \leq 2\sqrt{n} D f^{-1}(1/m^n)$

When is $0 < k_\pi(\tau_{\Omega,H_f}) < +\infty$? III

$$
\begin{aligned}
k_\pi(\tau_{\Omega,H_f}) \le |[P_m, \tau_{\Omega,H_f}]|_\pi &\le \max_j \|[P_m^{(\ell)}, T_j]\| \Phi_\pi(\underbrace{1, 1, \ldots, 1}_{m^n \text{ times}}, 0, 0, \ldots) \\
&\le 2D\sqrt{n} f^{-1}(1/m^n) \Phi_\pi(\underbrace{1, 1, \ldots, 1}_{m^n \text{ times}}, 0, 0, \ldots) \\
&= 2D\sqrt{n} f^{-1}(1/m^n) \sum_{k=1}^{m^n} \pi_k \\
&\le 2\alpha D\sqrt{n} \sup_m m\pi_m f^{-1}(1/m) < +\infty
\end{aligned}
$$

Finding the correct π

Need: $f^{-1}(1/m^n)\sum_{k=1}^{m^n}\pi_k < +\infty$

What if: $f^{-1}(1/m^n)\sum_{k=1}^{m^n}\pi_k \to 1$?

Note:

$$
\begin{aligned}
& f^{-1}(1/m^n)\sum_{k=1}^{m^n}\left(\frac{1}{f^{-1}(1/x)}\right)'\Bigg|_{x=k} \\
& \le f^{-1}\left(\frac{1}{m}\right)\int_1^m\left(\frac{1}{f^{-1}(1/x)}\right)'dx \\
& = f^{-1}\left(\frac{1}{m}\right)\left[\frac{1}{f^{-1}(1/x)}\right|_1^m \longrightarrow 1
\end{aligned}
$$

The sequence ρ

Does the proposed sequence work? Not quite.

Lemma

Let $f : [0, +\infty) \to [0, +\infty)$ be a logarithmically-concave function with property (R_s). Let $h(x) = 1/f^{-1}(1/x)$ and set $\rho_k := h'(k)$, $k \in \mathbb{N}$. Then for some positive integer N, $\rho := \{\rho_k\}_{k=N}^{\infty}$ defines a symmetric-norming function Φ_ρ.

The maximum obstruction ideal for τ_{Ω,H_f}

Theorem

Let $f : [0,\infty) \to [0,\infty)$ be a logarithmically-concave function with property (R_s), and let $\Omega \subset \mathbb{R}^n$ be a Borel set so that $0 < H_f(\Omega) < +\infty$. Then $\mathfrak{S}_\rho$ is an obstruction ideal for τ_{Ω,H_f}. Furthermore, if $\Phi \in \mathcal{SN}$, then $k_\Phi(\tau_{\Omega,H_f}) \neq 0$ if and only if $J_\Phi \subset J_\rho$.

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Motivation

Theorem (Voiculescu, 2021)

Let τ be an n-tuple of commuting self-adjoint operators. If m is the multiplicity function for τ, then

$$(k_p^-(\tau))^p = C \int_{\sigma(\tau)} m(x) dH_p(x).$$

Goal 2

Is there a similar formula for k_ρ?

Ampliation homogeneity for k_p^- I

One of the main ingredients of the proof for the exact formula for k_p^- is the following ampliation homogeneity result:

Theorem (Voiculescu, 2021)

If τ is an n-tuple of commuting, self-adjoint operators, then for $1 \leq p \leq +\infty$

$$k_p^-(\tau \otimes I_m) = m^{1/p} k_p^-(\tau)$$

Note: $T \in \mathfrak{S}_p^-$,

$$\begin{aligned} |T \otimes I_m|_p^- =& 1^{-(1-1/p)} s_1 + 2^{-(1-1/p)} s_1 + \cdots + m^{-(1-1/p)} s_1 \\ &+ (m+1)^{-(1-1/p)} s_2 + \cdots + (2m)^{-(1-1/p)} s_2 \\ &+ (2m+1)^{-(1-1/p)} s_3 + \dots \end{aligned}$$

Ampliation homogeneity for k_p^- II

$$
\begin{aligned}
|T \otimes I_m|_p^- &= \sum_{k=1}^{\infty} s_k \sum_{\ell=(k-1)m+1}^{km} \ell^{-(1-1/p)} \\
&= \sum_{k=1}^{\infty} s_k \left(\sum_{\ell=1}^{km} \ell^{-(1-1/p)} - \sum_{\ell=1}^{(k-1)m} \ell^{-(1-1/p)} \right) \\
&= \sum_{k=1}^{\infty} (s_{k+1} - s_k) \sum_{\ell=1}^{km} \ell^{-(1-1/p)}
\end{aligned}
$$

Ampliation homogeneity for k_p^- III

$$\begin{aligned}
|T \otimes I_m|_p^- &\leq \sum_{k=1}^{\infty}(s_{k+1} - s_k) \int_1^{mk} x^{-(1-1/p)} dx \\
&\approx \sum_{k=1}^{\infty}(s_{k+1} - s_k)(mk)^{1/p} \\
&= m^{1/p} \sum_{k=1}^{\infty}(s_{k+1} - s_k) k^{1/p} \\
&= \ldots \\
&= m^{1/p} |T|_p^-
\end{aligned}$$

Moral of the story: multiplicativity of the gauge function was central, but multiplicativity of f was not assumed.

Motivation for $\rho^{(\varepsilon)}$

... but "approximate multiplicativity" is!

Definition

Let $f : [0,\infty) \to [0,\infty)$ be a logarithmically-concave function with property (R_s). For a fixed $\varepsilon > 0$, define $\rho^{(\varepsilon)} = \{\rho_k\}_{k=N}^{\infty}$ with N chosen so that for every $k > N$ and for every positive integer m we have

$$\left| \frac{f^{-1}(1/k)}{f^{-1}(1/mk)} - m^{1/s} \right| < \varepsilon.$$

A homogeneity lemma

Lemma

If X_j is a sequence of operators such that $||X_j|| \to 0$ and $|X_j|_{\rho(\varepsilon)} < C < +\infty$, then

$$\lim_{j\to\infty} \left| |X_j \otimes I_m|_{\rho(\varepsilon)} - m^{1/s} |X_j|_{\rho(\varepsilon)} \right| \leq \varepsilon C.$$

Note: From this lemma, we see that

$$|k_{\rho(\varepsilon)}(\tau \otimes I_m) - m^{1/s} k_{\rho(\varepsilon)}(\tau)| < \varepsilon$$

...but this isn't k_ρ.

The key lemma

Lemma

Let τ be an n-tuple of commuting self-adjoint operators, and let π be a regular sequence. Then $k_\pi(\tau) = k_{S\pi}(\tau)$.

Proof sketch of lemma

Sketch of proof: $T \in \mathfrak{S}_\pi$

$$|T|_\pi = \sum_{k=1}^{\infty} s_k \pi_k \geq \sum_{k=1}^{\infty} s_k \pi_{k+1} = |T|_{S\pi}$$

Likewise, choose A_j so that $k_{S\pi}(\tau) = \lim_{j\to\infty} |[A_j, \tau]|_{S\pi}$

$$k_\pi(\tau) \leq \lim_j |[A_j, \tau]|_\pi = \lim_j \sum_{k=1}^{\infty} s_k^{(j)} \pi_k = \lim_j \left(s_1^{(j)} \pi_1 + \sum_{k=2}^{\infty} s_k^{(j)} \pi_k \right)$$

$$\leq \lim_j \left(s_1^{(j)} \pi_1 + \sum_{k=1}^{\infty} s_k^{(j)} \pi_{k+1} \right) = \lim_j \sum_{k=1}^{\infty} s_k^{(j)} \pi_{k+1} = k_{S\pi}(\tau)$$

Thus $k_\pi(\tau) = k_{S\pi}(\tau)$.

Ampliation Homogeneity for k_ρ

Theorem

Let $f : [0,\infty) \to [0,\infty)$ be a function with property (R_s) for some $s \geq 1$. Then

$$k_\rho(\tau_{\Omega,H_f} \otimes I_m) = m^{1/s} k_\rho(\tau_{\Omega,H_f}).$$

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Voiculescu's formula

Recall:

Theorem (Voiculescu, 2021)

Let τ be an n-tuple of commuting self-adjoint operators. If m is the multiplicity function for τ, then

$$(k_p^-(\tau))^p = C \int_{\sigma(\tau)} m(x) dH_p(x).$$

Formula for k_ρ

Theorem

Let τ be an n-tuple of commuting self-adjoint operators. If m is the multiplicity function for τ, then

$$(k_\rho(\tau))^s = C \int_{C_f} m(x) dH_f(x)$$

Lemma

Let f be a logarithmically-concave gauge function with property (R_s), and let τ be an n-tuple of commuting self-adjoint operators with $\sigma(\tau) \subset \Omega$. If E_τ is singular with respect to H_f, then $k_\rho(\tau) = 0$.

Proof sketch for formula

Sketch of proof: We can assume $\tau = \tau_{\Omega,H_f}$ for some $\Omega \subset C_f$. The "cutting-up" trick used before allows us to reduce to the case of showing $(k_\rho(\tau_{C_f^w,H_f}))^s = C \cdot H_f(C_f^w)$ for every w. Note that:

$$k_\rho(\tau_{C_f,H_f}) = k_\rho(\tau_{C_f^w,H_f} \otimes I_{2^{nL}}) = 2^{nL/s} k_\rho(\tau_{C_f^w,H_f})$$

Letting $\kappa_s^{(f)} := (k_\rho(\tau_{C_f,H_f}))^s / H_f(C_f)$

$$(k_\rho(\tau_{C_f^w,H_f}))^s = (2^{-nL/s} k_\rho(\tau_{C_f,H_f}))^s = 2^{-nL} \kappa_s^{(f)} H_f(C_f) = \kappa_s^{(f)} H_f(C_f^w)$$

THANK YOU!!!!!!!