A NEW APPROACH TO STRONG CONVERGENCE II

Joint work with C.F. Chen and R. van Handel

- <u>Notation</u>: We will use $\mathbb{C}\langle x_1, \dots, x_r, x_1^*, \dots, x_r^*\rangle$ to denote the space of non-commutative polynomials in x_i and x_i^* and abbreviate it as $\mathbb{C}\langle x, x^*\rangle$. Oftentimes we will work with $P \in M_D(\mathbb{C}) \otimes \mathbb{C}\langle x, x^*\rangle$, i.e.

$$P(x, x^*) = \sum_{w \text{ words}} A_w \otimes w(x, x^*),$$

for $A_w \in M_D(\mathbb{C})$. We will use $\mathrm{tr}(\cdot)$ to denote the normalized trace.

- <u>DEF</u> (Strong convergence) We say that a seq. of tuples of random matrices $X^N = (X_1^N, \dots, X_r^N)$ converges strongly to a tuple of bdd operators $x = (x_1, \dots, x_r)$ if

$$\|P(X^N)\| \xrightarrow[\text{Prob}]{} \|P(x)\| \quad \forall \text{ n.c.p. } P \quad (D=1)$$

- <u>THM</u> (Haagerup - Thorbjørnsen 02) If $G^N = (G_1^N, \dots, G_r^N)$ are $N \times N$ independent GUE's and $s = (s_1, \dots, s_r)$ is a std semicircular family then G^N converges strongly to s.

• <u>Today</u>: Non-asymptotic version of HT and other s.c. results.

High probability upper bounds $\left| \|P(X_1^N, \dots, X_r^N)\| - \|P(x_1, \dots, x_r)\| \right|$ as a function of N and D.

<u>RESULTS</u>

• THM (Chen, GV, van Handel 24) Let G^N and s be as above, and $\varepsilon > 0$. Then, $\forall\, P \in M_D(\mathbb{C}) \otimes \mathbb{C}\langle x, x^* \rangle$ we have

$$\mathbb{P}\left(\|P(G^N)\| \geq (1+\varepsilon) \|P(s)\| \right) \leq \frac{N}{c\varepsilon} e^{-cN\varepsilon^2}$$

as long as $D \leq e^{cN\varepsilon^2}$, where $c = c(\deg(P), r)$.

• <u>Remark</u>.- We obtain the same result for GOE/GSE and, up to logarithmic factors, we also obtain this for Haar from $O(N)/U(N)/Sp(N)$.

• <u>COR 1</u>.- Fix $D \in \mathbb{Z}_{\geq 1}$ $\overset{\text{THM}}{\Longrightarrow}$ with high probability

$$\|P(G^N)\| \leq \left(1 + O\left(\sqrt{\frac{\log N}{N}} \right) \right) \|P(s)\|$$

• <u>Remark</u>.- [Parraud 22-23] proves this in the case $D=1$ and for GUE and $U(N)$, and allows the polynomial P to be evaluated in arbitrary deterministic matrices.

- *Open problem*: Show that $\|P(G^N)\| \leq (1 + O(N^{-2/3}))\, \|P(s)\|$ w.h.p.

- *COR 2*.- ε as going to zero very very slowly $\overset{\text{THM}}{\Longrightarrow}$

$$\|P(G^N)\| \leq (1 + o(1))\, \|P(s)\|$$

as long as $D = \exp(o(N))$.

- *Remark*.- Pisier showed that the above is false in the regime $D = \exp(\Omega(N^2))$

- *Open question*: What happens when $D \in [\exp(cN), \exp(CN^2)]$?

OUR APPROACH

- Hereon fix $P \in M_D(\mathbb{C}) \otimes \mathbb{C}\langle x, x^* \rangle$, let G^N and s as above and set $X^N := P(G^N)$ and $X_\infty := P(s)$.

- *Input* ($\frac{1}{N}$-expansions) For any polynomial $h(x)$ we have that

$$\mathbb{E}[\operatorname{tr} h(X_N)] = \sum_{k=0}^{\infty} \frac{\nu_k(h)}{N^k}$$

(same is true in the GOE/GSE cases and for Haar in $O(N)$ / $U(N)$ / $Sp(N)$, and reps of certain groups)

- *Note*.- For each k, the functional $h \longmapsto \nu_k(h)$ is linear.

- <u>THM</u> (Smooth $\frac{1}{N}$-expansion) Each $v_k(\cdot)$ can be continuously extended to a compactly supported Schwarz distribution (now defined on all $C^\infty(\mathbb{R})$). Moreover, $\forall m \in \mathbb{Z}_+$ and $\forall h \in C^\infty(\mathbb{R})$ with $\|h\|_\infty < \infty$ one has

$$\left| \mathbb{E}[\operatorname{tr} h(X_N)] - \sum_{k=0}^{m-1} \frac{v_k(h)}{N^k} \right| \leq C_{P,h,m} \left(\frac{1}{N^m} + \exp(-cN) \right).$$

- <u>Proof</u>: Via the polynomial method.
- <u>Remark 1</u>: We obtain the same for other classical ensembles.
- <u>Remark 2</u>: Similar results are obtained by Parraud in the GUE and $U(N)$, but with a different dependence on D.

<u>PROVING STRONG CONVERGENCE</u>

- <u>Proof of HT</u>: Fix $D \in \mathbb{Z}_{\geq 1}$ and $P \in M_D(\mathbb{C}) \otimes \mathbb{C}\langle x, x^* \rangle$ and WTS $\|P(G^N)\| \xrightarrow[\text{Prob}]{} \|P(s)\|$ as $N \to \infty$.

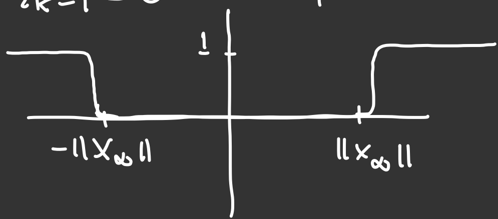
Apply theorem with $m=2$:

$$\left| \mathbb{E}[\operatorname{tr} h(X_N)] - \cancelto{0}{v_0(h)} - \cancelto{0}{\frac{v_1(h)}{N}} \right| \leq C_{P,h,2} \left(\frac{1}{N^2} + \exp(-cN) \right)$$

Note that $v_0(h) = \lim_{N\to\infty} \sum_{k=0}^{\infty} \frac{v_k(h)}{N^k} = \lim_{N\to\infty} \mathbb{E}[\mathrm{tr}\, h(\underset{\overset{\shortparallel}{P(G^N)}}{X_N})] = \tau(h(\underset{\overset{\shortparallel}{P(s)}}{X_\infty}))$

true for any polynomial h, so v_0 is the spectral dist. of X_∞.

For GUE's $v_{2k-1} \equiv 0$ so $v_1(h) = 0 \ \forall h$.

Take $h =$ 

$\Rightarrow v_0(h)$

$$|\mathbb{E}[\mathrm{Tr}\, h(X_N)]| \le C_{P,2}\left(\frac{D}{N} + DN\exp(-cN)\right) = o(1)$$

$\mathrm{Tr}\, h(X_N) = \sum_{i=1}^{N} \mathbb{E}[h(\lambda_i(X_N))]$

$\ge \mathbb{E}[\#\text{ outliers}]$

- <u>Proof of [Schultz 03]</u>. $G^N_{\mathbb{R}} = (G^N_{1,\mathbb{R}}, \dots, G^N_{r,\mathbb{R}})$ tuple of GOE and

$G^N_{\mathbb{H}} = (G^N_1, \dots, G^N_r)$ tuple of GSE's, and $X_N^{\mathbb{R}} := P(G^N_{\mathbb{R}})$, $X_N^{\mathbb{H}} := P(G^N_{\mathbb{H}})$.

<u>Fact</u> (Supersymmetry) For any $h(x)$, one has $\mathbb{E}[\mathrm{tr}\, h(X_N^{\mathbb{R}})] = \sum_{k=0}^{\infty} \frac{v_k(h)}{N^k}$

then $\mathbb{E}[\mathrm{tr}\, h(X_N^{\mathbb{H}})] = \sum_{k=0}^{\infty} \frac{v_k(h)}{(-2N)^k}$ where the $v_k(h)$ are the same.

THM (Schultz 03) $\mathrm{Supp}(\nu_1) \subseteq [-\|x_\infty\|, \|x_\infty\|]$.

Proof. - Our thm implies that $\forall\, h \in C^\infty(\mathbb{R})$ bdd

$$0 \leq \mathbb{E}[\operatorname{tr} h(X_N^{\mathbb{R}})]\cdot N = \cancel{\nu_0(h)}^{0} + \nu_1(h) + O\left(\frac{1}{N}\right) \qquad (1)$$

$$0 \leq \mathbb{E}[\operatorname{tr} h(X_N^{\mathbb{H}})]\cdot \overset{\text{supersymmetry}}{N} = \cancel{\nu_0(h)}^{0} - \frac{\nu_1(h)}{2} + O\left(\frac{1}{N}\right) \qquad (2)$$

If $h \geq 0$ on all $\mathbb{R}$ then the LHS's are non-negative, and if $h \equiv 0$ on $[-\|x_\infty\|, \|x_\infty\|] \Rightarrow \nu_0(h) = 0$. If $\nu_1(h) \neq 0$ then either (1) or (2) become strictly negative as $N \to \infty$! Therefore $\nu_1(h) = 0$ $\forall\, h \overset{\geq 0}{} $ vanishing $[-\|x_\infty\|, \|x_\infty\|]$. $\Rightarrow$ $\mathrm{supp}(\nu_1) \subseteq [-\|x_\infty\|, \|x_\infty\|]$ ∎

BOOTSTRAPPING

- G^N and S as above and let ν_k be s.t. $\mathbb{E}[\operatorname{tr} h(X_N)] = \sum_{k=0}^{\infty} \frac{\nu_k(h)}{N^k}$

for $X_N = P(G^N)$, and let $x_\infty = P(S)$.

THM - $\mathrm{supp}(\nu_k) \subseteq [-\|x_\infty\|, \|x_\infty\|]$ for all $k \in \mathbb{Z}_+$. (Parraud proved this for $D=1$ and GUE and $U(N)$)

Proof - By induction on k. When $k=0$ we know $\operatorname{supp}(\nu_0) \subseteq [-\|x_\infty\|, \|x_\infty\|]$

Now assume this is true for $k=1,\dots,m-1$, then use THM to get

$$\left| \mathbb{E}[\operatorname{tr} h(X_N)] - \cancel{\sum_{k=0}^{m-1} \frac{\nu_k(h)}{N^k}}^{\,0} - \frac{\nu_m(h)}{N^m} \right| \le C_{P,h,m}\left(\frac{1}{N^{m+1}} + \exp(-cN) \right)$$

Now take $h \equiv 0$ on $[-\|x_\infty\|, \|x_\infty\|] \implies$

$$|\nu_m(h)| \le C_{P,h,m}\left(\cancel{\frac{1}{N}}\to 0 + \cancel{N^m \exp(-cN)} \to 0 \right) + \overbrace{\cancel{\left|\mathbb{E}[\operatorname{tr} h(X_N)]\right|} \cdot N^m}^{O(\exp(-cN)) \to 0}$$

(1) (Concentration of measure) $\mathbb{P}(\|X_N\| \geq \operatorname{med}(\|X_N\|) + \varepsilon) \le \exp(-cN\varepsilon^2)$

(2) (Strong convergence of G^N to s) $\lim_{N\to\infty} \operatorname{med}(\|X_N\|) = \|x_\infty\|$

Putting both together $\mathbb{P}(\|X_N\| \geq \|x_\infty\| + \varepsilon') \le \exp(-cN(\varepsilon')^2)$

Since $\|h\|_\infty < \infty$ and $h \equiv 0$ on $[-\|x_\infty\|, \|x_\infty\|]$, then

$$\mathbb{E}[\operatorname{tr} h(X_N)] = \frac{1}{ND} \sum_{i=1}^{N} \underbrace{h(\lambda_i(X_N))}_{0 \text{ for } \lambda_i \in [-\|x_\infty\|, \|x_\infty\|] \text{ and } \underbrace{< \|h\|_\infty \text{ otherwise}}_{\text{happens with exp small proba}}}$$

$$= O(\exp(-cN))$$

$\Rightarrow \quad \nu_m(h) = 0 \quad \forall\, h$ vanishing on $[-\|X_\infty\|, \|X_\infty\|]$. $\blacksquare$

<u>APPLICATION</u> :- By our THM we have G

$$\Big|\, \mathbb{E}[\underbrace{\operatorname{tr}}_{=\frac{1}{DN}\operatorname{Tr}} h(X_N)] - \cancel{\sum_{k=0}^{m-1} \frac{\nu_k(h)}{N^k}} \,\Big| \leq C_{P,h,m} \left(\frac{1}{N^m} + \exp(-cN) \right)$$

If we take $h =$ 

$$\Rightarrow \quad \big|\mathbb{E}[\operatorname{Tr} h(X_N)]\big| \leq C_{P,h,m} \left(\frac{D}{N^{m-1}} + DN \exp(-cN) \right)$$

(true for all m), and it matters what $C_{P,h,m}$ is and we take $m = \Theta(N)$.