LARGE DEVIATIONS FOR MACROSCOPIC OBSERVABLES OF HEAVY-TAILED RANDOM MATRICES

Charles Bordenave (CNRS - Aix Marseille University)

Joint work with Alice Guionnet and Camille Male

arXiv: 2409.14027

FOREWORD :

BOLTZMANN'S MICROSTATE ENTROPY

DISTRIBUTION OF A VECTOR

Let $\mathcal{X}$ be a finite set.

For $f \in \mathcal{X}^N$, its empirical distribution is

$$L_f = \frac{1}{N} \sum_{i=1}^{N} \delta_{f_i} \in \mathcal{P}(\mathcal{X})$$

i.e. $\forall x \in \mathcal{X}$, $\quad L_f(x) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}(f_i = x)$.

BOLTZMANN – SHANNON THEOREM

For $p = (p_x)_{x \in X} \in \mathcal{P}(X)$, define :

Lévy distance (for example)

$$H_N(p, \varepsilon) = \left| \left\{ f \in X^N : d_L \left(L_f , p \right) \leq \varepsilon \right\} \right|$$

$=$ "nb of configurations of N particles ε-close to p"

BOLTZMANN - SHANNON THEOREM

For $p = (p_x)_{x \in X} \in \mathcal{P}(X)$, define :

Lévy distance (for example)

$$H_N(p,\varepsilon) = \left| \left\{ f \in X^N : d_L(L_f, p) \leq \varepsilon \right\} \right|$$

= "nb of configurations of N particles ε-close to p"

Theorem (Shannon 1948)

$$\lim_{\varepsilon \downarrow 0} \varliminf_N \frac{1}{N} \log H_N(p,\varepsilon) = \lim_{\varepsilon \downarrow 0} \varlimsup_N \frac{1}{N} \log H_N(p,\varepsilon)$$

$$= - \sum_x p_x \ln p_x =: H(p).$$

LARGE DEVIATIONS PRINCIPLE

Let (X,d) Polish space, $P_N \in \mathcal{P}(X)$ satisfies a LDP with rate/speed v_N and rate function $I : X \to [0,\infty]$ lower semi-continuous if $\forall\, B \subseteq X$ Borel

$$-\inf_{x \in B^\circ} I(x) \leq \liminf_{N} \frac{1}{v_N} \log P_N(B) \leq \limsup_{N} \frac{1}{v_N} \log P_N(B) \leq -\inf_{x \in \overline{B}} I(x).$$

LARGE DEVIATIONS PRINCIPLE

Let (X,d) Polish space, $P_N \in \mathcal{P}(X)$ satisfies a LDP with rate/speed v_N and rate function $I : X \to [0,\infty]$ lower semi-continuous if $\forall B \subseteq X$ Borel

$$-\inf_{x \in B^\circ} I(x) \leq \varliminf_{N} \frac{1}{v_N} \log P_N(B) \leq \varlimsup_{N} \frac{1}{v_N} \log P_N(B) \leq -\inf_{x\in \overline{B}} I(x).$$

Remark that

$$H_N(p,\varepsilon) = \left|\{ f \in X^N : d_L(L_f, p) \leq \varepsilon\}\right|$$

$$= |X|^N \cdot P_N\left(d_L(L_f, p) \leq \varepsilon\right).$$

Uniform measure on X^N

$$\Rightarrow \quad \frac{1}{N} \log H_N(p,\varepsilon) = \log|X| + \frac{1}{N} \log P_N\left(d_L(L_f, p) \leq \varepsilon\right).$$

SANOV'S THEOREM

(X, d) Polish space, $P \in \mathcal{P}(X)$, $\xi_N \in X^N$ with law $P^{\otimes N}$

Th (Sanov 1957)

$\frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_N^{(i)}} = L_{\xi_N} \in \mathcal{P}(X)$ satisfies a LDP with rate N and rate function

$$I(Q) = D_{KL}(Q \| P) = \begin{cases} \int g \log g \, dP & \text{if } g = \frac{dQ}{dP} \text{ exists.} \\ \infty & \text{otherwise} \end{cases}$$

Kullback-Leibler divergence
aka relative entropy

$\Rightarrow$ This is a far-reaching extension of Shannon's theorem

BEYOND $S(X)$

- Kolmogorov-Sinai entropy of dynamical systems,
- Voiculescu's microstate free entropy,
- Bowen, Kerr-Li sofic entropy of group actions,
- Entropy of random rooted graphs, B.-Caputo,
- Austin's entropy of unitary representation.
- ...

LDP for heavy-tailed matrices

In this talk, we obtain LDP's for families of random matrices with $O(1)$ non-negligeable entries per row.

It gives non-trivial entropies associated to Male's traffic distribution which are additive for the free traffic convolution.

LARGE DEVIATIONS PRINCIPLE FOR A SINGLE MATRIX

PREVIOUS RESULTS FOR ESD

$Y \in M_N^{s.a.}(\mathbb{C})$ $\quad L_Y := \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i(Y)} \in \mathcal{P}(\mathbb{R})$ (empirical distribution of eigenvalues)

For $Y_N \in M_N^{s.a.}(\mathbb{C})$ random, $L_{Y_N} \xrightarrow{w} L_\infty$ a.s., known LDP of L_{Y_N}

+ β-Ensembles: $dP \propto \exp(-\beta N \operatorname{Tr}(V(X))\, dX$, speed N^2 — Ben-Arous - Guionnet (1997)

  (for example GUE/GOE/GSE).

+ $A + UBU^*$ — $U \sim$ Haar on $\mathbb{U}_N$, $A, B \in M_N^{s.a}(\mathbb{C})$ speed N^2 — Belinschi - Guionnet - Huang (2020)

+ Wigner, stretched exponential — $Y_N = \frac{X}{\sqrt{N}}$, $\mathbb{P}(|X_{ij}| \geq t) \sim \exp(-ct^\alpha)$, $0<\alpha<2$, speed $N^{1+\frac{\alpha}{2}}$

  $(X_{ij})_{i\geq j}$ iid — B. Caputo (2014)

+ Sparse Wigner: $Y_N = \frac{A \odot X}{\sqrt{d}}$, $(X_{ij})_{i\geq j}$ iid bounded, speed Nd — Augeri (2024)

  $(A_{ij})_{i\geq j}$ iid $\operatorname{Ber}(\frac{d}{N})$

  $\log N \ll d \ll N$

+ Erdős-Rényi bounded average degree — $Y_N = A$ $\quad A_{ij} \sim \operatorname{Ber}(\frac{d}{N})$, speed N — B. Caputo (2015)

+ Open problem: Wigner $Y_N = \frac{X}{\sqrt{N}}$ bounded entries.

today: more speed N LDP.

HEAVY-TAILED RANDOM MATRICES

Let Λ be a Radon measure on $\mathbb{C}\setminus\{0\}$, $\Lambda(\mathbb{C}\setminus B(0,1)) < +\infty$

Let $Y_N = (Y_{ij})_{i,j} \in M_N^{s.a}(\mathbb{C})$ such that $(Y_{ij})_{i\geq j}$ is iid and

$N \cdot \mathbb{P}(Y_{ij} \in B) \longrightarrow \Lambda(B) \quad \forall B \subseteq \mathbb{C}\setminus\{0\}.$

Ex 1. $Y_N = A \odot X$ $\quad A_{ij} \sim Ber\left(\frac{d}{N}\right)$ $\quad \Lambda = d\gamma$

X law $\gamma \in \mathcal{P}(\mathbb{C})$

Ex 2. (Lévy matrices) $\quad Y_N = \frac{X}{N^{1/\alpha}}$ $\quad \underbrace{X_{ij} \in \mathbb{R}}$,

$\mathbb{P}(|X_{ij}| \geq t, X_{ij} > 0) \xrightarrow[t\to+\infty]{} p\, t^{-\alpha}$

$\mathbb{P}(|X_{ij}| \geq t, X_{ij} < 0) \longrightarrow (1-p)\, t^{-\alpha}$

$$\boxed{\Lambda = p\alpha x^{-\alpha-1} \mathbb{1}_{x>0}\, dx + (1-p)\alpha |x|^{-\alpha-1} \mathbb{1}_{x<0}\, dx.}$$

$0 < \alpha < 2$

$L_{Y_N} \longrightarrow L_\Lambda$ a.s. $\quad$ Ben Arous - Guionnet (2006).

HEAVY-TAILED RANDOM MATRICES

Let Λ be a Radon measure on $\mathbb{C} \setminus \{0\}$, $\Lambda(\mathbb{C} \setminus B(0,1)) < +\infty$

Let $Y_N = (Y_{ij})_{i,j} \in M_N^{s.a}(\mathbb{C})$ such that $(Y_{ij})_{i \geq j}$ is iid and

$$N \cdot \mathbb{P}(Y_{ij} \in B) \longrightarrow \Lambda(B) \quad \forall B \subseteq \mathbb{C} \setminus \{0\}.$$

Ex 1. $Y_N = A \odot X$ $\quad A_{ij} \sim Ber(\frac{d}{N})$ $\quad \Lambda = d\gamma$

X law $\gamma \in \mathcal{P}(\mathbb{C})$

Ex 2. $Y_N = \frac{X}{N^{1/\alpha}}$ $\quad X_{ij} \in \mathbb{R}$, $\quad \mathbb{P}(|X_{ij}| \geq t, X_{ij} > 0) \xrightarrow[t \to +\infty]{} p\, t^{-\alpha}$

$\mathbb{P}(|X_{ij}| \geq t, X_{ij} < 0) \longrightarrow (1-p)\, t^{-\alpha}$

$$\Lambda = p \alpha x^{-\alpha-1} \mathbb{1}_{x>0}\, dx + (1-p)\alpha |x|^{-\alpha-1} \mathbb{1}_{x<0}\, dx.$$

$L_{Y_N} \longrightarrow L_\Lambda$ a.s. $\qquad$ Ben Arous - Guionnet (2006).

Th $\quad L_{Y_N}$ satisfies a LDP with good rate function $J_\Lambda : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ with a unique minimizer.

$\{x : I(x) \leq t\}$ is cpct $\forall t \geq 0$.

if Ex1, Ex2 OR $\Lambda(\{x : |x| \geq t\}) \leq c\, t^{-\beta}$ in a neighborhood of 0, $\quad 0 < \beta < 2$

RANDOM WEIGHTED GRAPH

Let $D_N = (D_N(1), \dots, D_N(N))$ an integer sequence such that $\sum D_N(i)$ even, $\|D_N\|_\infty \leq \Theta$

and $L_{D_N} = \frac{1}{N}\sum_{i=1}^{N} \delta_{D_N(i)} \xrightarrow[N\to+\infty]{} \pi$, $\pi \in \mathcal{P}(\mathbb{Z}_+)$

For example $D_N(i) = d$ and $\pi = \delta_d$.

A = adjacency matrix of a uniformly sampled graph on vertex set $\{1, \dots, N\}$ and degree sequence D_N.

$Y_N = A \odot X$ $\quad (X_{ij})_{i>j}$ iid with law $\gamma \in \mathcal{P}(\mathbb{C})$
$\in M_N^{s.a.}(\mathbb{C})$

Th $\quad L_{Y_N}$ satisfies a LDP with good rate function $J_{\gamma,\pi}$ with a unique minimizer.

LARGE DEVIATIONS PRINCIPLE FOR MULTIPLE MATRICES

PROBABILITISC VIEW ON VOICULESCU'S MICROSTATE ENTROPY

Let (A, τ) $*$-algebra $x = (x_1, \dots, x_d) \in A^d$, their distribution $L_x \in \mathbb{C}[F_d]^*$ is

$$L_x : \mathbb{C}[F_d] \to \mathbb{C}$$
$$P \longmapsto \tau(P(x_1, \dots, x_d, x_1^*, \dots, x_d^*))$$

We equip $\mathbb{C}[F_d]^*$ of the topology of point wise convergence.

PROBABILITISC VIEW ON VOICULESCU'S MICROSTATE ENTROPY

let (A, τ) $*$-algebra $x = (x_1, \dots, x_d) \in A^d$, their distribution $L_x \in \mathbb{C}[F_d]^*$ is

$$L_x : \mathbb{C}[F_d] \to \mathbb{C}$$
$$P \mapsto \tau(P(x_1, \dots, x_d, x_1^*, \dots, x_d^*))$$

We equip $\mathbb{C}[F_d]^*$ of the topology of point wise convergence.

(cond. $\|Y_N^i\| \leq \theta$)

Open problem: let $Y_N = (Y_N^1, \dots, Y_N^d)$ independent GUE matrices, prove that L_{Y_N} satisfies a LDP with speed N^2 and good rate function χ_θ, $\chi = \lim_{\theta \to +\infty} \chi_\theta$

Voiculescu defines the micro-state entropy χ by taking a limsup

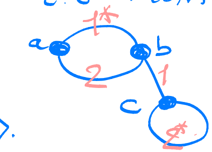
Notably, it is not known, in general, that: if $(x_1, \dots, x_k)$ and $(y_1, \dots, y_\ell)$ are free

$$\chi(x_1, \dots, x_k, y_1, \dots, y_\ell) = \chi(x_1, \dots, x_k) + \chi(y_1, \dots, y_\ell)$$

(we write $\chi(x)$ in place of $\chi(L_x)$)

MALE'S TRAFFIC DISTRIBUTION

Let J a finite set, **test graph** = connected graph $H=(V,E,\ell,\varepsilon)$ with $\ell : E \to J$, $\varepsilon : E \to \{1,*\}$



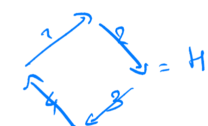
$\mathcal{H}\langle J\rangle$ the set of test graphs.

A **traffic distribution** is an element of $\mathbb{C}[\mathcal{H}\langle J\rangle]^* =: \mathrm{Traf}\langle J\rangle$.

If $Y=(Y_j)_{j\in J} \in M_N(\mathbb{C})^J$, $\tau_Y \in \mathrm{Traf}\langle J\rangle$ is defined by:

$$\tau_Y : \mathcal{H}\langle J\rangle \longrightarrow \mathbb{C}$$
$$H=(V,E,\ell,\varepsilon) \longmapsto \frac{1}{N} \sum_{\substack{(i_v)_{v\in V} \\ 1\le i_v\le N}} \prod_{e=(u,v)} \left(Y_{\ell(e)}^{\varepsilon(e)}\right)_{i_u i_v}$$

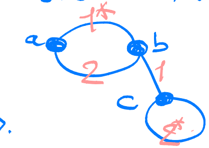
We equip $\mathrm{Traf}\langle J\rangle$ with the topology of pointwise convergence.



$\tau_Y(H) = \frac{1}{N} \mathrm{Tr}(Y_1 Y_2 Y_3 Y_4)$

MALE'S TRAFFIC DISTRIBUTION

Let J a finite set, test graph = connected graph $H=(V,E,\ell,\varepsilon)$ with $\ell : E \to J$, $\varepsilon : E \to \{1,*\}$



$\mathcal{H}\langle J\rangle$ the set of test graphs.

A traffic distribution is an element of $\mathbb{C}[\mathcal{H}\langle J\rangle]^* =: \mathrm{Traf}\langle J\rangle$.

If $Y=(Y_j)_{j\in J} \in M_N^J(\mathbb{C})$, $\tau_Y \in \mathrm{Traf}\langle J\rangle$ is defined by:

$$\tau_Y : \mathcal{H}\langle J\rangle \longrightarrow \mathbb{C}$$
$$H=(V,E,\ell,\varepsilon) \longmapsto \frac{1}{N} \sum_{\substack{(i_v)_{v\in V} \\ 1\le i_v \le N}} \prod_{e=(u,v)} \left(Y_{\ell(e)}^{\varepsilon(e)}\right)_{i_u i_v}$$

We equip $\mathrm{Traf}\langle J\rangle$ with the topology of pointwise convergence.

Th (Male) Let $Y_1^N = (Y_{1,j}^N)_{j\in J_1}$, $Y_2^N = (Y_{2,j}^N)_{j\in J_2} \in M_N(\mathbb{C})$ such that $\tau_{Y_i^N} \to \tau_i$, $\|Y_{i,j}^N\| \le \Theta$

Set $Y^N = \left((Y_{1,j}^N)_{j\in J_1}, (S\, Y_{2,j}^N\, S^*)_{j\in J_2} \right)$ then $\tau_{Y^N} \longrightarrow \underbrace{\tau_1 * \tau_2}_{\text{traffic free product}}$ in probability.

S uniform permutation matrix.

ENTROPY FOR TRAFFIC DISTRIBUTION

Th (simplified) If $Y = (Y_j)_{j \in J}$ are independent heavy-tailed with parameter Λ or (π, γ)
cond. on $\| \ \| \leq 8$
Then τ_Y satisfies a LDP on $\mathrm{Traf}\langle J \rangle$ with speed N and good rate function χ_Λ with a unique minimizer.

Cor. We have $\chi_\Lambda(\underbrace{\tau_1 * \tau_2}_{\text{traffic free product}}) = \chi_\Lambda(\tau_1) + \chi_\Lambda(\tau_2)$, $\forall \tau_i \in \mathrm{Traf}\langle J_i \rangle$

WHERE DO THESE LDP's COME FROM ?

Contraction Principle

Lemma.
Let X, Y Polish and $f : X \longrightarrow Y$ continuous.

If $Z_N \in X$ satisfies a LDP with speed v_N on X and rate function I

Then $f(Z_N)$ ______________________ Y ______________ $J(y) = \inf \{ I(x) : f(x) = y \}$

$\Rightarrow$ to prove a LDP for $Y_N \in Y$, it is often easier to find a "larger" space X and prove a LDP for Z_N, with $Y_N = f(Z_N)$ and f continuous.

OVERVIEW OF PROOFS

* We represent a family of matrices $Y = (Y_j)_{j \in J} \in M_N^{sa}(\mathbb{C})$ (or rather their eq class by conjugation by permutation) by a probability measure on rooted graphs $U(G(Y)) \in \mathcal{P}(\mathcal{G}^\bullet)$.

* If those matrices have $O(1)$ non-negligeable entries per row then $U(G(Y))$ is tight for the local weak topology on $\mathcal{P}(\mathcal{G}^\bullet)$.

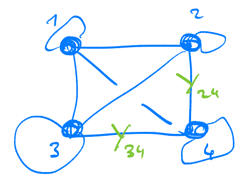
* We prove a LDP for $U(G(Y))$ with speed N and explicit rate function.

* We show that the map : $L : \begin{cases} \mathcal{P}(\mathcal{G}^\bullet) \longrightarrow \mathrm{Traf}\langle J \rangle \\ \mu \longmapsto L_\mu \end{cases}$ is continuous

* We apply the contraction principle.

MARKED GRAPHS

$Y = (Y_j)_{j \in J} \in M_N^{s.a.}(\mathbb{C})$



$N = 4$

$Y_{ab} = (Y_{j,ab})_{j \in J}$.

$G(Y)$ on N vertices

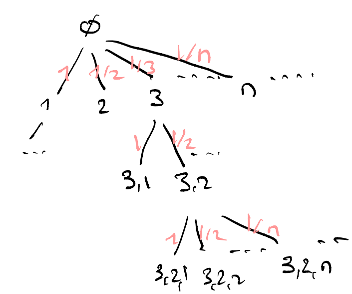
It defines a marked graph $G = (V, E, \xi)$

$$\begin{cases} \xi : E \to \mathbb{C}^J. \\ V = \{1, \dots, N\}. \\ E = \{ \{a,b\} : Y_{ab} \neq 0 \}. \end{cases}$$

A marked graph is locally finite, if $\forall v \in V$ $\deg(v) < +\infty$

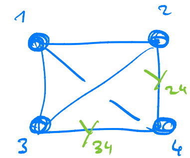
A marked graph is a network if for $\forall \varepsilon > 0$: G^ε is locally finite

$(V, E^\varepsilon, \xi^\varepsilon)$, $E^\varepsilon = \{ \{a,b\} : \|Y_{ab}\| > \varepsilon \}$.



MARKED GRAPHS

$Y = (Y_j)_{j \in J} \in M_N^{s.a.}(\mathbb{C})$



$N = 4$

$Y_{ab} = (Y_{j,ab})_{j \in J}$.

$G(Y)$ on N vertices

It defines a **marked graph** $G = (V, E, \xi)$

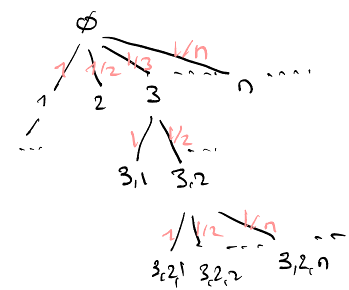
$$\begin{cases} \xi : E \to \mathbb{C}^J \\ V = \{1, \dots, N\} \\ E = \{ \{a,b\} : Y_{ab} \neq 0 \} \end{cases}$$

A marked graph is **locally finite**, if $\forall v \in V$ $\deg(v) < +\infty$

A marked graph is a **network** if for $\forall \varepsilon > 0$: G^ε is locally finite

$G^\varepsilon = (V, E^\varepsilon, \xi^\varepsilon)$, $E^\varepsilon = \{ \{a,b\} : \|Y_{ab}\| > \varepsilon \}$.

A **rooted marked graph** = connected marked graph + root vertex.



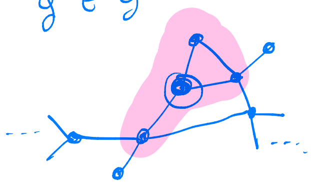
$\mathcal{G}^\bullet$ = set of locally finite unlabeled rooted marked graph → eq. class up to bijections

$\mathcal{W}^\bullet$ = __ unlabeled rooted networks

LOCAL WEAK TOPOLOGY

If $g \in \mathcal{G}^{\bullet}$ and $h \geq 0$, $(g)_h$ = restriction of g to h-neighborhood $\in \mathcal{G}^{\bullet}_h$

point-wise topology



Projective topology: on $\mathcal{G}^{\bullet}$: $g_N \to g$ iff $(g_N)_h \to (g)_h$. $\forall h$

on $\mathcal{W}^{\bullet}$: $g_N \to g$ iff $(g_N^{\varepsilon})_h \to (g^{\varepsilon})_h$ $\forall h$, $\forall \varepsilon > 0$ along subsequence

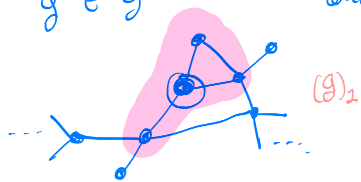
This defines Polish metric spaces $(\mathcal{G}^{\bullet}, d_{loc})$, $(\mathcal{W}^{\bullet}, d_{loc})$

In particular: $(\mathcal{P}(\mathcal{G}^{\bullet}), d_{Lévy})$, $(\mathcal{P}(\mathcal{W}^{\bullet}), d_{Lévy})$ are Polish spaces.

LOCAL WEAK TOPOLOGY

If $g \in \mathcal{G}^\circ$ and $h \geq 0$, $(g)_h$ = restriction of g to h-neighborhood $\in \mathcal{G}^\circ_h$

point-wise topology



Projective topology: on $\mathcal{G}^\circ$: $g_N \to g$ iff $(g_N)_h \to (g)_h$ $\forall h$

on $\mathcal{W}^\circ$: $g_N \to g$ iff $(g_N^\varepsilon)_h \to (g^\varepsilon)_h$ $\forall h$, $\forall \varepsilon > 0$ along subsequence

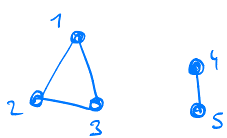
This defines Polish metric spaces $(\mathcal{G}^\circ, d_{loc})$, $(\mathcal{W}^\circ, d_{loc})$

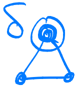
In particular: $(\mathcal{P}(\mathcal{G}^\circ), d_{\text{Lévy}})$, $(\mathcal{P}(\mathcal{W}^\circ), d_{\text{Lévy}})$ are Polish spaces.

For G finite: $U(G) = \frac{1}{|V|} \sum_v \delta_{[G(v), v]} \in \mathcal{P}(\mathcal{G}^\circ)$. (Benjamini-Schramm).

= law of G rooted uniformly.

for ex: G



$U(G) = \frac{3}{5}\,\delta$  $+ \frac{2}{5}\,\delta$  .

MICRO STATE ENTROPY FOR NETWORKS

Let Λ be a Radon measure on $\mathbb{C} \setminus \{0\}$, $\Lambda(\mathbb{C} \setminus B(0,1)) < +\infty$

Let $Y_N = (Y_{ij})_{i,j} \in M_N^{s.a}(\mathbb{C})$ such that $(Y_{ij})_{i \geq j}$ is iid and

$N \cdot \mathbb{P}(Y_{ij} \in B) \longrightarrow \Lambda(B) \quad \forall B \subseteq \mathbb{C} \setminus \{0\}$

$Y_N = A \odot X$ $(X_{ij})_{i \geq j}$ iid with law γ and A = adjacency operator of a uniform random graph with degree sequence $D_N = (D_N(1), \dots, D_N(n))$ s.t. $L_{D_N} \to \pi$

Th If $Y = (Y_j)_{j \in J}$ are independent heavy-tailed with parameter Λ or (π, γ)

Then $U(G(Y))$ satisfies a LDP in $\mathcal{P}(\mathcal{N}^\bullet)$ with speed N and an explicit rate function $\Sigma_\Lambda(\mu)$. or $\Sigma_{\pi,\gamma}(\mu)$.

THE RATE FUNCTION FOR (π, γ) case

$Y_N = A \odot X$ $\quad (X_{ij})_{i>j}$ iid with law γ and A = adjacency operator of a uniform random graph with degree sequence $D_N = (D_N(1), \dots, D_N(N))$ s.t. $L_{D_N} \to \pi$

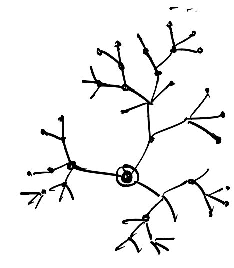
$|J| = 1$ without real loss of generality (up to enlarging the mark space)

edge rooted

root edge

$\mu \in \mathcal{P}(\mathcal{G}^\bullet)$ is unimodular, if $\forall f : \vec{\mathcal{G}}^\bullet \to \mathbb{R}_+$: $\mathbb{E}_\mu \sum_v f(G, o, v) = \mathbb{E}_\mu \sum_v f(G, v, o)$

(If G is finite, $U(G)$ is unimodular)



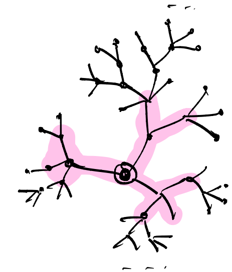
We have $\Sigma_{\pi,\gamma}(\mu) = \infty$ if

$$\begin{cases} \mu \text{ is not unimodular} \\ \mu \text{ is not supported on trees} \\ \text{Law}(\deg(o)) \neq \pi \\ D_{KL}\left((G)_h \mid (G_\gamma)_h \right) = +\infty \end{cases}$$

$G \sim \text{law}(\mu)$ $\quad$ h-neighborhood

G_γ: same graph with iid marks with law γ

THE RATE FUNCTION FOR (π,γ) case



$$\Sigma_{\pi,\gamma}(\mu) = \sup_{h\geq 0} \Sigma_{\pi,\gamma}(\mu_h, h), \qquad \mu \in \mathcal{P}(\mathcal{G}^\bullet)$$

(Dawson-Gärtner Thm)

law of h-neighborhood.

$\mu_h \in \mathcal{P}(\mathcal{G}_h^\bullet)$

$$\Sigma_{\pi,\gamma}(\nu,h) = \Sigma^0_{\pi,\gamma}(\nu,h) + \Sigma^1_{\pi,\gamma}(\nu,h), \qquad \nu \in \mathcal{P}(\mathcal{G}_h^\bullet)$$

$$\Sigma^1_{\pi,\gamma}(\nu,h) = D_{KL}(G \mid G_\gamma) - \frac{d}{2} D_{KL}(\vec{G} \mid \vec{G}_\gamma), \qquad d = \sum_k k\pi(k)$$

law ν

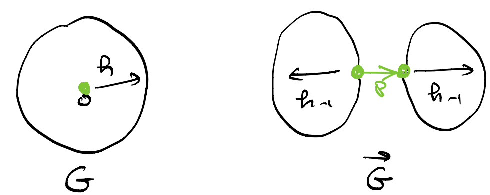
same graph iid marks with law γ

edge rooted version of $\nu \sim$ law(G)

$$\mathbb{P}(\vec{G} \in \cdot) = \frac{1}{\mathbb{E}\deg(o)} \mathbb{E} \sum_{v\sim o} \mathbb{1}((G,o,v)\in \cdot)$$

$$\Sigma^0_{\pi,\gamma}(\nu,h) = -H(G^\circ) + \frac{d}{2} H(\vec{G}^\circ) + cte(\pi)$$

Graph G without marks.



B.–Caputo 15, Delgosha–Anantharam 19, Backhausz–B.–Szegedy 22, Ramanan–Yasodharan 23.

CONTINUITY OF DISTRIBUTION

If $G = (V, E, \xi) \in \mathcal{G}^\circ$ such that $\forall\, u \in V$, $\sum_v |\xi(u,v)|^2 < +\infty$

we define Y_G operator on compactly supported function, $\ell^2_c(V)$:

$$Y_G \delta_v = \sum_u \xi(u,v)\, \delta_u$$

If Y_G is ess. self-adjoint, we set $L^{\delta_0}_{Y_G}$ = spectral measure at δ_0.

If $\mu \in \mathcal{P}(\mathcal{G}^\circ)$ such that Y_G is ν. a.s. ess. self adjoint, we set: $\underset{\mathcal{P}(\mathbb{R})}{L_\mu} = \mathbb{E}\left[L^{\delta_0}_{Y_G}\right]$.

Th. $0<\varepsilon<1$, $0<\beta<2$, Set $K_{\beta,\varepsilon} = \{\mu \in \underset{\text{unimodular}}{\mathcal{P}(\mathcal{G}^\circ)} : \mathbb{E}_\mu [\sum_{v \neq o} |\xi(o,v)|^\beta \mathbb{1}_{|\xi(o,v)| \le \varepsilon}) \le 1\}$.

Then $L : \begin{cases} K_{\beta,\varepsilon} \to \mathcal{P}(\mathbb{R}) \\ \mu \longmapsto L_\mu \end{cases}$ is well-defined and continuous. (same result for $\underset{\text{unimodular}}{\mu \in \mathcal{P}(\mathcal{G}^\circ)}$)

(If $Y \in M_N^{sa}(\mathbb{C})$, $L_{U(G(Y))} = \frac{1}{N} \sum_{x=1}^N L^{\delta_x}_Y = L_Y = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i(Y)}$).

Similarly, the Traffic distribution is continuous (with some care).

CONCLUDING WORDS

IN SUMMARY

- For matrix ensembles with $O(1)$ non-negligeable entries per row the local weak topology on rooted graphs / networks gives a good framework for limits of matrix algebras.

- Explicit LDP are known for heavy-tailed random matrices and random graphs in these local weak topologies.

COMMENTS & PERSPECTIVES

* There is a unique and explicit minimizer of the entropy Σ.

* More properties of Σ and $\chi(\tau) = \inf\{\Sigma(\mu) : \tau_\mu = \tau\}$?

* On $\mathcal{G}^\circ$, the traffic distribution of weighted adjacency operators $\Leftrightarrow$ local weak topology.

* Aldous–Lyons (2007) define a W^*-algebra associated to any $\mu \in \mathcal{P}(\mathcal{G}^\circ)$ unimodular.

* A similar analysis could be done for $(Y_1, \dots, Y_d)$ iid GUE and non-tracial state: $\langle \varphi, \cdot\, \varphi \rangle$.

THANK YOU FOR ATTENTION !