Asymptotics of Harish-Chandra transform and Infinitesimal Free Probability

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Outline

- Unitarily invariant random matrices and the Fourier transform approach
- Infinitesimal free probability via Fourier transform
- Finite-rank perturbations and BBP phase transitions
- Higher order infinitesimal free probability
- Schur generating functions of random signatures
- Quantized infinitesimal free probability
- Outliers in random domino tilings and asymptotic representation theory of $U(N)$.

Sums of random matrices

- The empirical spectral distribution of a $N \times N$ Hermitian matrix is

$$m[A_N] := \frac{1}{N}\sum_{i=1}^{N} \delta(\lambda_i(A_N)), \text{ where } \lambda_1(A_N) \geq \cdots \geq \lambda_N(A_N) \text{ are the eigenvalues of } A_N.$$

- Let A_N, B_N be uniformly random Hermitian matrices with deterministic spectra, i.e.

$$A_N = U_N \mathrm{A}_N U_N^*, \quad B_N = V_N \mathrm{B}_N V_N^*, \quad \text{where} \quad U_N, V_N \quad \text{are Haar distributed}$$

and $\mathrm{A}_N, \mathrm{B}_N$ are deterministic and diagonal.

Question. What can we say for the random spectrum of $A_N + B_N$?

Theorem **(Voiculescu, 1991)** *Assume that A_N, B_N are independent and such that as $N \to \infty$, $m[A_N], m[B_N]$ weakly converge to probability measures $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2$ respectively. Then the random measure $m[A_N + B_N]$ converges weakly, in probability to a deterministic probability measure $\boldsymbol{\mu}_1 \boxplus \boldsymbol{\mu}_2$ which is the free convolution of $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$.*

- A **non-commutative probability space** is a pair $(\mathcal{A},\varphi)$ where $\mathcal{A}$ is a unital algebra and $\varphi\colon\mathcal{A}\to\mathbb{C}$ is a linear functional such that $\varphi(1)=1$.

Given Hermitian random matrices A_N, global asymptotic results for the spectrum emerge by the limits of the ncps $\left(\mathbb{C}[\mathbf{x}], P\overset{\varphi_{A_N}}{\longmapsto}\frac{1}{N}\mathbb{E}[\mathrm{Tr}P(A_N)]\right)$. A type of limit is the pair $(\mathbb{C}[\mathbf{x}], P\overset{\varphi}{\longmapsto}\int_{\mathbb{R}}P(t)\boldsymbol{\mu}(dt))$ where

$$\lim_{N\to\infty}\varphi_{A_N}(P)=\varphi(P).$$

There is a notion of cumulants which expresses in a simpler way how the limit of $\varphi_{A_N+B_N}$ is related to the limits of $\varphi_{A_N},\varphi_{B_N}$.

- The **free cumulants** $(\kappa_n(a))_{n\in\mathbb{N}}$ of the random variable $a\in(\mathcal{A},\varphi)$ are uniquely determined by

$$\varphi(a^k)=\sum_{\pi\in\mathrm{NC}(k)}\prod_{V\in\pi}\kappa_{|V|}(a),\quad\text{for every } k\in\mathbb{N}.$$

  where $\mathrm{NC}(k)$ denotes the lattice of non-crossing partitions of $\{1,\ldots,k\}$.

The ***R*-transform** of a is $R_a(z)=\sum_{n\geq 1}\kappa_n(a)z^{n-1}$.

We recall that the free convolution is an operation of probability measures characterized by.

$$\kappa_n(\boldsymbol{\mu}_1\boxplus\boldsymbol{\mu}_2)=\kappa_n(\boldsymbol{\mu}_1)+\kappa_n(\boldsymbol{\mu}_2),\quad\text{for every } n\geq 1.$$

The **Harish-Chandra-Itzykson-Zuber integral** is defined by

$$HC(A_N; B_N) = \int_{U(N)} \exp(\mathrm{Tr}(A_N U_N B_N U_N^*)) dU_N.$$

- $HC(A_N; B_N)$ depends only on the eigenvalues of A_N and B_N.
- $HC(A_N; B_N)$ is symmetric in $\{\lambda_1(A_N), \ldots, \lambda_N(A_N)\}$ and $\{\lambda_1(B_N), \ldots, \lambda_N(B_N)\}$ separately.

It is due to Harish-Chandra that

$$HC(a_1, \ldots, a_N; b_1, \ldots, b_N) = c_N \frac{\det(e^{a_i b_j})_{1 \le i,j \le N}}{\prod_{i<j}(a_i - a_j) \prod_{i<j}(b_i - b_j)} \quad \text{where} \quad c_N = \prod_{i=1}^{N-1} i!.$$

Definition *Let A_N be Hermitian random matrices of size N. The Harish-Chandra transform of A_N is*

$$(x_1, \ldots, x_N) \longmapsto \mathbb{E}[HC(A_N; x_1, \ldots, x_N)].$$

Theorem (Bufetov-Gorin, 2013) *Let A_N be Hermitian random matrices such that*

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}[HC(A_N;x_1,\ldots,x_r,0^{N-r})]=\sum_{i=1}^{r}\Psi(x_i)$$

where Ψ is analytic and r is arbitrary and it does not depend on N. Then $m[N^{-1}A_N]$ converges as $N\to\infty$ in probability, in the sense of moments to a probability measure $\boldsymbol{\mu}$ with moments

$$\boldsymbol{\mu}_k=\sum_{m=0}^{k-1}\binom{k}{m}\frac{1}{(m+1)!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}(\Psi'(x))^{k-m}\Big|_{x=0}.$$

The R-transform of $\boldsymbol{\mu}$ is Ψ'.

Theorem (Bufetov-Z., 2024) *Let A_N be Hermitian random matrices such that*

$$\lim_{N\to\infty}\frac{1}{N^l}\log\mathbb{E}[HC(A_N;x_1,\ldots,x_r,0^{N-r})]=\sum_{i=1}^{r}\Psi(x_i)$$

where Ψ is analytic, r is arbitrary and does not depend on N and $l\geq 2$ fixed. Then $m[N^{-l}A_N]$ converges as $N\to\infty$ in probability, in the sense of moments to $\delta_{\Psi'(0)}$.

- Guionnet-Maida (2004) proved that for matrices A_N with deterministic spectrum, convergence of $m_N[N^{-1}A_N]$ implies asymptotic additivity for the normalized logarithm of the Harish-Chandra integral.
- We can use the previous theorem and the results of Guionnet-Maida to prove the theorem of Voiculescu.

Example. Let $A_N = (a_{i,j})_{1\le i,j\le N}$ be the Hermitian Gaussian random matrix where $a_{i,i} \sim \mathcal{N}(0, 2\gamma)$, $\Re a_{i,j} \sim \mathcal{N}(0,\gamma)$, $\Im a_{i,j} \sim \mathcal{N}(0,\gamma)$ and $\{a_{i,j}\}_{i\ge j}$ are independent. Then

$$\mathbb{E}[HC(A_N; x_1, \ldots, x_N)] = \prod_{i=1}^{N} \exp(\gamma x_i^2).$$

Example. Let $A_N = \alpha\, X_1 X_1^* + \cdots + \alpha\, X_M X_M^*$, where $(X_i)_{i=1}^{M}$ are independent standard complex Gaussian vectors. Then,

$$\mathbb{E}[HC(A_N; x_1, \ldots, x_N)] = \prod_{i=1}^{N} (1 - \alpha x_i)^{-M}.$$

In the previous examples we have additivity for the logarithm of the Harish-Chandra integral, for every N and not just asymptotically. These examples are related to ergodic unitarily invariant measures on the space of infinite Hermitian matrices.

$$H = \{\infty \times \infty \text{ Hermitian matrices}\}$$

$$H(\infty) = \{\infty \times \infty \text{ Hermitian matrices with finitely many entries different from } 0\}$$

$$U(\infty) = \{\infty \times \infty \text{ unitary matrices with finitely many entries different from } \delta_{i,j}\}$$

Olshanski-Vershik (1996) proved that the ergodic $U(\infty)$-invariant Borel probability measures $\boldsymbol{M}$ on H are characterized by the multiplicativity of the Fourier transform, i.e. for every $A \in H(\infty)$

$$f_{\boldsymbol{M}}(A) := \int_H \exp(i\mathrm{Tr}(AB))\boldsymbol{M}(dB) = \prod_{a \in \mathrm{Spec}(A)} F(a), \quad \text{for some function } F\colon \mathbb{R} \to \mathbb{C} \text{ with } F(0) = 1.$$

The functions F are of the form

$$F(a) = e^{i\gamma_1 a - \gamma_2 a^2/2} \prod_k \frac{e^{-iy_k a}}{1 - iy_k a}$$

where $\gamma_1 \in \mathbb{R}$, $\gamma_2 \geq 0$ and $(y_k)_{k \in \mathbb{N}}$ is a sequence of real numbers such that $\sum_k y_k^2 < \infty$.

The classification of ergodic $U(\infty)$-invariant probability measures is related to a limit regime for the Harish-Chandra integral, different from the previous ones.

Theorem (**Olshanski-Vershik, 1996**) *We have the following equivalence*

$$\log HC(\lambda_1,\dots,\lambda_N;x_1,\dots,x_r,0^{N-r}) \longrightarrow \Phi(x_1)+\cdots+\Phi(x_r)$$

$$\Longleftrightarrow \sum_{i=1}^{N}\delta\left(\frac{\lambda_i}{N}\right)\longrightarrow \mu, \quad \textit{in the sense of moments.}$$

In other words,

$$\frac{\lambda_1}{N}\longrightarrow y_1',\dots,\frac{\lambda_i}{N}\longrightarrow y_i',\dots,$$

$$\frac{\lambda_N}{N}\longrightarrow y_1'',\dots,\frac{\lambda_{N-i+1}}{N}\longrightarrow y_i'',\dots$$

where $y_k = y_k' + y_k''$. *There are also precise expressions for* γ_1,γ_2, *as limits of* (λ_i) *related terms.*

Limit regime for the unnormalized trace

Different limit regimes for the logarithm of the Harish-Chandra transform lead to different type of asymptotic results for the empirical distribution of the corresponding matrix. The following limit regime leads to asymptotic results for the unnormalized trace

Theorem (**Bufetov-Z., 2024**) *Let A_N be Hermitian random matrices such that*

$$\lim_{N\to\infty} \log \mathbb{E}[HC(A_N; x_1, \ldots, x_r, 0^{N-r})] = \sum_{i=1}^{r} \Phi(x_i),$$

where Φ is analytic and r is arbitrary and does not depend on N. Then for every $k \in \mathbb{N}$, the k-th moment of $Nm[N^{-1}A_N]$ converges in probability to $\Phi^{(k)}(0)/(k-1)!$.

Studying limits of $\mathbb{E}[\mathrm{Tr} A_N^k]$ as $N \to \infty$, fits into the framework of infinitesimal free probability. The tools of infinitesimal free probability explain the dependence of the limits from the function Φ.

- An **infinitesimal ncps** is a triple $(\mathcal{A}, \varphi, \varphi')$ where $(\mathcal{A}, \varphi)$ is a ncps and $\varphi' : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi'(1) = 0$.

In certain cases the extra functional φ' arises naturally when $(\mathcal{A}, \varphi)$ is the limit of a sequence $(\mathcal{A}, \varphi_N)$ but we also have an infinitesimal limit, i.e.

$$\varphi(a) = \lim_{N \to \infty} \varphi_N(a) \quad \text{and} \quad \varphi'(a) = \lim_{N \to \infty} N(\varphi_N(a) - \varphi(a)), \quad \text{for every } a \in \mathcal{A}.$$

In this sense φ' plays the role of the derivative.

- The **infinitesimal free cumulants** $(\kappa'_n(a))_{n \in \mathbb{N}}$ of a random variable $a \in (\mathcal{A}, \varphi, \varphi')$ are uniquely determined by

$$\varphi'(a^k) = \sum_{\pi \in \mathrm{NC}(k)} \sum_{V \in \pi} \kappa'_{|V|}(a) \prod_{\substack{W \in \pi \\ W \neq V}} \kappa_{|W|}(a), \quad \text{for every } k \in \mathbb{N}.$$

The **infinitesimal $\boldsymbol{R}$-transform** of a is $R'_a(z) = \sum_{n \geq 1} \kappa'_n(a) z^{n-1}$.

The sequence $(\kappa'_n(a))_{n \in \mathbb{N}}$ emerges differentiating one time the moment-cumulant relations, e.g.

$$\varphi'(a) = (\varphi(a))' = (\kappa_1(a))' = \kappa'_1(a) \quad \text{and} \quad \varphi'(a^2) = (\kappa_2(a) + \kappa_1(a)\kappa_1(a))' = \kappa'_2(a) + 2\kappa'_1(a)\kappa_1(a).$$

The limit regimes

$$\frac{1}{N^{\varepsilon}}\log\mathbb{E}[HC(A_N;x_1,\dots,x_r,0^{N-r})]\longrightarrow\Phi_{\varepsilon}(x_1)+\cdots+\Phi_{\varepsilon}(x_r),\quad\text{for }\varepsilon=0,1$$

have concrete and different impacts on the empirical distribution of A_N from the side of infinitesimal free probability.

We recall

$$\varphi_{A_N}(P)=\frac{1}{N}\mathbb{E}[\mathrm{Tr}P(A_N)],\quad\text{for }P\in\mathbb{C}[\mathbf{x}].$$

- The regime $\varepsilon=1$ gives a limit of $(\mathbb{C}[\mathbf{x}],\varphi_{N^{-1}A_N})$ which depends on Φ_1 concretely, in the sense that Φ_1' is the R-transform.
- The regime $\varepsilon=0$ implies that the limit of $(\mathbb{C}[\mathbf{x}],\varphi_{N^{-1}A_N})$ is 0 and it also gives an infinitesimal limit which depends on Φ_0. The dependence from Φ_0 is also specific because in this case

  $$\varphi'(\mathbf{x}^n)=\kappa_n'(\mathbf{x}),\quad\text{for every }n.$$

  This means that Φ_0' is the infinitesimal R-transform.

Theorem (Bufetov-Z., 2024) *Let A_N be Hermitian random matrices such that*

$$\lim_{N\to\infty}\left(\log\mathbb{E}[HC(A_N;x_1,\dots,x_r,0^{N-r})]-N\sum_{i=1}^{r}\Psi(x_i)\right)=\sum_{i=1}^{r}\Phi(x_i)$$

where Ψ,Φ are analytic and r is arbitrary and does not depend on N. Then we have that

$$\varphi_{N^{-1}A_N}(P)=\varphi(P)+\frac{1}{N}\varphi'(P)+o(N^{-1}),$$

where $(\mathbb{C}[\mathbf{x}],\varphi,\varphi')$ is an incps such that

$$\varphi(\mathbf{x}^k)=\sum_{m=0}^{k-1}\binom{k}{m}\frac{1}{(m+1)!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}(\Psi'(x))^{k-m}\bigg|_{x=0}$$

and

$$\varphi'(\mathbf{x}^k)=\sum_{m=0}^{k-1}\binom{k}{m+1}\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\Phi'(x)(\Psi'(x))^{k-m-1}\bigg|_{x=0}.$$

Moreover the R-transform is Ψ' and the infinitesimal R-transform is Φ'.

It was shown by Shlyakhtenko (2015) that infinitesimal free probability can provide information about outlier eigenvalues of a random matrix, that are not visible in the limit. Such outlier eigenvalues can emerge by considering additive perturbations of classical random matrix ensembles by finite-rank matrices. It was proved by Olshanski-Vershik (1996) that for matrices of finite rank the (unnormalized) logarithms of the Harish-Chandra integrals are additive asymptotically.

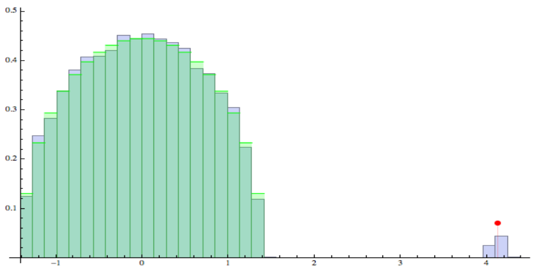
- For $\theta \in \mathbb{R}$ the fact that $(u_{i,j})_{1 \leq i,j \leq N} \in U(N) \mapsto (|u_{11}|^2, \dots, |u_{1N}|^2)$ has a Dirichlet distribution with parameters $(1, \dots, 1)$ implies that

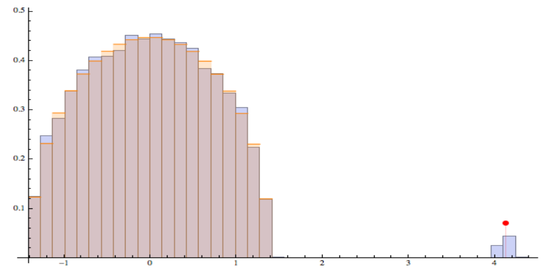
$$\log HC(x_1, \dots, x_r, 0^{N-r}, N\theta, 0^{N-1}) \longrightarrow \sum_{i=1}^{r} \log(1 - \theta x_i)^{-1}.$$

  As a consequence the matrix $N \cdot A_N + \mathrm{diag}(N\theta, 0^{N-1})$, where A_N is a GUE, fits into the framework of our previous theorem and the $1/N$ correction of $A_N + \mathrm{diag}(\theta, 0^{N-1})$ is given by the signed measure

$$\boldsymbol{\mu}'_\theta(dt) = \mathbf{1}_{|\theta| \geq 1} \delta_{\theta + \theta^{-1}}(dt) - \frac{\theta(t - 2\theta)}{2\pi(\theta(t - \theta) - 1)\sqrt{4 - t^2}} dt.$$

From the $1/N$ correction of a rank one perturbation of a GUE, we deduce that the extremal eigenvalue exhibits a phase transition depending on the magnitude $|\theta|$. This is called a BBP transition and this phenomenon was originally showed by Baik-Ben Arous-Peche (2005) for complex sample covariance matrices.


(1a)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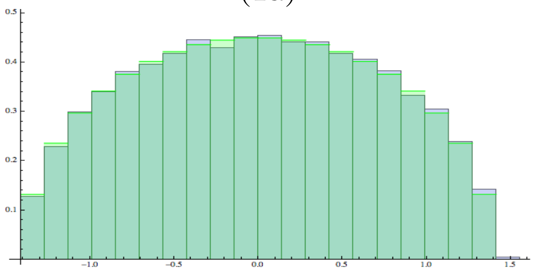
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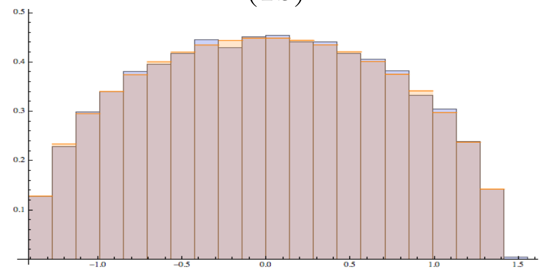

(1d)

Additional asymptotic conditions for the Harish-Chandra transform of A_N lead to higher order infinitesimal limits of $(\mathbb{C}[\mathbf{x}], \varphi_{N^{-1}A_N})$ as $N \to \infty$. This is related to higher order infinitesimal free probability, introduced by Fevrier (2010).

Given a second order correction to the limit of the logarithm of the Harish-Chandra transform, i.e.

$$N^2\left(\frac{1}{N}\log\mathbb{E}[HC(A_N; x_1, \ldots, x_r, 0^{N-r})] - \sum_{i=1}^{r} \Psi(x_i) - \frac{1}{N}\sum_{i=1}^{r} \Phi(x_i)\right) \longrightarrow \frac{1}{2}\sum_{i=1}^{r} \mathrm{T}(x_i),$$

we get a "second order Taylor expansion"

$$\varphi_{N^{-1}A_N}(P) = \varphi^{(0)}(P) + \frac{1}{N}\varphi^{(1)}(P) + \frac{1}{N^2}\frac{\varphi^{(2)}(P)}{2} + \frac{1}{N^2}\phi^{(2)}(P) + o(N^{-2}), \quad P \in \mathbb{C}[\mathbf{x}],$$

where $(\mathbb{C}[\mathbf{x}], \varphi^{(0)}, \varphi^{(1)}, \varphi^{(2)})$ is an incps of order 2.

The function T' plays the role of the second order infinitesimal R-transform. Second order infinitesimal free cumulants $(\kappa''_n(a))_{n\in\mathbb{N}}$ of $a \in (\mathcal{A}, \varphi, \varphi', \varphi'')$ can be defined, differentiating twice the moment cumulant relations.

For example, $\varphi''(a) = \kappa''_1(a)$ and

$$\varphi'(a^2) = \kappa'_2(a) + 2\kappa'_1(a)\kappa_1(a) \Longrightarrow \varphi''(a^2) = \kappa''_2(a) + 2\kappa''_1(a)\kappa_1(a) + 2(\kappa'_1(a))^2.$$

$$\varphi_{N^{-1}A_N}(P) = \varphi^{(0)}(P) + \frac{1}{N}\varphi^{(1)}(P) + \frac{1}{N^2}\frac{\varphi^{(2)}(P)}{2} + \frac{1}{N^2}\phi^{(2)}(P) + o(N^{-2}), \quad P \in \mathbb{C}[\mathbf{x}].$$

There are explicit formulas for $\varphi^{(2)}$,

$$\varphi^{(2)}(\mathbf{x}^k) = \sum_{m=0}^{k-1} \binom{k}{m+1} \frac{1}{m!} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \mathrm{T}'(x)(\Psi'(x))^{k-m-1}\bigg|_{x=0} + \sum_{m=0}^{k-2} \frac{k!}{m!(m+1)!(k-m-2)!} \frac{\mathrm{d}^m}{\mathrm{d}x^m}(\Phi'(x))^2(\Psi'(x))^{k-m-2}\bigg|_{x=0}.$$

- For the extra functional $\phi^{(2)}$, there are explicit formulas and the values $\phi^{(2)}(\mathbf{x}^k)$ depend only on Ψ.
- One can thing that for the second order limit regime our random matrix has the form $A_N = A_N^{(0)} + A_N^{(1)} + A_N^{(2)}$, where $A_N^{(0)}$ is the leading matrix that satisfies the free probability limit regime and $A_N^{(1)}, A_N^{(2)}$ are perturbations. The perturbations $A_N^{(1)}, A_N^{(2)}$ are not visible in the limit, but in the $1/N$ and $1/N^2$ correction respectively.
- The leading matrix $A_N^{(0)}$ gives rise to $\phi^{(2)}$ because the free probability limit regime leads to an expansion

$$\frac{1}{N^{k+1}}\mathbb{E}\big[\operatorname{Tr}\big(A_N^{(0)}\big)^k\big] = M_{N,k}^{(0)} + \frac{1}{N^2}M_{N,k}^{(1)} + \frac{1}{N^4}M_{N,k}^{(2)} + \cdots.$$

The limit regime for the $1/N$ correction detects if there are finitely many outliers in the spectrum, possibly coming from finite rank perturbations.

Perturbations of rank that depend on N may cause infinitely many outliers. This phenomenon cannot be captured by the $1/N$ correction and the corresponding limit regime. Instead, we have

Theorem (**Bufetov-Z., 2024**) *Let A_N be Hermitian random matrices and $0 \leq \varepsilon \leq 1$, such that*

$$\lim_{N \to \infty} N^{\varepsilon}\left(\frac{1}{N}\log \mathbb{E}[HC(A_N; x_1, \ldots, x_r, 0^{N-r})] - \sum_{i=1}^{r} \Psi(x_i)\right) = \sum_{i=1}^{r} \Phi(x_i)$$

where Ψ, Φ are analytic and r is arbitrary and does not depend on N. Then we have that

$$\varphi_{N^{-1}A_N}(P) = \varphi(P) + \frac{1}{N^{\varepsilon}}\varphi'(P) + o(N^{-\varepsilon}),$$

where $(\mathbb{C}[\mathbf{x}], \varphi, \varphi')$ is an incps such that $(\varphi(\mathbf{x}^k))$ are as before and

$$\varphi'(\mathbf{x}^k) = \sum_{m=0}^{k-1} \binom{k}{m+1} \frac{1}{m!} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \Phi'(x)(\Psi'(x))^{k-m-1}\bigg|_{x=0}.$$

The intermediate limit regime gives an interpolation between the free probability limit regime and the Olshanski-Vershik limit regime.

Example. The $\lfloor N^{1-\varepsilon} \rfloor$-rank perturbation of the GUE

$$A_N = N \cdot \mathrm{GUE} + \theta \sum_{i=1}^{\lfloor N^{1-\varepsilon} \rfloor} X_i X_i^*, \quad \text{where } (X_i) \text{ are independent standard complex Gaussian vectors,}$$

fits into the intermediate limit regime.

Our theorem implies that the $1/N^{\varepsilon}$ correction of $N^{-1}A_N$ is given by the same signed measure as for the $1/N$ correction of $\mathrm{GUE} + \mathrm{diag}(\theta, 0^{N-1})$.

This leads to the same BBP phase transition for $N^{-1}A_N$, where depending on $|\theta|$ we might have $N^{1-\varepsilon}$ outliers $\theta + \theta^{-1}$.

Higher order generalizations of the intermediate limit regime can describe the situation where we have finitely many perturbations. The first one will have rank $N^{1-\varepsilon}$ and the outliers that creates will be visible from the $1/N^{\varepsilon}$ correction. The second one will have rank $N^{1-2\varepsilon}$ and the outliers that creates will be visible from the $1/N^{2\varepsilon}$ correction, and so forth.

Theorem (**Bufetov-Z., 2024**) *Let* A_N *be Hermitian random matrices and* $0 < \varepsilon < n^{-1}$*, such that*

$$\lim_{N\to\infty} N^{n\varepsilon}\left(\frac{1}{N}\log\mathbb{E}[HC(A_N;x_1,\dots,x_r,0^{N-r})] - \sum_{j=0}^{n-1}\sum_{i=1}^{r}\frac{1}{N^{j\varepsilon}}\frac{\Psi_j(x_i)}{j!}\right) = \sum_{i=1}^{r}\frac{\Psi_n(x_i)}{n!},$$

where $\Psi_0,\dots,\Psi_n$ *are analytic and* r *is arbitrary and does not depend on* N*. Then, we have that*

$$\varphi_{N^{-1}A_N}(P) = \varphi^{(0)}(P) + \frac{1}{N^{\varepsilon}}\varphi^{(1)}(P) + \cdots + \frac{1}{N^{n\varepsilon}}\varphi^{(n)}(P) + o(N^{-n\varepsilon}),$$

where $(\mathbb{C}[\mathbf{x}], \varphi^{(0)}, \varphi^{(1)}, \dots, \varphi^{(n)})$ *is an incps of order* n*. The values* $\varphi^{(i)}(\mathbf{x}^k)$ *can be computed explicitly and the function* Ψ_i' *is the* i*-th order infinitesimal* R*-transform.*

- A signature of length N is a N-tuple of integers $\lambda = (\lambda_1, \ldots, \lambda_N)$. We denote by $\mathrm{Sign}(N)$ the set of such N-tuples. For example $\lambda = (5, 3, 3, 1, -2, -2)$ is a signature of length 6.
- It is known that all irreducible representations of $U(N)$ are parametrized by signatures. Let π^λ be an irreducible representation of $U(N)$ corresponding to λ.
- The character π^λ is the Schur function

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det\left(x_i^{\lambda_j + N - j}\right)_{1 \leq i, j \leq N}}{\prod_{1 \leq i < j \leq N} (x_i - x_j)}.$$

- We will encode a representation π^λ and a signature λ by the counting measure $m[\lambda]$:

$$m[\lambda] = \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right).$$

Decomposition into irreducibles

- Given a finite-dimensional representation π of $U(N)$ we can decompose it into irreducible components

$$\pi = \bigoplus_{\nu \in \mathrm{Sign}(N)} c_\lambda \, \pi^\lambda, \quad \text{where the non-negative integers } c_\lambda \text{ are multiplicities.}$$

- This decomposition can be identified with a probability measure ρ^π on signatures of length N, such that

$$\rho^\pi(\lambda) := \frac{c_\lambda \dim(\pi^\lambda)}{\dim(\pi)}.$$

- The pushforward of ρ^π with respect to the map $\lambda \mapsto m[\lambda]$ is a random probability measure on $\mathbb{R}$. We denote this by $m[\lambda]$.

Example. Let $\pi = \pi^{(3,2)} \oplus \pi^{(3,1)}$. Because $\dim(\pi^{(3,2)}) = 2, \dim(\pi^{(3,1)}) = 3$, $m[\pi]$ is the random probability measure which takes values

$$\begin{cases} \frac{1}{2}\delta(2) + \frac{1}{2}\delta(1), & \text{with probability } \frac{2}{5}, \\ \frac{1}{2}\delta(2) + \frac{1}{2}\delta\left(\frac{1}{2}\right), & \text{with probability } \frac{3}{5}. \end{cases}$$

- Let λ, μ be signatures of length N. The decomposition of the tensor product $\pi^\lambda \otimes \pi^\mu$ is given by the Littlewood-Richardson coefficients.
- One can write the decomposition of tensor products in terms of Schur functions

$$s_\lambda(x_1,\ldots,x_N)s_\mu(x_1,\ldots,x_N)=\sum_\eta c_\eta^{\lambda,\mu}\, s_\eta(x_1,\ldots,x_N).$$

- The explicit formula for the measure on signatures

$$\rho^{\pi^\lambda\otimes\pi^\mu}(\eta)=\frac{c_\eta^{\lambda,\mu}\, s_\eta(1,\ldots,1)}{s_\lambda(1,\ldots,1)s_\mu(1,\ldots,1)}.$$

We are interested in the asymptotic behaviour of the decomposition of the tensor product into irreducibles. We will extract information studying the asymptotic behaviour of $m[\pi^\lambda \otimes \pi^\mu]$.

Theorem (**Bufetov-Gorin, 2013**) *Let $\lambda^1, \lambda^2 \in \mathrm{Sign}(N)$, $N \in \mathbb{N}$, which satisfy some technical assumptions including that*

$$m[\lambda^i] \xrightarrow{w} \boldsymbol{\mu}_i, \quad as\ N \to \infty.$$

Let $\pi_N = \pi^{\lambda^1} \otimes \pi^{\lambda^2}$. Then, $m[\pi_N]$ converges in probability, in the sense of moments to a deterministic probability measure $\boldsymbol{\mu}_1 \otimes \boldsymbol{\mu}_2$ such that

$$(\boldsymbol{\mu}_1 \otimes \boldsymbol{\mu}_2) \boxplus u[0,1] = \boldsymbol{\mu}_1 \boxplus \boldsymbol{\mu}_2$$

where $u[0,1]$ is the uniform measure on $[0,1]$.

- The operation $(\mu_1, \mu_2) \mapsto \mu_1 \otimes \mu_2$ is defined by the linearization of an analytic function. This analytic function is not the usual R-transform but its quantized version:

$$R_{\boldsymbol{\mu}}^{quant}(z) := R_{\boldsymbol{\mu}}(z) - R_{u[0,1]}(z).$$

  In other words

$$R_{\boldsymbol{\mu} \otimes \boldsymbol{\nu}}^{quant}(z) = R_{\boldsymbol{\mu}}^{quant}(z) + R_{\boldsymbol{\nu}}^{quant}(z).$$

- Comparing the theorems of Voiculescu (1991)and Bufetov-Gorin (2013), there is the following parallelism between the frameworks of Hermitian random matrices and representations of $U(N)$: In the discrete setting, uniformly random Hermitian matrices with deterministic spectrum are replaced by irreducible representations of $U(N)$, the sum is replaced by the tensor product and $\boxplus$ is replaced by $\otimes$.

- There is a notion of Fourier transform for discrete N-particle systems. Analogously to random matrices, the transform of the tensor product of irreducibles should be the product of the corresponding transforms.

Recall that

$$\frac{s_\lambda(x_1,\ldots,x_N)}{s_\lambda(1,\ldots,1)}\,\frac{s_\mu(x_1,\ldots,x_N)}{s_\mu(1,\ldots,1)} = \sum_{\nu\in\mathrm{Sign}(N)} \rho^{\pi^\lambda\otimes\pi^\mu}(\nu)\,\frac{s_\nu(x_1,\ldots,x_N)}{s_\nu(1,\ldots,1)}.$$

Given probability measures ρ_N on $\mathrm{Sign}(N)$ the **Schur generating function** is given by

$$S_{\rho_N}(x_1,\dots,x_N) = \sum_{\lambda\in\mathrm{Sign}(N)} \rho_N(\lambda)\,\frac{s_\lambda(x_1,\dots,x_N)}{s_\lambda(1,\dots,1)}.$$

Theorem (**Bufetov-Gorin, 2013**) *Let ρ_N be a sequence of probability measures on $\mathrm{Sign}(N)$ such that*

$$\lim_{N\to\infty}\frac{1}{N}\log S_{\rho_N}(x_1,\dots,x_r,1^{N-r}) = \Psi(x_1)+\cdots+\Psi(x_r),$$

where Ψ is analytic and r does not depend on N. Then $m[\rho_N]$ converges in probability, in the sense of moments to a probability measure with moments

$$\mu_k = \sum_{m=0}^{k}\binom{k}{m}\frac{1}{(m+1)!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(x^k(\Psi'(x))^{k-m}\right)\Big|_{x=1},$$

and R-transform

$$R_\mu(z) = e^z\Psi'(e^z) + R_{u[0,1]}(z).$$

The Olshanski-Vershik limit regime for $\lambda_N \in \mathrm{Sign}(N)$ will be

$$\frac{s_{\lambda_N}(x_1,\dots,x_r,1^{N-r})}{s_{\lambda_N}(1,\dots,1)} \longrightarrow \Phi(x_1)\dots\Phi(x_r).$$

- Similarly to random matrices, the Olshanski-Vershik limit regime for discrete λ_N is related to the classification of extreme characters of $U(\infty)$ (Vershik-Kerov (1982), Okounkov-Olshanski (1998)).

A character $\boldsymbol{\chi}\colon U(\infty)\to\mathbb{C}$ is a central and positive-definite function that maps unit to the unit.

For a character $\boldsymbol{\chi}\colon U(\infty)\to\mathbb{C}$, there exist probability measures ϱ_N on $\mathrm{Sign}(N)$ such that

$$\boldsymbol{\chi}(\mathrm{diag}(z_1,\dots,z_N,1,1,\dots)) = S_{\varrho_N}(z_1,\dots,z_N).$$

The classification of the extreme characters is due to the Edrei-Voiculescu theorem (Edrei (1953), Voiculescu (1976), Vershik-Kerov (1982), Okounkov-Olshanski (1998), Borodin-Olshanski (2012)). Each extreme character corresponds to a unique $\omega\in(\mathbb{R}_+^\infty)^4\times(\mathbb{R}_+)^2$. For arbitrary ω, the extreme character $\boldsymbol{\chi}^\omega\colon U(\infty)\to\mathbb{C}$ has the multiplicative form

$$\boldsymbol{\chi}^\omega(U) = \prod_{u\in\mathrm{Spectrum}(U)} \Phi^\omega(u).$$

The function $\Phi^\omega\colon\{u\in\mathbb{C}: |u|=1\}\to\mathbb{C}$ is called **Voiculescu function** and it has an explicit form.

Infinitesimal free probability limit regime

- Similarly to random matrices, a generalization of the Olshanski-Vershik limit regime for random $\lambda_N \in \mathrm{Sign}(N)$ can lead to asymptotic results for the $1/N$ correction to the Law of Large Numbers of Bufetov-Gorin (2013). This $1/N$ correction can reveal outliers and has an interpretation from the side of infinitesimal free probability.

Given a sequence ρ_N of probability measures on $\mathrm{Sign}(N)$, consider

$$\phi_{\rho_N}(P) = \mathbb{E}_{\lambda \sim \rho_N}\left[\int_{\mathbb{R}} P(t) m[\lambda](dt)\right], \quad P \in \mathbb{C}[\mathbf{x}].$$

From asymptotics of Schur generating functions we extract infinitesimal limits for $(\mathbb{C}[\mathbf{x}], \phi_{\rho_N})$ and a quantized analogoue of the infinitesimal free convolution.

Theorem (Bufetov-Z., 2024) *Let ϱ_N be a sequence of probability measures on $\mathrm{Sign}(N)$ such that*

$$\lim_{N\to\infty}\left(\log S_{\varrho_N}(x_1,\ldots,x_r,1^{N-r}) - N\sum_{i=1}^{r}\Psi(x_i)\right) = \sum_{i=1}^{r}\Phi(x_i),$$

where Ψ,Φ are analytic and r is arbitrary and does not depend on N. Then we have that

$$\phi_{\varrho_N}(P) = \phi(P) + \frac{1}{N}\phi'(P) + o(N^{-1}), \quad P\in\mathbb{C}[\mathbf{x}],$$

where $(\mathbb{C}[\mathbf{x}],\phi,\phi')$ is an incps such that

$$\phi(\mathbf{x}^k) = \sum_{m=0}^{k}\binom{k}{m}\frac{1}{(m+1)!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^k(\Psi'(x))^{k-m})\Big|_{x=1},$$

and

$$\phi'(\mathbf{x}^k) = \sum_{m=0}^{k-1}\binom{k}{m+1}\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(x^k\left(\Phi'(x)-\frac{1}{2x}\right)(\Psi'(x))^{k-m-1}\right)\Big|_{x=1}.$$

Moreover, the infinitesimal R-transform of $(\mathbb{C}[\mathbf{x}],\phi,\phi')$ is $z\mapsto e^z\Phi'(e^z)-\frac{1}{2}$.

Example. Consider the extreme character in $U(\infty)$ that creates the Schur generating function

$$S_{\rho_N}(x_1,\ldots,x_N)=\prod_{i=1}^{N}\exp(\gamma N(u_i-1))\frac{1}{1-\alpha(u_i-1)}.$$

Such a character is the product of the one-sided Plancherel character (discrete analog of a GUE) and another character that plays the role of a rank-one matrix.

The limit shape is known from Biane (2001) and Bufetov-Borodin-Olshanski (2015).

The $1/N$ correction of $m[\rho_N]$ is given by the signed measure

$$\mu'_{\alpha,\gamma}(dt)=\mathbf{1}_{\alpha>\sqrt{\gamma}}\,\delta\Big(\alpha+1+\frac{\gamma}{\alpha}+\gamma\Big)+\frac{1}{2}\mathbf{1}_{\alpha=\sqrt{\gamma}}\,\delta\Big(\alpha+1+\frac{\gamma}{\alpha}+\gamma\Big)$$

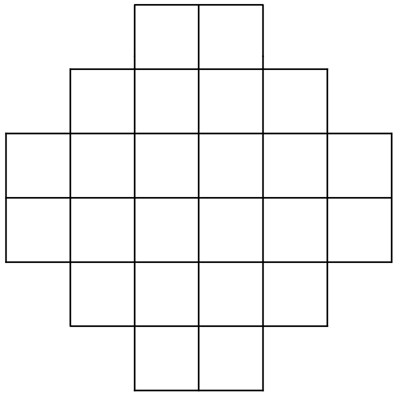
$$+\left(\frac{\alpha(t-2\alpha-\gamma-1)}{\gamma+a^2+\alpha\gamma+\alpha-\alpha t}-\frac{t+1-\gamma}{2t}\right)\frac{dt}{2\pi\sqrt{(\gamma+1+2\sqrt{\gamma}-t)(t-\gamma-1+2\sqrt{\gamma})}}.$$

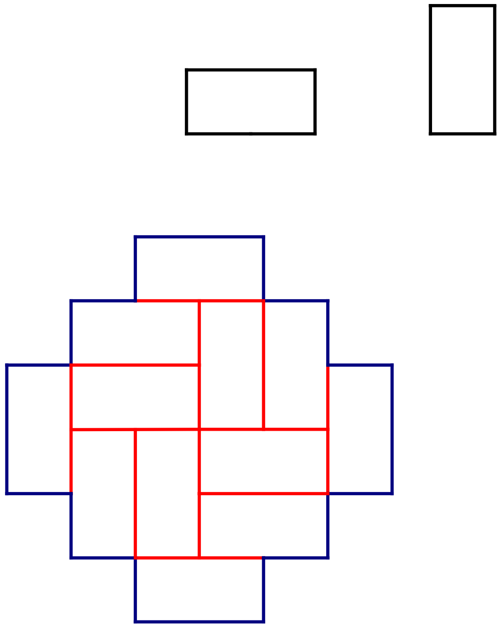
We see that depending on whether $\alpha>\sqrt{\gamma}$, the outlier does or does not appear in the model.

Domino tilings of Aztec diamond

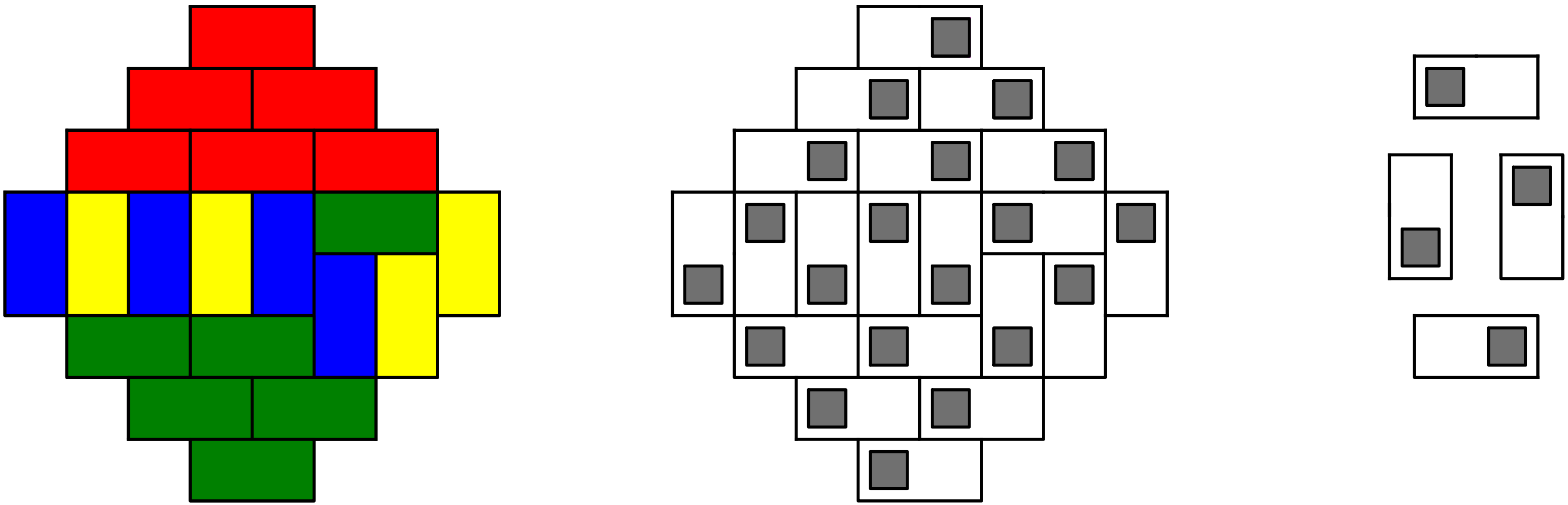
- The Aztec diamond of size N is all lattice squares which are fully contained in $\{(x,y): |x|+|y| \leq N+1\}$.
- Domino tilings of Aztec diamond where introduced by Elkies-Kuperberg-Larsen-Propp (1992). They proved that the number of tilings is equal to $2^{N(N+1)/2}$.

Aztec diamond of size 3





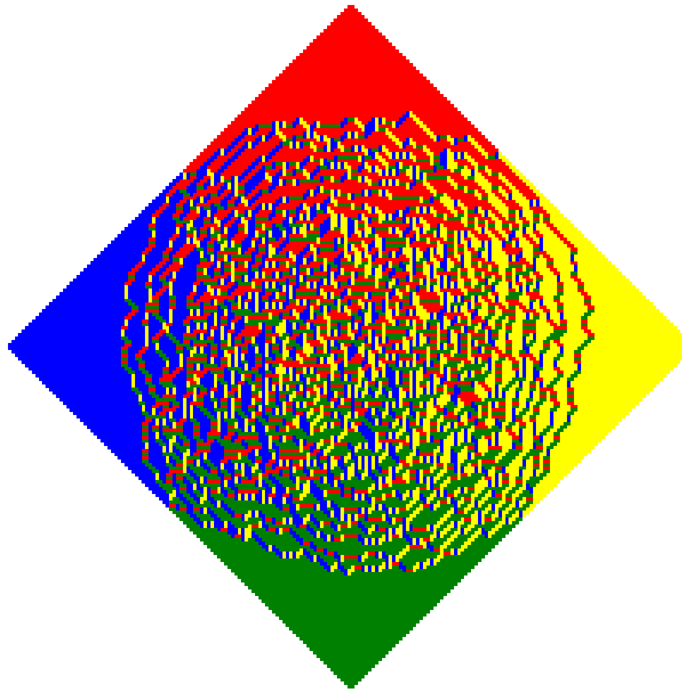
Let us consider a chessboard coloring of Aztec diamond. It is useful to distinguish not two, but four different types of dominoes.



- Each domino tiling is uniquely determined by dominoes of two types (one vertical and one horizontal).

How does a uniformly random domino tiling of a large Aztec diamond look like?

- Random tilings of the Aztec diamond is a very well studied model Jockusch-Propp-Schor (1998), Johansson (2005), ...



Theorem (**Jockusch-Propp-Schor, 1998**) *Asymptotically a uniformly random tiling becomes frozen outside of a certain circle.*

- We say that a signature λ of length N and a signature μ of length $N-1$ interlace $(\lambda \succ \mu)$ if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{N-1} \geq \mu_{N-1} \geq \lambda_N.$$

Example. $(6,3,2,0) \succ (3,3,2)$.

- We say that signatures λ, μ of length N differ by a vertical strip $(\lambda \succ_\nu \mu)$ if

$$\lambda_i - \mu_i \in \{0,1\}, \quad \text{for any } i = 1, \ldots, N.$$

Example. $(3,3,2,1,1,1) \succ_\nu (3,3,1,0,0,0)$.

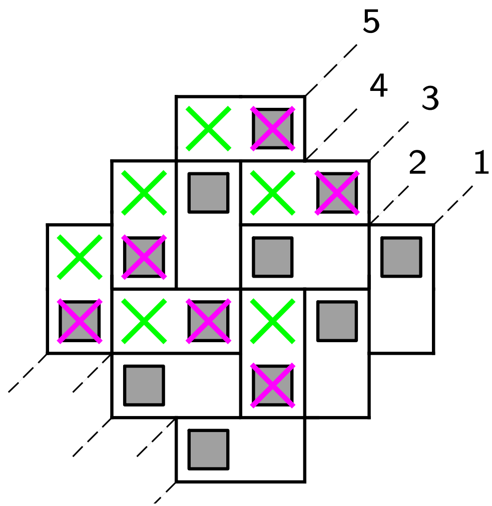
Theorem *The set of collection of signatures*

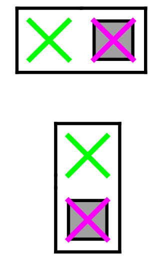
$$\{(\lambda^1, \ldots, \lambda^N, \mu^1, \ldots, \mu^{N-1}) : \lambda^1 \succ_\nu \mu^1 \prec \lambda^2 \succ_\nu \mu^2 \prec \cdots \prec \lambda^N \succ_\nu (0, \ldots, 0)\},$$

is is bijection with domino tilings of Aztec diamond of size N.

Each signature λ^r describes the positions of domino tilings along a one-dimensional slice of the Aztec diamond.

$$\lambda^1 \succ_\nu \mu^1 \prec \lambda^2 \succ_\nu \mu^2 \prec \cdots \prec \lambda^N \succ_\nu (0, \ldots, 0).$$





On each level boxes are numbered starting from 0. Distinct particles are in positions

$$1; 1; (1,3); (0,2); (0,1,3); (0,1,2).$$

Making shifts we obtain signatures:

$$1 \succ_\nu 1 \prec (1,2) \succ_\nu (0,1) \prec (0,0,1) \succ_\nu (0,0,0).$$

BBP for domino tilings of Aztec diamond

- We consider a perturbation of the uniform measure. This is

$$\mathbb{P}[\text{domino tiling}] = Z\,\theta^{\text{number of horizontal dominoes in the up level}},$$

  where $\theta > 1$.
- In terms of sequences of signatures $(\lambda^N, \mu^{N-1}, \lambda^{N-1}, \dots, \lambda^1)$, this is

$$\mathbb{P}[(\lambda^N, \mu^{N-1}, \dots, \lambda_1)] = Z\,\theta^{|\lambda^N|}.$$

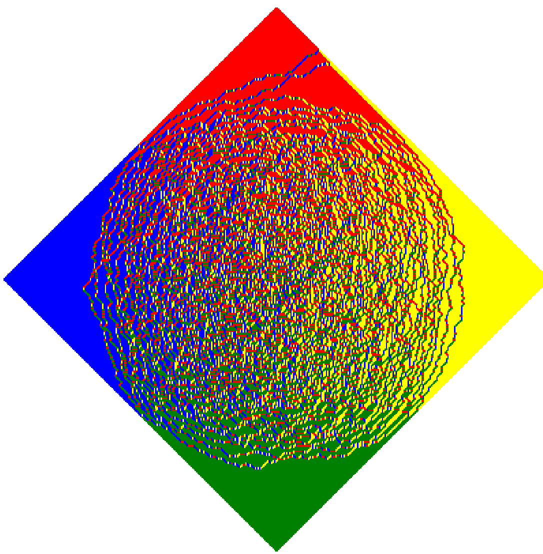
- For $\alpha < 1$, the Schur generating function of the distribution of $\lambda^{\lfloor \alpha N \rfloor}$ can be computed explicitly and it satisfies the infinitesimal free probability limit regime.

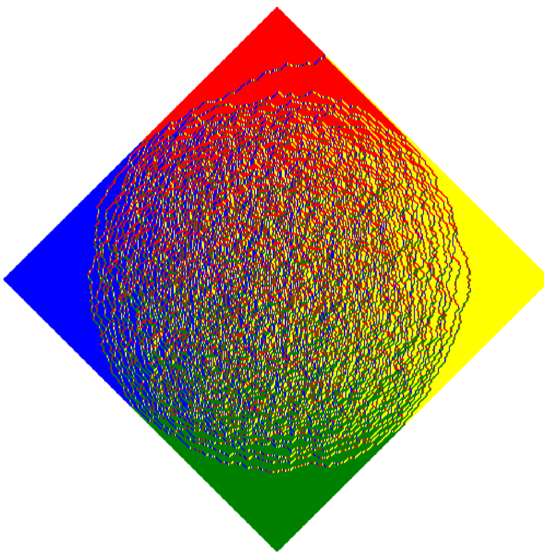
BBP for domino tilings of Aztec diamond

The $1/N$ correction to the limit of $\lambda^{\lfloor \alpha N \rfloor}$ is given by the signed measure

$$\mu'_{\alpha,\gamma}(dt) = -\mathbf{1}_{\alpha > \frac{(\theta+1)^2}{2\theta^2+2}} \delta\left(\frac{-1+2\alpha\theta-\theta}{\alpha\theta^2-\alpha}\right) - \frac{1}{2}\mathbf{1}_{\alpha = \frac{(\theta+1)^2}{2\theta^2+2}} \delta\left(\frac{-1+2\alpha\theta-\theta}{\alpha\theta^2-\alpha}\right)$$

$$+\left(\alpha + \frac{-2\alpha+1}{4t} - \frac{2\alpha^2-\alpha}{4\alpha t-4} + \frac{2\alpha^2\theta^2-\alpha\theta^2-2\alpha\theta+2\alpha^2-\alpha}{-2\alpha t+2\alpha\theta^2 t-4\alpha\theta+2+2\theta}\right)\frac{dt}{\pi\sqrt{1-(2\alpha-1)^2-(2\alpha t-1)^2}}.$$





Some references

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