Free Cumulants For Random Tensors

Răzvan Gurău (POAS, 2025)

joint work with Benoît Collins and Luca Lionni

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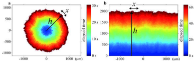
Random matrices

[Wishart '28, Wigner '55]

- Theory of strong interactions ['t Hooft, etc.]
- Random surfaces [David, Kazakov, Fröhlich, etc.]
- Growing interfaces fluctuations [Kardar, Parisi, Zhang, etc.]



- Birds perched on wires, parked cars
- ...
- **Free probability theory** [Voiculescu, Speicher, Guionnet, Collins etc.]

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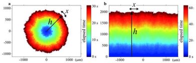

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How about tensors?

Random matrices (M_{ab}) generalize to random higher order tensors (T_{abc}) [Ambjørn Durhuus Jonsson '90, Sasakura '90, Boulatov '92, Ooguri '92, ...] and [2010: RG, Rivasseau, Oriti, Bonzom, Carrozza, Benedetti, Lionni, Tanasa, Ben Geloun, Ramgoolam, Dartois, Sasakura...]

- $1/N$ expansion (like random matrices)
- new large N limit (different from random matrices)
- large N field theory [Witten, Klebanov, etc.], spin glasses, [Zdeborová, Ros, etc.], tensor PCA [Ben Arous, etc.] ...

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Lately – increased efforts to generalize "freeness" to tensors

- Identify the right objects at finite N, take the limit $N \to \infty$ and find their intrinsic defining properties
- Self contained formulation of the limit theory, without going through finite N first

This talk

No freeness for tensors (almost)!

But I will introduce the building blocks:

- free cumulants: the right objects that describe the limit regime
- asymptotic moments / free cumulants relations

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Strategy – mimic what works for matrices, start at finite N and take the limit

Large / small scale

Small scales – quantum field theory (QFT):

- Feynman path integral:

$$Z = \int [d\phi][d\psi]\; e^{-S(\phi,\psi)}$$

- $S(\phi, \psi)$ matter action (field content: leptons, quarks, gauge bosons, Higgs boson).

Sum random configurations of dynamical fields.

“So God does play dice with the universe. All the evidence points to him being an inveterate gambler, who throws the dice on every possible occasion.” Stephen Hawking

Large scales – general relativity (GR):

- Einstein–Hilbert action:

$$S = \frac{c^4}{16\pi G} \int \sqrt{g}(-R + 2\Lambda) + S_m$$

- S_m matter action (matter content: visible, dark, dark energy).

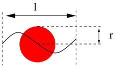
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Geometry is dynamical.

“Spacetime tells matter how to move; matter tells spacetime how to curve.” John Archibald Wheeler

The Planck scale

Photon of frequency ν



QFT:

- quantum behavior at the scale of the wave length $\lambda = \frac{c}{\nu}$

GR:

- energy $E = h\nu$, equivalent mass $M = \frac{h\nu}{c^2}$
- GR effects at the scale of the Schwarzschild radius of a black hole of mass M, $r_S = \frac{GM}{c^2}$

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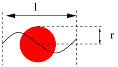

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same order of magnitude at the Planck scale

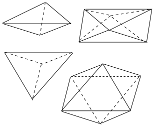
$$\ell_{Planck} = \sqrt{r_S \cdot \lambda} = \sqrt{\frac{Gh}{c^3}} \sim 10^{-35}\mathrm{m}$$

$$Z \sim \underbrace{\sum_{\text{topologies}} \int \mathcal{D}g_{(\text{metrics})}\, \mathcal{D}X_{\text{matter}}\ \exp}_{\text{QFT}} \left\{ \underbrace{-\frac{c^4}{16\pi G} \int \sqrt{g}(-R + 2\Lambda) - S_m}_{\text{GR}} \right\}$$

Random discrete geometries

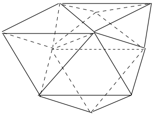
QFT + GR = summing random geometries

Build the geometry by gluing discrete blocks, “space time quanta”.



$$\sum_{\text{topologies}} \int \mathcal{D}g_{(\text{metrics})} \rightarrow \sum_{\text{random discretizations}}$$

Fundamental interactions of few “quanta” lead to effective behavior of an ensemble of “quanta”.



Random matrices and tensors for random discrete spaces

Random Matrices

Partition function of an invariant probability measure for a $N \times N$ matrix $X_{a^1 a^2}$:

$$Z = \int [dX d\bar{X}] \; e^{-NS(X,\bar{X})}$$

Free energy is a sum over two dimensional triangulations

$$\frac{\ln Z}{N^2} = \sum_{\text{triangulations}} N^{-2g} \; ,$$

genus $g \geq 0$.

$g = 0$, planar (spherical topology).

Random tensors

Partition function for an invariant probability measure for a $N \times \ldots N$ tensor $T_{a^1 \ldots a^D}$

$$Z = \int [dT d\bar{T}] \; e^{-N^{D-1} S(T,\bar{T})}$$

Free energy is a sum over D dimensional triangulations

$$\frac{\ln Z}{N^D} = \sum_{\text{triangulations}} N^{-\omega} \; ,$$

degree $\omega \geq 0$.

$\omega = 0$, *melonic* (spherical topology)

Random matrices ⇔ invariant probability measures for N^2 numbers:

$$Z = \int [dH] \;\; e^{-N \,\mathrm{Tr}[w(H)]} \;, \qquad H \to UHU^\dagger \quad U \in \mathcal{U}(N)$$

$$Z = \int [dX d\bar{X}] \;\; e^{-N \,\mathrm{Tr}[w(XX^\dagger)]} \;, \qquad X \to UXV \;, \;\; X^\dagger \to V^\dagger X^\dagger U^\dagger \quad U, V \in \mathcal{U}(N) \;,$$

Random matrices

Random matrices $\Leftrightarrow$ invariant probability measures for N^2 numbers:

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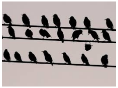
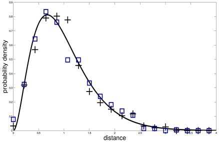
So where are the birds?

$$w(H) = H^2/2 \xrightarrow{\text{eigenvalues}} Z = \int \left(\prod_i d\lambda_i\right) \underbrace{\left(\prod_{i<j} |\lambda_i - \lambda_j|^2\right)}_{\text{eigenvalue repulsion}} \quad \underbrace{e^{-\frac{N}{2}\sum_i \lambda_i^2}}_{\text{confining potential}}$$





Gap distribution at large N fits the spacing between perched birds (squares) and parked cars (crosses) [Šeba 2013]

Basic building block $\rightarrow$ complex tensor T

N^D complex numbers $(T_{a^1,\dots a^D}\ ,\ \bar{T}_{a^1,\dots a^D})$, $a^c = 1, \dots N$

Probability measure for $(T_{a^1,\dots a^D}\ ,\ \bar{T}_{a^1,\dots a^D})$ with expectations invariant under local unitary transformations

$$T'_{b^1\dots b^D} = \sum_a U^{(1)}_{b^1a^1} \dots U^{(D)}_{b^Da^D} T_{a^1\dots a^D}\ , \quad \bar{T}'_{p^1\dots p^D} = \sum_q \bar{U}^{(1)}_{p^1q^1} \dots \bar{U}^{(D)}_{p^Dq^D} \bar{T}_{q^1\dots q^D}$$

$$\mathbb{E}[f(T,\bar{T})] = \mathbb{E}[f(T',\bar{T}')]$$

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Invariant “traces” $\sum_{a^1,q^1} \delta_{a^1q^1} \dots T_{a^1\dots a^D} \bar{T}_{q^1\dots q^D} \dots \to$ colored graphs

Tensor invariants, colored graphs and permutations

$$T'_{b^1\dots b^D} = \sum_a U^{(1)}_{b^1 a^1} \dots U^{(D)}_{b^D a^D} T_{a^1\dots a^D} \ , \quad \bar{T}'_{p^1\dots p^D} = \sum_q \bar{U}^{(1)}_{p^1 q^1} \dots \bar{U}^{(D)}_{p^D q^D} \bar{T}_{q^1\dots q^D}$$

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$$D = 3 \ , \quad \sum \delta_{a^1 p^1} \delta_{a^2 q^2} \delta_{a^3 r^3} \quad \delta_{b^1 r^1} \delta_{b^2 p^2} \delta_{b^3 q^3} \quad \delta_{c^1 q^1} \delta_{c^2 r^2} \delta_{c^3 p^3}$$
$$T_{a^1 a^2 a^3} T_{b^1 b^2 b^3} T_{c^1 c^2 c^3} \bar{T}_{p^1 p^2 p^3} \bar{T}_{q^1 q^2 q^3} \bar{T}_{r^1 r^2 r^3}$$

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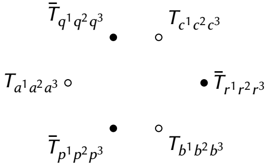

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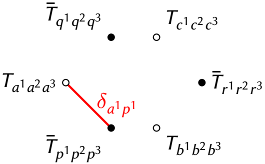

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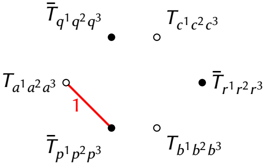

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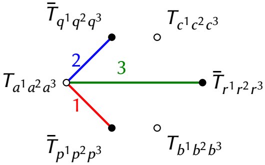

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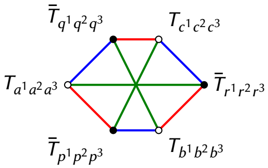

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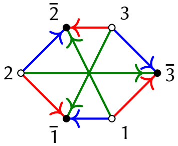
Invariant "traces" $\sum_{a^1,q^1} \delta_{a^1q^1} \dots T_{a^1\dots a^D} \bar T_{q^1\dots q^D} \dots \to$ colored graphs $\to$ D-tuples of permutations

$$\mathrm{Tr}_{\boldsymbol{\sigma}}(T,\bar T) = \sum_{i,j} \Big(\prod_{s=1}^n T_{i^1_s\dots i^D_s} \bar T_{j^1_{\bar s}\dots j^D_{\bar s}} \Big) \prod_{s=1}^n \prod_{c=1}^D \delta_{i^c_s j^c_{\overline{\sigma^c(s)}}}$$

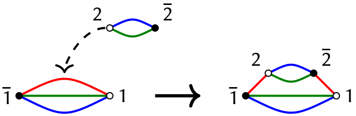
White (black) vertices for T ($\bar T$).

edges of color $c \to$ pairing $\{s, \overline{\sigma^c(s)}\}$
associated to the permutation σ^c



$$\sigma^{1;\mathrm{red}} = (132) \qquad \{1,\bar 3\}\{3,\bar 2\}\{2,\bar 1\}$$
$$\sigma^{2;\mathrm{blue}} = (1)(2)(3) \qquad \{1,\bar 1\}\{2,\bar 2\}\{3,\bar 3\}$$
$$\sigma^{3;\mathrm{green}} = (123) \qquad \{1,\bar 2\}\{2,\bar 3\}\{3,\bar 1\}$$

Examples of graphs: the melons

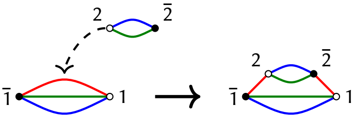


$$\begin{cases} \sigma^1 & = (1) \\ \sigma^2 & = (1) \\ \sigma^3 & = (1) \end{cases} \to \begin{cases} \sigma^1 & = (1\,2) \\ \sigma^2 & = (1)(2) \\ \sigma^3 & = (1)(2) \end{cases}$$

Well labelled melon at step n:

- insert n in a cycle in one σ^c
- append fixed point (n) to $\sigma^{c', c' \neq c}$

Examples of graphs: the melons



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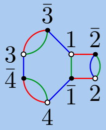
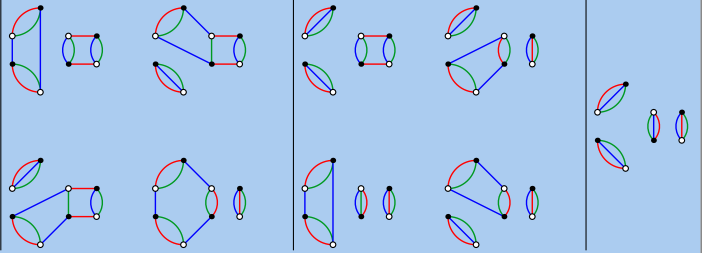
$\sigma^1 = (12)(3)(4)$

$\sigma^2 = (134)(2)$

$\sigma^3 = (1)(2)(3)(4)$





Lemma (Ben Geloun, Ramgoolam; Collins, RG, Lionni)

Denote $\boldsymbol{\sigma} = (\sigma^1, \dots \sigma^D), \sigma^c \in S(n)$. *For* $N > (n!)^{D-2}$, *the family:*

$$Tr_{\boldsymbol{\sigma}}(T, \bar{T}) = \sum_{i,j} \left(\prod_{s=1}^{n} T_{i_s^1 \dots i_s^D} \bar{T}_{j_{\bar{s}}^1 \dots j_{\bar{s}}^D} \right) \prod_{s=1}^{n} \prod_{c=1}^{D} \delta_{i_s^c j^c_{\overline{\sigma^c(s)}}} \ ,$$

up to relabeling $\boldsymbol{\sigma} \to \eta \boldsymbol{\sigma} \nu$ *is a basis in the space of homogeneous invariant polynomials of degree* n *in* T *and* $\bar{T}$.

Basis and decomposition

Lemma (Ben Geloun, Ramgoolam; Collins, RG, Lionni)

Denote $\boldsymbol{\sigma} = (\sigma^1, \dots \sigma^D), \sigma^c \in S(n)$. For $N > (n!)^{D-2}$, the family:

$$Tr_{\boldsymbol{\sigma}}(T, \bar{T}) = \sum_{i,j} \left(\prod_{s=1}^{n} T_{i_s^1 \dots i_s^D} \bar{T}_{j_{\bar{s}}^1 \dots j_{\bar{s}}^D} \right) \prod_{s=1}^{n} \prod_{c=1}^{D} \delta_{i_s^c j^c_{\overline{\sigma^c(s)}}} \,,$$

up to relabeling $\boldsymbol{\sigma} \to \eta \boldsymbol{\sigma} \nu$ is a basis in the space of homogeneous invariant polynomials of degree n in T and $\bar{T}$.

Decomposition $\to$ averaging over $\mathbf{U} = U^{(1)} \otimes \dots U^{(D)}$:

$$f(T, \bar{T}) = \int d\mathbf{U} \, f(\mathbf{U}T, \bar{T}\mathbf{U}^\dagger)$$

$$\int dU \; U_{a_1 i_1} \dots U_{a_n i_n} \overline{U_{b_{\bar{1}} j_{\bar{1}}} \dots U_{b_{\bar{n}} j_{\bar{n}}}} = \sum_{\sigma, \tau \in S(n)} \prod_{s=1}^{n} \delta_{a_s b_{\overline{\tau(s)}}} \, \delta_{i_s j_{\overline{\sigma(s)}}} \; \underbrace{W(\sigma\tau^{-1})}_{\text{Weingarten functions}}$$

Expectations and connected expectations

Random variables $x_1, \dots x_n, \dots$

$$\underbrace{\mathbb{E}[x_1, \dots x_n]}_{\text{expectation}} = \underbrace{\sum_{\pi \leq 1_n}}_{\text{Partitions of } n \text{ elements}} \prod_{B \in \pi} \underbrace{k[\{x_i, i \in B\}]}_{\text{connected expectation}}$$

Partitions are lattice for the refinement order $\rightarrow$ Möbius inversion

$$k[x_1, \dots x_n] = \sum_{\pi \leq 1_n} \underbrace{\lambda_\pi}_{\text{Möbius function } (-1)^{|\pi|-1}(|\pi|-1)!} \prod_{B \in \pi} \mathbb{E}[\{x_s, s \in B\}]$$

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Multiplicative extensions $\mathbb{E}_\pi = \prod_{B \in \pi} \mathbb{E}[B]$ and $k_\pi = \prod_{B \in \pi} k[B]$

$$\underbrace{\mathbb{E}_{1_n} = \sum_{0_n \leq \pi \leq 1_n} k_\pi \qquad k_{1_n} = \sum_{0_n \leq \pi \leq 1_n} \lambda_\pi \, \mathbb{E}_\pi}_{\text{moments cumulants relations in any lattice}}$$

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Main Message

We identify large N **asymptotic moments**, **free cumulants** and a **lattice** such that **asymptotic moments cumulants relations** for random tensors hold.

Wishart matrices

Gaussian i.i.d. matrix entries:

$$\mathbb{E}[f(X,\bar{X})] = \int [dXd\bar{X}]\ f(X,\bar{X})\ e^{-N(\sum_{a,b} X_{ab}\bar{X}_{ab})_{\nwarrow \mathrm{Tr}[XX^{\dagger}]}}\ ,$$

the only non zero connected expectation $k(X_{i^1 i^2}\bar{X}_{j^1 j^2}) = N^{-1}\delta_{i^1 j^1}\delta_{i^2 j^2}$

$$\mathbb{E}[X_{i^1_1 i^2_1} \dots X_{i^1_n i^2_n} \bar{X}_{j^1_{\bar{1}} j^2_{\bar{1}}} \dots \bar{X}_{j^1_{\bar{n}} j^2_{\bar{n}}}] = \sum_{\eta \in S(n)} \prod_{s=1}^{n} \frac{1}{N} \delta_{i^1_s j^1_{\overline{\eta(s)}}}\ \delta_{i^2_s j^2_{\overline{\eta(s)}}}$$

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But we are interested in other "connected" expectations:

$$\Phi[\mathrm{Tr}(XX^{\dagger})\mathrm{Tr}(XX^{\dagger})] = \mathbb{E}[\mathrm{Tr}(XX^{\dagger})\mathrm{Tr}(XX^{\dagger})] - \mathbb{E}[\mathrm{Tr}(XX^{\dagger})]\;\mathbb{E}[\mathrm{Tr}(XX^{\dagger})] = \text{``}\mathrm{Tr}(XX^{\dagger})\mathrm{Tr}(XX^{\dagger})\text{``} = 1$$

Φ_{σ} "classical cumulants" defined via moments cumulants relations in an appropriate lattice!

Classical cumulants $\rightarrow$ same definition as connected expectations, but respecting the connected components of $\boldsymbol{\sigma}$

$\Pi(\boldsymbol{\sigma})$ the partition of the vertices $\{1, \dots n, \bar{1}, \dots \bar{n}\}$ of $\boldsymbol{\sigma}$ in connected components $\boldsymbol{\sigma}_1, \dots \boldsymbol{\sigma}_q$:

$$\Phi_{\boldsymbol{\sigma}}[T, \bar{T}] = \sum_{\Pi(\boldsymbol{\sigma}) \leq \Pi \leq 1_{n,\bar{n}}} \lambda_\Pi \underbrace{\prod_{B \in \Pi} \mathbb{E}[\mathrm{Tr}_{\boldsymbol{\sigma}_{|B}}(T, \bar{T})]}_{\mathbb{E}_{\Pi,\boldsymbol{\sigma}}}$$

$$\Phi_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3} = \mathbb{E}[\mathrm{Tr}_{\boldsymbol{\sigma}_1}\mathrm{Tr}_{\boldsymbol{\sigma}_2}\mathrm{Tr}_{\boldsymbol{\sigma}_3}] - \mathbb{E}[\mathrm{Tr}_{\boldsymbol{\sigma}_1}\mathrm{Tr}_{\boldsymbol{\sigma}_2}]\mathbb{E}[\mathrm{Tr}_{\boldsymbol{\sigma}_3}] - \dots + 2\mathbb{E}[\mathrm{Tr}_{\boldsymbol{\sigma}_1}]\mathbb{E}[\mathrm{Tr}_{\boldsymbol{\sigma}_2}]\mathbb{E}[\mathrm{Tr}_{\boldsymbol{\sigma}_3}]$$

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$\Pi(\boldsymbol{\sigma}) \leq \Pi \leq 1_{n,\bar{n}}$ lattice interval in the lattice of partitions $\to$ Möebius inversion works with the same Möebius function λ_Π:

$$\mathbb{E}[\mathrm{Tr}_{\boldsymbol{\sigma}}(T, \bar{T})] = \sum_{\Pi(\boldsymbol{\sigma}) \leq \Pi \leq 1_{n,\bar{n}}} \underbrace{\prod_{B \in \Pi} \Phi_{\boldsymbol{\sigma}|_B}[T, \bar{T}]}_{\Phi_{\Pi,\boldsymbol{\sigma}}}$$

$$\mathbb{E}[\mathrm{Tr}_{\boldsymbol{\sigma}_1}\mathrm{Tr}_{\boldsymbol{\sigma}_2}\mathrm{Tr}_{\boldsymbol{\sigma}_3}] = \Phi_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3} + \Phi_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}\Phi_{\boldsymbol{\sigma}_3} + \Phi_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_3}\Phi_{\boldsymbol{\sigma}_2} + \Phi_{\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3}\Phi_{\boldsymbol{\sigma}_1} + \Phi_{\boldsymbol{\sigma}_1}\Phi_{\boldsymbol{\sigma}_2}\Phi_{\boldsymbol{\sigma}_3}$$

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Equivalently, in terms of partitions among the connected components $\boldsymbol{\sigma}_1 \dots \boldsymbol{\sigma}_q$:

$$\mathbb{E}[\mathrm{Tr}_{\boldsymbol{\sigma}_1} \dots \mathrm{Tr}_{\boldsymbol{\sigma}_q}] = \sum_{\pi \leq 1_q} \prod_{B \in \pi} \Phi_{\bigcup_{j \in B} \boldsymbol{\sigma}_j}$$

Finite N free cumulants

The generating function of connected expectations is invariant:

$$W(J,\bar{J}) = \ln \mathbb{E}[e^{N^{D/2}\sum_a (\bar{J}_{a^1\dots a^D}\bar{T}_{a^1\dots a^D} + J_{a^1\dots a^D}T_{a^1\dots a^D})}] = \sum_{\boldsymbol{\sigma}} \mathrm{Tr}_{\boldsymbol{\sigma}}(J,\bar{J})\, \mathcal{K}_{\boldsymbol{\sigma}}[T,\bar{T}]$$

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Theorem (Collins, RG, Lionni)

The finite N free cumulants write in terms of the classical cumulants as:

$$\mathcal{K}_{\boldsymbol{\sigma}}[T,\bar{T}] = N^{nD}\sum_{\boldsymbol{\tau}}\sum_{\Pi(\boldsymbol{\tau})\vee\Pi(\boldsymbol{\sigma})\le\Pi''}\lambda_{\Pi''}\underbrace{\Big(\sum_{\Pi(\boldsymbol{\tau})\le\Pi'\le\Pi''}\Phi_{\Pi',\boldsymbol{\tau}}[T,\bar{T}]\Big)}_{\mathbb{E}_{\Pi'',\boldsymbol{\tau}}[T,\bar{T}]}\prod_{B\in\Pi''}\prod_{c=1}^{D}W\big(\sigma^{c}|_B(\tau^{c}|_B)^{-1}\big)$$

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Proof $\rightarrow$ $\ln(e^x) = \ln(1+(e^x-1)) = \sum_{q\ge1} \frac{(-1)^q}{q}(e^x-1)^q \dots$ hence:

$$\ln \mathbb{E}[e^{f(T,\bar{T})}] = \sum_{n\ge1} \frac{1}{n!} \sum_{\pi\le 1_n} \lambda_\pi \prod_{B\in\pi} \mathbb{E}\Big[\int d\mathbf{U}\, f(\mathbf{U}T, \bar{T}\mathbf{U}^\dagger)^{|B|}\Big]$$

Large N factorization and scaling assumptions

Large N factorization $\mathbb{E}[\prod_j \mathrm{Tr}_{\boldsymbol{\sigma}_j}] \sim \prod_j \Phi_{\boldsymbol{\sigma}_j}$

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$$\lim_{N\to\infty} \frac{1}{N^{\underbrace{r(\boldsymbol{\sigma})}_{\nwarrow \text{choice}}}} \Phi_{\boldsymbol{\sigma}}[T,\bar{T}] \to \varphi_{\boldsymbol{\sigma}}(t,\bar{t})$$

- Expand rescaled expectations on classical cumulants

$$\frac{1}{N^{\sum_{i=1}^q r(\boldsymbol{\sigma}_i)}} \mathbb{E}[\mathrm{Tr}_{\boldsymbol{\sigma}_1} \dots \mathrm{Tr}_{\boldsymbol{\sigma}_q}] = \sum_{\pi \le 1_q} \frac{1}{N^{\sum_{i=1}^q r(\boldsymbol{\sigma}_i) - \sum_{B\in\pi} r(\bigcup_{j\in B} \boldsymbol{\sigma}_j)}} \prod_{B\in\pi} \frac{1}{N^{r(\bigcup_{j\in B}\boldsymbol{\sigma}_j)}} \Phi_{\bigcup_{j\in B}\boldsymbol{\sigma}_j}$$

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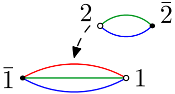
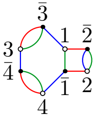
We choose the same scaling $r(\boldsymbol{\sigma})$ as in the Gaussian case[a] which we conjecture to be strictly sub additive, but we make no assumptions on $\varphi_{\boldsymbol{\sigma}}(t,\bar{t})$.

[a] $[d\bar{T}dT]\, \exp\{-N^{D-1}\sum T_{a^1\dots a^D}\bar{T}_{a^1\dots a^D}\}$, $\quad r(\boldsymbol{\sigma}) = n - \min_{\eta\in S_n, \Pi(\boldsymbol{\sigma})\vee\Pi(\eta)=1_{n,\bar{n}}} \sum_{c=1}^{D} |\sigma^c\eta^{-1}|$.

The melon strikes back



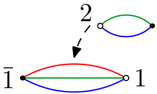


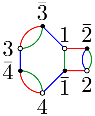
Theorem (RG)

We have $r(\boldsymbol{\sigma}) = D - (D-1)|\Pi(\boldsymbol{\sigma})| - \Omega(\boldsymbol{\sigma})$ *where* $|\Pi(\boldsymbol{\sigma})|$ *is the number of connected components of* $\boldsymbol{\sigma}$ *and* $\Omega(\boldsymbol{\sigma}) \geq 0$. *Furthermore,* $\Omega(\boldsymbol{\sigma}) = 0$ *if and only if* $\boldsymbol{\sigma}$ *is melonic.*[a]

The **leading invariants are connected, melonic** *and at fixed* $|\Pi(\boldsymbol{\sigma})|$ *the leading invariants are melonic.*

[a] $r(\boldsymbol{\sigma})$ is sub additive on melons.





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For $D = 2$ all the invariants are melonic (bi colored cycles)!

The flip partial order on the melons

A flip on a well labelled melon $\boldsymbol{\sigma}$ – split a cycle of one σ^c into two:

$$(i_1 \dots i_p i_{p+1} \dots i_q i_{q+1} \dots i_l) \to (i_1 \dots i_p i_{q+1} \dots i_l)(i_{p+1} \dots i_q)$$

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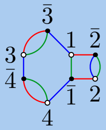
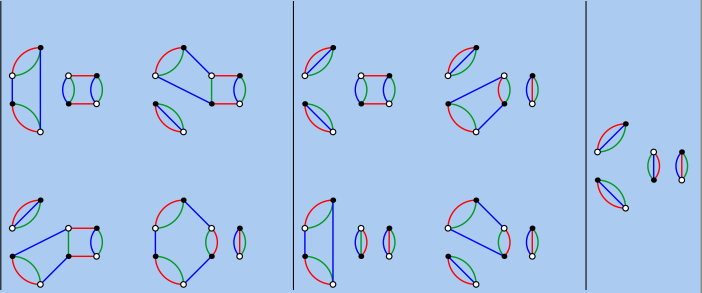
Cut two edges in a 2 cut, reconnect respecting coloring

$$\sigma^1 = (12)(3)(4)$$
$$\sigma^2 = (134)(2)$$
$$\sigma^3 = (1)(2)(3)(4)$$





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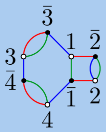

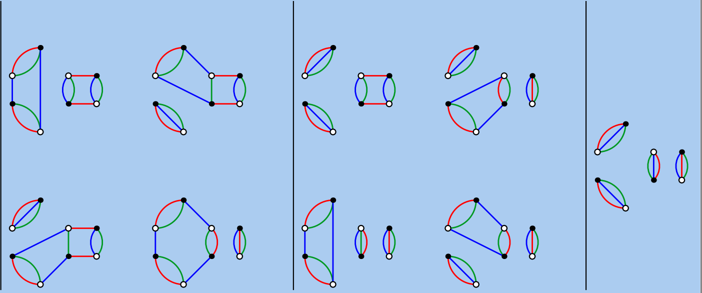

Flip partial order: $\boldsymbol{\tau} \preceq \boldsymbol{\sigma}$ if $\boldsymbol{\tau}$ can be obtained from $\boldsymbol{\sigma}$ by a sequence of flips ($\boldsymbol{\tau}$ is melonic)

$\boldsymbol{\tau} \preceq \boldsymbol{\sigma}$ if and only if τ^c is non-crossing on σ^c for all c:

- $\boldsymbol{\tau} \preceq \boldsymbol{\sigma}$ is isomorphic to a sub-lattice in the D-fold Cartesian product of lattices of non-crossing partitions hence has the same, known, Möebius function $\mathrm{M}(\boldsymbol{\sigma}\boldsymbol{\tau}^{-1})$

Melonic cumulants

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Theorem (Collins, RG, Lionni)

For **well labeled melonic, connected** *invariants $\boldsymbol{\sigma}$ in $D \geq 3$ the free cumulants are:*

$$\kappa_{\boldsymbol{\sigma}}(t,\bar{t}) = \lim_{N\to\infty} \frac{\mathcal{K}_{\boldsymbol{\sigma}}[T,\bar{T}]}{N^{r(\boldsymbol{\sigma})=1}} = \sum_{\boldsymbol{\tau}\preceq\boldsymbol{\sigma}} \mathrm{M}(\boldsymbol{\sigma}\boldsymbol{\tau}^{-1}) \prod_{i=1}^{|\Pi(\boldsymbol{\tau})|} \varphi_{\boldsymbol{\tau}_i}(t,\bar{t}) \quad , \quad \varphi_{\boldsymbol{\sigma}}(t,\bar{t}) = \sum_{\boldsymbol{\tau}\preceq\boldsymbol{\sigma}} \prod_{i=1}^{|\Pi(\boldsymbol{\tau})|} \kappa_{\boldsymbol{\tau}_i}(t,\bar{t}) \ ,$$

$\preceq$ *is* **the flip partial order** *on the set of melons and $\mathrm{M}(\boldsymbol{\sigma}\boldsymbol{\tau}^{-1})$ is its Möebius function.*

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Asymptotic moments $\varphi_{\boldsymbol{\sigma}}$ - free cumulants $\kappa_{\boldsymbol{\sigma}}$ relations.

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$\boldsymbol{\sigma}$ connected $\Rightarrow \Pi(\boldsymbol{\sigma}) = 1_{n,\bar{n}} \Rightarrow \Pi'' = 1_{n,\bar{n}}$, and $\lambda_{1_{n,\bar{n}}} = 1$

Melonic cumulants

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$W(\nu) \sim N^{-n-|\nu|} M(\nu)$, and $\Phi_{\Pi',\tau} \sim N^{\sum_{B'\in\Pi'} r(\boldsymbol{\tau}|_{B'})} \prod_{B'} \varphi_{\boldsymbol{\tau}|_{B'}} \Rightarrow \Pi' = \Pi(\boldsymbol{\tau}), \boldsymbol{\tau} \preceq \boldsymbol{\sigma}$

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$$M(\nu) = \underbrace{\prod_{p\geq 1} \left[(-1)^{p-1} \frac{1}{p}\underbrace{\binom{2p-2}{p-1}}_{\text{Catalan number}} \right]^{\overbrace{d_p(\nu)}^{\text{number of cycles of length } p \text{ of } \nu}}}_{\text{Möebius function for lattice of non-crossing partitions}} \ , \quad M(\boldsymbol{\nu}) = \prod_c M(\nu_c)$$

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To do list:

- prove strict sub additivity of $r(\boldsymbol{\sigma})$.
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  - non melonic connected
  - arbitrary
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We have barely scratched the surface of tensor freeness at large N!

What is an infinite random tensor?

Single trace measures

$$Z(t_{\boldsymbol{\sigma}}) = \int [d\bar{T} dT] \; e^{-N^{D-1} S(T,\bar{T})}$$

$S(T, \bar{T})$ is a "single trace" invariant:

$$S(T,\bar{T}) = \sum T_{b^1 \dots b^D} \bar{T}_{q^1 \dots q^D} \prod_{c=1}^{D} \delta_{b^c q^c} + \sum_{\substack{\text{connected graphs } \boldsymbol{\sigma} \\ \text{with D colors}}} t_{\boldsymbol{\sigma}} \mathrm{Tr}_{\boldsymbol{\sigma}}(T,\bar{T})$$

- invariant observables $\mathrm{Tr}_{\boldsymbol{\sigma}}(T, \bar{T})$ for all $\boldsymbol{\sigma}$
- compute $\ln Z$, $\langle \mathrm{Tr}_{\boldsymbol{\sigma}_1}(T, \bar{T}) \dots \mathrm{Tr}_{\boldsymbol{\sigma}_q}(T, \bar{T}) \rangle$

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Feynman expansion:

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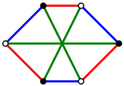
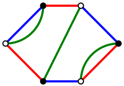
Feynman expansion:

- Taylor expand in $t_{\boldsymbol{\sigma}} \to$ graphs with D colors

$$Z(t_{\boldsymbol{\sigma}}) = \sum \int_{T,\bar{T}} e^{-N^{D-1}T\cdot\bar{T}} \mathrm{Tr}_{\boldsymbol{\sigma}_1}(T,\bar{T})\mathrm{Tr}_{\boldsymbol{\sigma}_2}(T,\bar{T})\dots$$

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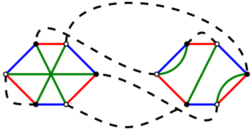

$$\ln Z(t_{\boldsymbol{\sigma}}) = \sum_{\substack{\text{connected graphs } \mathcal{G} \\ \text{with } D+1 \text{ colors}}} A(\mathcal{G})$$

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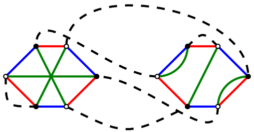
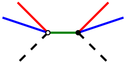


Each graph $\mathcal{G}$ is embedded in a D dimensional space
(Poincaré dual to a triangulation)

Colored graphs and vertex colored triangulations

White and black $D+1$ valent vertices connected by edges with colors $0, 1 \ldots D$.



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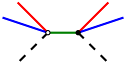
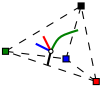
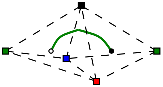


Vertex $\leftrightarrow$ D simplex with colored vertices .



Edges $\leftrightarrow$ gluings along $D-1$ simplices respecting all the colorings



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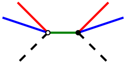

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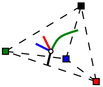

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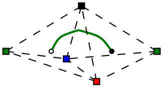

Invariants $\mathrm{Tr}_{\boldsymbol{\sigma}}$ encode boundary triangulations and expectations:

$$\Big\langle \mathrm{Tr}_{\boldsymbol{\sigma}_1} \mathrm{Tr}_{\boldsymbol{\sigma}_2} \ldots \mathrm{Tr}_{\boldsymbol{\sigma}_q} \Big\rangle = \frac{1}{Z(t_{\boldsymbol{\sigma}})} \int [d\bar{T} dT] \; \mathrm{Tr}_{\boldsymbol{\sigma}_1} \mathrm{Tr}_{\boldsymbol{\sigma}_2} \ldots \mathrm{Tr}_{\boldsymbol{\sigma}_q} \; e^{-N^{D-1} S(T,\bar{T})}$$

sums over all bulk triangulations compatible with the boundary.

Random tensors provide generating functions for random triangulations.

$$\Big\langle \mathrm{Tr}_{\boldsymbol{\sigma}_1} \mathrm{Tr}_{\boldsymbol{\sigma}_2} \ldots \mathrm{Tr}_{\boldsymbol{\sigma}_q} \Big\rangle = N^{\text{Leading Order}} \sum_{\text{triangulations}} N^{-\text{either 0 or suppression}}$$

The $1/N$ expansion provides a selection criterion for the dominant triangulations.

Partitions and permutations

Set partitions of $\{1,\dots n\}$ into blocks $\{1,2,3,4\}$: $\big\{\{1,2\},\{3\},\{4\}\big\}$, $\big\{\{1,2\},\{3,4\}\big\}$

- poset ordered by refinement $\big\{\{1,2\},\{3\},\{4\}\big\} \leq \big\{\{1,2\},\{3,4\}\big\}$, in fact lattice with global sup $1_n = \big\{\{1,\dots n\}\big\}$ and inf $0_n = \big\{\{1\},\{2\},\dots\{n\}\big\}$
- if $1 < 2 < \dots < n$, π is **non-crossing** if there exist **no** $i < j < k < l$ with $i, k \in B$ and $j, l \in B'$ e.g. $\big\{\{1,2\},\{3,4\}\big\}$ is non-crossing while $\big\{\{1,3\},\{2,4\}\big\}$ is crossing
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Permutations are bijections $\sigma : \{1, \ldots n\} \to \{1, \ldots n\}$

- decompose into cycles: $(1\,2)(3\,4)$, $(1\,3\,2)(4)$
- cycles of σ yield a partition $\pi(\sigma)$ of $\{1, \ldots n\}$, e.g. $\big\{\{1, 2\}, \{3, 4\}\big\}$, $\big\{\{1, 3, 2\}, \{4\}\big\}$
- poset $\tau \preceq \sigma$, τ **non-crossing** on σ if $\pi(\tau) \leq \pi(\sigma)$ and non-crossing and τ respects the orientation of σ

$$(135)(2)(4) \text{ non-crossing on } (12345)\ ; \qquad (135)(24)\,,\ (153)(2)(4) \text{ are not}$$