Rectangular cumulants in finite free probability and high temp. singular values

Cesar Cuenca

Ohio State University

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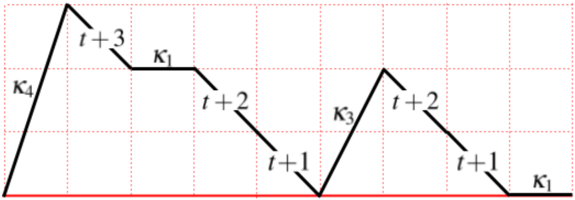

Plan of the talk

Free convolution and free cumulants

One parameter deformations: degree d polynomial convolutions VS random β-sum of matrices

Rectangular convolutions and cumulants

Outlook

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If $(a_1^{(N)}, \dots, a_N^{(N)}),\ (b_1^{(N)}, \dots, b_N^{(N)}) \in \mathbb{R}^N$ satisfy

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{a_i^{(N)}} \to \mu, \quad \frac{1}{N}\sum_{i=1}^{N} \delta_{b_i^{(N)}} \to \nu, \quad \text{as } N \to \infty,$$

then the real eigenvalues $(c_1^{(N)}, \dots, c_N^{(N)})$ of

$$\mathrm{diag}(a_1^{(N)}, \dots, a_N^{(N)}) + U^* \mathrm{diag}(b_1^{(N)}, \dots, b_N^{(N)}) U,$$

where $U \in U(N)$ is Haar-distributed, converge weakly a.s:

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{c_i^{(N)}} \to \kappa, \quad \text{as } N \to \infty,$$

for some $\kappa \in \mathcal{M}_1(\mathbb{R})$ [Voiculescu '91, '98].

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The free convolution is defined by $\boxed{\mu \boxplus \nu := \kappa}$

Free cumulants

For simplicity, assume that $\mu, \nu \in \mathcal{M}_1^c(\mathbb{R})$ have compact support.
$\Rightarrow \mu \boxplus \nu \in \mathcal{M}_1^c(\mathbb{R})$.
In particular, μ, ν, $\mu \boxplus \nu$ are uniquely determined by their moments.

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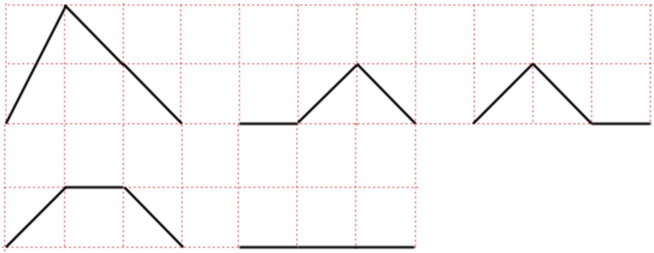
Definition (Voiculescu '85, Speicher '94)

For any moment sequence $(m_1, m_2, \dots)$, the free cumulants $(\kappa_1, \kappa_2, \dots)$ are uniquely defined by:

$$m_n = \sum_{\Gamma \in \mathbf{Luk}(n)} \prod_{\text{steps } (1,\ell) \text{ of } \Gamma} \kappa_{\ell+1}, \quad n = 1, 2, \cdots,$$

where the sum is over Lukasiewicz paths of length n, and $\kappa_0 := 1$.



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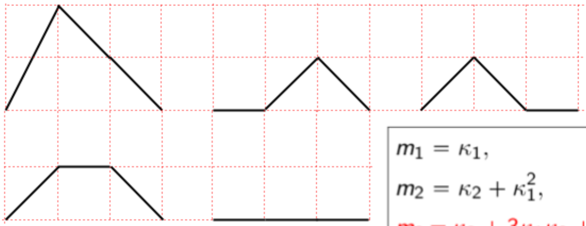

$m_1 = \kappa_1,$

$m_2 = \kappa_2 + \kappa_1^2,$

$m_3 = \kappa_3 + 3\kappa_1\kappa_2 + \kappa_1^3.$

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$$\kappa_\ell(\mu \boxplus \nu) = \kappa_\ell(\mu) + \kappa_\ell(\nu), \ \textit{for all } \ell \geqslant 1.$$

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Finite free convolution

Given fixed $d \times d$ complex Hermitian matrices A_d, B_d, let

$$p_d(x) := \det(xI - A_d), \quad q_d(x) := \det(xI - B_d).$$

Then the finite free convolution of $p_d(x)$ and $q_d(x)$ is

$$\big(p_d \boxplus_d q_d\big)(x) := \mathbb{E}_{U \in U(d)}\Big[xI - \big(A_d + U^* B_d U\big)\Big],$$

where $U \in U(d)$ is Haar-distributed.

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[Marcus–Spielman–Srivastava '22] studied $p_d \boxplus_d q_d$ arising from their studies on the Kadison–Singer problem (existence of Ramanujan graphs of every degree).

They found an explicit formula that showed the polynomial convolution $\boxplus_d$ had been studied by [Walsh 1922], who proved that if $p_d(x)$, $q_d(x)$ are real-rooted polys, then so is $\big(p_d \boxplus_d q_d\big)(x)$.

Finite free convolution

The operation $\boxplus_d$ gives birth to Finite Free Probability when regarding it as an operation between empirical measures:
If $p_d(x)$ a degree d polynomial with real roots $\alpha_1, \dots, \alpha_d$, then

$$\mu[p_d] := \frac{1}{d} \sum_{i=1}^{d} \delta_{\alpha_i}.$$

[MSS '22] proved that finite free convolution converges to free convolution, as $d \to \infty$:

$$\mu[p_d] \to \mu, \quad \mu[q_d] \to \nu \implies \mu[p_d \boxplus_d q_d] \to \mu \boxplus \nu.$$

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So, the operation $\boxplus_d$ and further developments on FFP can be regarded as 1-parameter deformations of FP, with deformation parameter $d \in \mathbb{N}$.

Mantra: FFP(d) $\xrightarrow{d \to \infty}$ FP.

Finite free cumulants

The combinatorial approach to FFP (akin to [Speicher '94] for FP) began with:

Definition (Arizmendi–Perales '18)

For the monic polynomial $p_d(x) = x^d + \sum_{j=1}^{d} a_j x^{d-j}$, define the finite free cumulants $K_1^d[p_d], K_2^d[p_d], \dots, K_d^d[p_d]$ as the first d numbers in the unique sequence that satisfies*

$$\exp\left(\sum_{\ell \geqslant 1} \frac{K_\ell^d[p_d]}{\ell} z^\ell\right) = 1 + \sum_{j=1}^{d} \frac{a_j}{(-d)_j} z^j,$$

where $(u)_j := (u)(u+1)\cdots(u+j-1)$.

Theorem (Arizmendi–Perales '18)

$$K_\ell^d[p_d \boxplus_d q_d] = K_\ell^d[p_d] + K_\ell^d[q_d], \quad \textit{for all } 1 \leqslant \ell \leqslant d.$$

* Their definition of $K_\ell^d[p_d]$ differs by a factor depending only on ℓ.

Finite free cumulants

To view $K^d_\ell[p_d]$ in terms of empirical measures (not of polys), let $m_1[p_d], m_2[p_d], \ldots$ be the moments of the empirical measures of the real-rooted polynomial $p_d(x) = x^d + \sum_{j=1}^{d} a_j x^{d-j}$, then

$$\exp\left(-d \sum_{n=1}^{\infty} \frac{m_n[p_d]}{n} z^n\right) = 1 + \sum_{j=1}^{d} a_j z^j.$$

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[AP '18] found moment/finite-free-cumulant formulas involving sums over pairs of set partitions $\pi, \sigma \in \mathcal{P}(n)$, and used them to prove that finite free cumulants tend to free cumulants:

$$\mu[p_d] \to \mu \implies K_\ell^d[p_d] \to \kappa_\ell[\mu], \quad \text{for all } \ell = 1, \ldots, d.$$

This gives an alternative proof that finite free convolution converges to free convolution [MSS '22].

Random β-sum of matrices

From a different point of view, we generalize the map

$$\mathrm{Spec}(A) \times \mathrm{Spec}(B) \to \mathrm{Spec}(A + UBU^*) \quad \big(U \in U(N) \text{ is Haar}\big)$$

with the deformation parameter $\beta > 0$ from Statistical Mechanics (inverse temperature). When $\beta = 1, 2, 4$, the desired map will recover this operation for groups $O(N), U(N), Sp(N)$, akin to similar constructions in Random Matrix Theory (beta ensembles).

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This involves the matrix Fourier transform (spherical integral)

$$\mathbb{E}_{U \in U(N)}\left[e^{\mathrm{Tr}(AUXU^*)}\right] =: B_N\big(\mathrm{Spec}\, A,\ \mathrm{Spec}\, X\big)$$

that satisfies

$$\mathbb{E}_{\vec{\mathbf{c}}}\left[B_N(\vec{\mathbf{c}}, \vec{\mathbf{x}})\right] = B_N(\vec{\mathbf{a}}, \vec{\mathbf{x}}) \cdot B_N(\vec{\mathbf{b}}, \vec{\mathbf{x}}), \quad \text{for all } \vec{\mathbf{x}} \in \mathbb{R}^N,$$

where $\vec{\mathbf{a}}, \vec{\mathbf{b}} \in \mathbb{R}^N$ are the spectra of A, B; and $\vec{\mathbf{c}}$ is the random spectra of $C = A + UBU^*$, for $U \in U(N)$ Haar-distributed.

Random β-sum of matrices

$\boldsymbol{\beta}$-Fourier transform based on Multivariate Bessel Function $B_N^{(\boldsymbol{\beta})}(\vec{\mathbf{a}}, \vec{\mathbf{x}})$:

- (Symmetry) $B_N^{(\boldsymbol{\beta})}(\vec{\mathbf{a}}, \vec{\mathbf{x}})$ is symmetric on $x_1, \dots, x_N$.
- (Normalization) $B_N^{(\boldsymbol{\beta})}(\vec{\mathbf{a}}, (\mathbf{0}, \dots, \mathbf{0})) = 1$.
- (Eigenrelations) $\mathcal{P}_k^{(\boldsymbol{\beta})} B_N^{(\boldsymbol{\beta})}(\vec{\mathbf{a}}, \vec{\mathbf{x}}) = \left(a_1^k + \cdots + a_N^k\right) \cdot B_N^{(\boldsymbol{\beta})}(\vec{\mathbf{a}}, \vec{\mathbf{x}})$,

for all $k \geqslant 1$, where:

$$\mathcal{P}_k^{(\boldsymbol{\beta})} := (\xi_1)^k + \cdots + (\xi_N)^k,$$

$$\xi_i := \frac{\partial}{\partial x_i} + \frac{\beta}{2} \sum_{j:\, j \neq i} \frac{1}{x_i - x_j}(1 - s_{i,j}) \quad \text{(Dunkl operators).}$$

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Example

If $\boldsymbol{\beta} = 0$*:* $\xi_i = \frac{\partial}{\partial x_i} \Rightarrow B_N^{(\boldsymbol{\beta}=0)}(\vec{\mathbf{a}}, \vec{\mathbf{x}}) = \frac{1}{N!} \sum_{\sigma \in S_N} e^{a_1 x_{\sigma(1)} + \cdots + a_N x_{\sigma(N)}}$.

If $\boldsymbol{\beta} = 2$*:* $B_N^{(\boldsymbol{\beta}=2)}(\vec{\mathbf{a}}, \vec{\mathbf{x}}) = \mathbb{E}_{U \in U(N)}\left[e^{Tr\left(diag(\vec{\mathbf{a}})\, U\, diag(\vec{\mathbf{x}})\, U^*\right)}\right]$.

Random β-sum of matrices

Definition (Gorin–Marcus '18; Benaych-Georges–C.–Gorin '22)

The β-sum of matrix spectra is the map

$$\mathbb{R}^N \times \mathbb{R}^N \xrightarrow{+_\beta} \mathcal{M}_1(\mathbb{R}^N)$$

that given $\vec{\mathbf{a}}, \vec{\mathbf{b}} \in \mathbb{R}^N$, produces the random tuple $\vec{\mathbf{c}} := \vec{\mathbf{a}} +_\beta \vec{\mathbf{b}}$ defined by

$$\mathbb{E}_{\vec{\mathbf{c}}}\left[B_N^{(\boldsymbol{\beta})}(\vec{\mathbf{c}}, \vec{\mathbf{x}})\right] = B_N^{(\boldsymbol{\beta})}(\vec{\mathbf{a}}, \vec{\mathbf{x}}) \cdot B_N^{(\boldsymbol{\beta})}(\vec{\mathbf{b}}, \vec{\mathbf{x}}), \text{ for all } \vec{\mathbf{x}} \in \mathbb{R}^N.$$

It is a conjecture that $\vec{\mathbf{c}}$ is a probability measure (it's related to the conjectural positivity of Jack-Littlewood-Richardson coefficients). However, the definition can be understood in terms of distributions (i.e. make the image of $+_{\boldsymbol{\beta}}$ be the space of distributions on $\mathbb{R}^N$).

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When $\beta > 0$ is fixed, the limit $N \to \infty$ gives the free convolution as when $\beta = 2$. But we have a new phenomenon when $\beta \to 0 \cdots$

Random β-sum of matrices and γ-convolution

Theorem (Benaych-Georges – C. – Gorin '22)

Assume the weak limits to compactly supported measures

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{a_i} \to \mu, \quad \frac{1}{N}\sum_{i=1}^{N}\delta_{b_i} \to \nu,$$

Let $\vec{\mathbf{c}} := \vec{\mathbf{a}} +_\beta \vec{\mathbf{b}}$. *In the high temperature regime*

$$N \to \infty, \quad \beta \to 0^+, \quad \frac{N\beta}{2} \to \gamma \in (0, \infty),$$

we have the weak convergence, in probability:

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{c_i} \to \mu \boxplus_\gamma \nu =: \gamma\text{-convolution of } \mu \text{ and } \nu.$$

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Moreover, γ-convolution $\xrightarrow{\gamma\to\infty}$ free convolution.

Mantra: HighTempBetaEnsembles(γ) $\xrightarrow{\gamma\to\infty}$ FP.

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For $\mu \in \mathcal{M}_1(\mathbb{R})$ with moments $m_1[\mu], m_2[\mu], \ldots$, define the γ-cumulants $\kappa_1^\gamma[\mu], \kappa_2^\gamma[\mu], \ldots$ by:

$$(*)\begin{cases} \exp\left(\sum_{\ell \geqslant 1} \frac{\kappa_\ell^\gamma[\mu]}{\ell} z^\ell\right) = 1 + \sum_{n \geqslant 1} \frac{a_n}{(\gamma)_n} z^n, \\ \exp\left(\gamma \sum_{k \geqslant 1} \frac{m_k[\mu]}{k} z^k\right) = 1 + \sum_{n \geqslant 1} a_n z^n. \end{cases}$$

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The amazing fact is that (*) exactly define the finite free cumulants upon the formal parameter identification $\gamma \leftrightarrow -d$.

So not only are FFP(d) and HTBE(γ) deformations of FP, but under $\boxed{\gamma \leftrightarrow -d}$, they are the same convolution/cumulants!

Moments/γ-cumulants formula

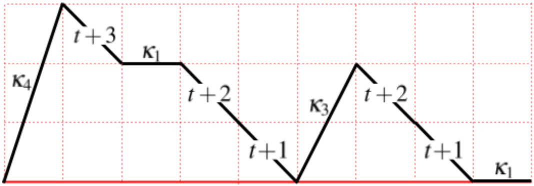
Theorem (C. '24; Benaych-Georges – C. – Gorin '22)

If $(m_1, m_2, \ldots)$ and $(\kappa_1, \kappa_2, \ldots)$ are related to each other via:

$$\begin{cases} \exp\left(\sum_{\ell \geqslant 1} \frac{\kappa_\ell}{\ell} z^\ell\right) = 1 + \sum_{n \geqslant 1} \frac{a_n}{(t)_n} z^n, \\ \exp\left(t \sum_{k \geqslant 1} \frac{m_k}{k} z^k\right) = 1 + \sum_{n \geqslant 1} a_n z^n, \end{cases}$$

then for all $n \geqslant 1$,

$$m_n = \sum_{\Gamma \in \mathbf{Luk}(n)} \prod_{\substack{\text{down-steps of } \Gamma \\ \text{from height } s \to s-1}} (t+s) \prod_{\text{steps } (1,\ell) \text{ of } \Gamma} \kappa_{\ell+1}.$$



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This formula (for $t = -d$) gives a direct proof to the result of [AP '18] that FFC $\to$ FC, as both are sums over Lukasiewicz paths.

Sketch of combinatorial proof in [C '24]

$$\begin{cases} \exp\left(\sum_{\ell\geqslant 1} \frac{\kappa_\ell}{\ell} z^\ell\right) = 1 + \sum_{n\geqslant 1} \frac{a_n}{(t)_n} z^n, \\ \exp\left(t\sum_{k\geqslant 1} \frac{m_k}{k} z^k\right) = 1 + \sum_{n\geqslant 1} a_n z^n. \end{cases}$$

The operator $\boxed{\partial_t(z^s) := \mathbf{1}_{\{s\geqslant 0\}} \cdot (t+s+1) z^{s-1}}$ gives

$$\partial_t \left[\frac{z^{n-1}}{(t)_n}\right] = \mathbf{1}_{n\geqslant 2} \cdot \frac{z^{n-2}}{(t)_{n-1}}.$$

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$$\partial_t\left[\frac{z^{n-1}}{(t)_n}\right] = \mathbf{1}_{n\geqslant 2} \cdot \frac{z^{n-2}}{(t)_{n-1}}.$$

Let $G(z) := \sum_{\ell\geqslant 1} \frac{\kappa_\ell}{\ell} z^\ell \implies \boxed{g(z) := G'(z) = \sum_{\ell\geqslant 1} \kappa_\ell z^{\ell-1}}$ then

$$g(z)e^{G(z)} = \sum_{n\geqslant 1} \frac{n a_n}{(t)_n} z^{n-1} \implies a_n = \frac{t}{n} \cdot [z^0]\partial_t^{n-1}\left(g(z)e^{G(z)}\right)$$

With some more work: $\boxed{m_n = [z^0](\partial_t + *_g)^n(1)}$

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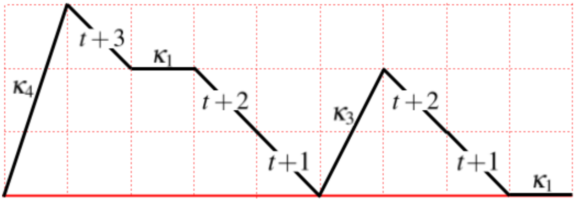

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$$\Longrightarrow m_n = \sum_{\Gamma\in\mathbf{Luk}(n)} \prod_{\substack{\text{down-steps of }\Gamma\\ \text{from height } s\to s-1}} (t+s) \prod_{\text{steps }(1,\ell)\text{ of }\Gamma} \kappa_{\ell+1}.$$

Remark

The operator ∂_t is in a sense the limit of Dunkl operators in the high-temp. regime, but the combinatorial proof is elementary, and leads to generalizations (later).

Plan of the talk

Free convolution and free cumulants

One parameter deformations: degree d polynomial convolutions VS random β-sum of matrices

Rectangular convolutions and cumulants

Outlook

Rectangular (free) q-convolution

The "rectangular" version of Voiculescu's theory was developed in [Benaych-Georges '09]. Now the matrices have size $M \times N$, $M \leqslant N$.

If $(a_1^{(M)}, \ldots, a_M^{(M)})$, $(b_1^{(M)}, \ldots, b_M^{(M)}) \in \left(\mathbb{R}_{\geqslant 0}\right)^M$ are the singular values of $A_{M,N}$, $B_{M,N}$, and

$$\frac{1}{2M} \sum_{i=1}^{M} \left(\delta_{a_i^{(M)}} + \delta_{-a_i^{(M)}}\right) \to \mu, \qquad \frac{1}{2M} \sum_{i=1}^{M} \left(\delta_{b_i^{(M)}} + \delta_{-b_i^{(M)}}\right) \to \nu,$$

$$\text{in the regime} \qquad N, M \to \infty, \qquad \frac{N}{M} \to q \in [1, \infty),$$

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then the singular values of $(c_1^{(M)}, \ldots, c_M^{(M)})$ of

$$C_{M,N} := A_{M,N} + U B_{M,N} V,$$

with Haar-distributed $U \in U(M)$, $V \in U(N)$, converge weakly a.s:

$$\frac{1}{2M}\sum_{i=1}^{M}\left(\delta_{c_i^{(M)}} + \delta_{-c_i^{(M)}}\right) \to \kappa, \quad \text{as } N, M \to \infty, \ \frac{N}{M} \to q,$$

for some $\kappa \in \mathcal{M}_1^{\text{sym}}(\mathbb{R})$, the rectangular q-convolution $\boxed{\mu \boxplus_q \nu := \kappa}$

Rectangular (free) q-cumulants

There are also rectangular versions of cumulants. Since measures are symmetric around zero, all odd moments vanish.

Assume all measures below are compactly supported, for simplicity.

Rectangular (free) q-cumulants

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Definition (Benaych-Georges '09)

For any even moment sequence $(m_2, m_4, \dots)$, the rectangular q-cumulants $(\kappa_2^q, \kappa_4^q, \dots)$ are uniquely determined by:

$$m_{2n} = \sum_{\Gamma \in \mathbf{Luk}^{\text{odd}}(2n)} q^{-\text{even}(\Gamma)} \prod_{\text{steps } (1, 2\ell - 1) \text{ of } \Gamma} \kappa_{2\ell}^q, \quad n = 1, 2, \cdots,$$

where the sum is over all odd Lukasiewicz paths of length $2n$ $\big[$any step $i \to j$ has $j - i \cong 1 \pmod 2\big]$, and even$(\Gamma)$ is the number of up-steps of Γ among the steps at positions $2, 4, \dots, 2n-2$.

Theorem (Benaych-Georges '09)

For all symmetric $\mu, \nu \in \mathcal{M}_1^c(\mathbb{R})$,

$$\kappa_{2\ell}^q[\mu \boxplus_q \nu] = \kappa_{2\ell}^q[\mu] + \kappa_{2\ell}^q[\nu], \qquad \textit{for all } \ell \geqslant 1.$$

β-sum of rectangular matrices

Q: Does the q-rectangular FP picture also admits one-parameter generalizations in the realms of FFP and HTBE?

β-sum of rectangular matrices

Q: Does the q-rectangular FP picture also admits one-parameter generalizations in the realms of FFP and HTBE?

For the point of view of HTBE, we need new *multivariate special functions*: for $M \leqslant N$, let

$$B^{(\beta)}_{M,N}(\vec{\mathbf{a}}_M, \vec{\mathbf{x}}_M) := \text{Multivariate Bessel Function of type BC.}$$

Definition (Jiaming Xu 2023)

The random M-tuple $\vec{\mathbf{c}}_M := \vec{\mathbf{a}}_M +_{\beta,N} \vec{\mathbf{b}}_M$ is defined by

$$\mathbb{E}_{\vec{\mathbf{c}}_M}\left[B^{(\beta)}_{M,N}(\vec{\mathbf{c}}_M, \vec{\mathbf{x}}_M)\right] = B^{(\beta)}_{M,N}(\vec{\mathbf{a}}_M, \vec{\mathbf{x}}_M) \cdot B^{(\beta)}_{M,N}(\vec{\mathbf{b}}_M, \vec{\mathbf{x}}_M), \ \forall\, \vec{\mathbf{x}}_M \in \mathbb{R}^M.$$

Again, the existence of $\vec{\mathbf{c}}_M$ as a probability measure is a conjecture, but it can be made sense as a distribution.

β-sum of rectangular matrices and (q, γ)-convolution

Theorem (Xu '23)

Assume the weak limits to compactly supported measures

$$\frac{1}{M}\sum_{i=1}^{M}\delta_{a_i} \to \mu, \quad \frac{1}{M}\sum_{i=1}^{M}\delta_{b_i} \to \nu.$$

Let $\vec{\mathbf{c}}_M := \vec{\mathbf{a}}_M +_{\beta,N} \vec{\mathbf{b}}_M$. *In the high temperature regime*

$$N \to \infty, \quad \beta \to 0^+, \quad \frac{M\beta}{2} \to \gamma, \quad \frac{N}{M} \to q \in [1, \infty),$$

we have the weak convergence, in probability:

$$\frac{1}{M}\sum_{i=1}^{M}\delta_{c_i} \to \mu \boxplus_{q,\gamma} \nu =: (q,\gamma)\text{-convolution of } \mu \text{ and } \nu.$$

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Further, [Xu '23] defined (q,γ)-cumulants and showed that (q,γ)-cumulants $\xrightarrow{\gamma\to\infty}$ rectangular q-cumulants of [BG '09].

Finite free rectangular convolution

On the side of FFP: given $d \times (n+d)$ matrices A_d, B_d, let

$$p(x),\ q(x) := \text{characteristic polynomials of } A_d A_d^*,\ B_d B_d^*,$$

the rectangular FF (n, d)-convolution [Gribinski–Marcus '22] is

$$\big(p \boxplus_d^n q\big)(x) := \mathbb{E}_{U,V}\Big[\text{char. poly. of } (A_d + UB_d V)(A_d + UB_d V)^*\Big]$$

where $U \in U(n)$, $V \in U(n+d)$ are Haar-distributed.

Theorem (GM '22)

If $p(x)$, $q(x)$ are degree d polynomials with all real nonnegative roots, then all roots of $\big(p \boxplus_d^n q\big)(x)$ are also real and nonnegative. Moreover, the rectangular FF (n, d)-convolution converges to rectangular q-convolution in the regime

$$n, d \to \infty, \qquad 1 + \frac{n}{d} \to q.$$

Finite free rectangular cumulants

Motivated by [Xu '23] $\big((q,\gamma)$-cumulants$\big)$, [Gribinski '24] (rectangular finite R-transform) and [AP '18]:

Definition (C. '24)

Given $p(x) = x^d + \sum_{j=1}^{d} a_{2j} x^{d-j}$, the rectangular (finite free) cumulants $K_2^{n,d}[p], K_4^{n,d}[p], \ldots, K_{2d}^{n,d}[p]$ are defined by:

$$\exp\left(\sum_{\ell \geqslant 1} \frac{K_{2\ell}^{n,d}[p]}{\ell} z^{2\ell}\right) = 1 + \sum_{j=1}^{d} \frac{a_{2j}}{(-d)_j(-d-n)_j} z^{2j}.$$

Theorem (C. '24)

For any degree d polynomials $p(x)$, $q(x)$, and $n \geqslant 0$,

$$K_{2\ell}^{n,d}\left[p \boxplus_d^n q\right] = K_{2\ell}^{n,d}[p] + K_{2\ell}^{n,d}[q], \ \textit{for all } \ell = 1, 2, \ldots, d.$$

Finite free rectangular cumulants/moments

The empirical measures $\mu\big[p(x^2)\big]$ have vanishing odd moments and the even moments denoted $M_2[p], M_4[p], \cdots$ satisfy:

$$\exp\left(-d\sum_{k\geqslant 1}\frac{M_{2k}[p]}{k}z^{2k}\right) = 1 + \sum_{j=1}^{d} a_{2j}z^{2j}.$$

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Theorem (C. '24)

$$M_{2n}[p] = \sum_{\Gamma\in \mathbf{Luk}^{odd}(2n)} \prod_{\substack{\text{down-steps of } \Gamma \\ \text{from height } 2s+1\to 2s}} (-d-n+s) \times \prod_{\substack{\text{down-steps of } \Gamma \\ \text{from height } 2s\to 2s-1}} (-d+s) \prod_{\text{steps } (1,2\ell-1) \text{ of } \Gamma} K_{2\ell}^{n,d}[p].$$

[C. '24] also gives as a corollary that FF (n,d)-convolution converges to rectangular q-convolution.

Asymptotic theory of polynomials

These results are useful in asymptotic theory of real-rooted polys, e.g. **Q:** Given $\{p_d(x)\}_{d\geqslant 1}$, find the weak limit of the empirical measures.

More precisely, **Q:** If $\mu[p_d] \xrightarrow{d\to\infty} \mu$, how do the roots of $p_d(x)$ evolve under repeated differentiation? (e.g. [Kabluchko '22] considered the backward heat flow $\exp\big(-\frac{s}{2}D^2\big)$, where $D := \frac{\partial}{\partial x}$).

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Theorem (C. '24)

Let $\{p_d(x)\}_{d\geqslant 1}$ be a sequence of polynomials with real nonnegative roots, and let $q > 1$, $n \in \mathbb{Z}_{\geqslant 0}$. Assume that $\mu[p_d(x^2)] \xrightarrow{d\to\infty} \mu \in \mathcal{M}_1^c(\mathbb{R})$.

If $q_d(x) := \exp\Big(-\dfrac{1}{n}x^{-n}Dx^{n+1}D\Big)p_d(x)$, then

$$\mu\big[q_d(x^2)\big] \to \mu \boxplus_q \lambda_q, \quad \text{as } n, d \to \infty,\ 1 + \frac{n}{d} \to q,$$

in the sense of moments, where λ_q is the q-rectangular analogue of the centered Gaussian distribution (its density is known).

Plan of the talk

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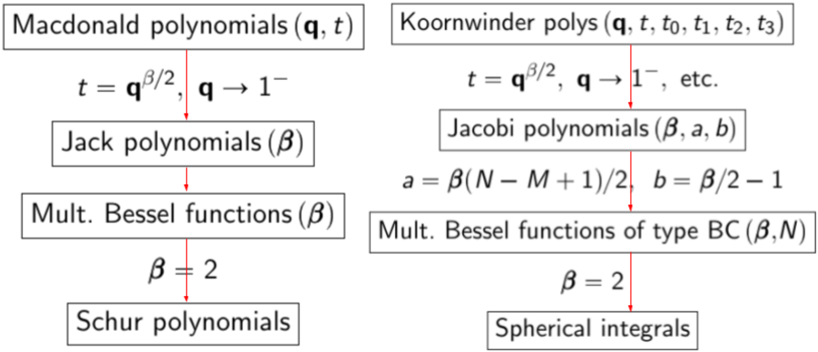
Outlook

Hierarchy of symmetric orthogonal polynomials

High temp. β-ensembles was understood by MBF of type A / BC:



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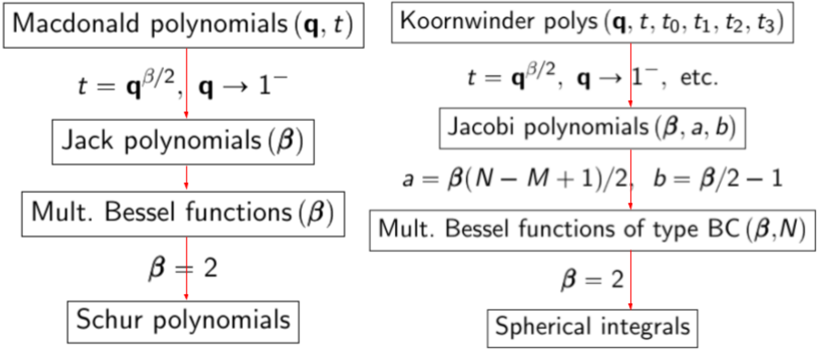

Q: Are there *high temp. β-ensembles* at higher hierarchy levels?

A1: **q**-analogue of formulas with Lukasiewicz paths in [C. '24].

A2: In recent/ongoing works [C.–Dolega–Moll '25], [C.–Dolega '25+], we treat discrete particle ensembles with the parameter $\beta > 0$, at the level of Jack polynomials.

Random partitions and Jack polynomials

There are two kinds of ensembles $\vec{\boldsymbol{\lambda}} = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N) \in \mathbb{Z}_{\geqslant 0}^N$: of fixed size $\lambda_1 + \lambda_2 + \ldots$, and of fixed length N.

E.g. for fixed length N, the natural "sum" (tensor product of representations) $\mathbb{Z}_{\geqslant 0}^N \times \mathbb{Z}_{\geqslant 0}^N \to \mathcal{M}\big(\mathbb{Z}_{\geqslant 0}^N\big)$, $\big(\vec{\boldsymbol{\mu}}, \vec{\boldsymbol{\nu}}\big) \mapsto \vec{\boldsymbol{\lambda}}$, is given by:

$$\boxed{\mathbb{E}_{\vec{\boldsymbol{\lambda}}}\Big[J_N^{(\beta)}\big(\vec{\boldsymbol{\lambda}}, \vec{\mathbf{x}}\big)\Big] = J_N^{(\beta)}\big(\vec{\boldsymbol{\mu}}, \vec{\mathbf{x}}\big) \cdot J_N^{(\beta)}\big(\vec{\boldsymbol{\nu}}, \vec{\mathbf{x}}\big), \quad \text{for all } \vec{\mathbf{x}} \in \mathbb{R}^N.}$$

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As a consequence, we discovered two new conjectural 1-parameter γ-convolutions of prob. measures and cumulants, e.g. if $m_1 = \frac{\gamma}{2}$:

$$m_n = \sum_{\Gamma \in \mathbf{Luk}(n)} \frac{\prod_{\text{horizontal steps in } \Gamma \text{ at height } s} (s+\gamma)}{1 + \#\,\text{horizontal steps at height 0 in } \Gamma} \times \prod_{\substack{\text{down-steps in } \Gamma \\ \text{from height } s \to s-1}} (s+\gamma) \prod_{\text{up-steps } (1,\ell) \text{ in } \Gamma} (\kappa_\ell + \kappa_{\ell+1}).$$

Open Q: Are there connections to polynomial convolutions?

Finite Free Probability ⇒ High temperature β-world?

Recall:

Theorem (Walsh 1922; Marcus–Spielman–Srivastava '22)

If $p(x)$, $q(x)$ are degree d real-rooted polynomials, then so is $p(x) \boxplus_d q(x)$. Similar result holds for $\boxplus_d^n$.

Q: Can similar ideas be used to prove:

Conj: If $\kappa_n^\gamma[\mu], \kappa_n^\gamma[\nu]$ are γ-cumulants of $\mu, \nu \in \mathcal{M}_1^c(\mathbb{R})$, then there exists $\tau \in \mathcal{M}_1^c(\mathbb{R})$ with γ-cumulants $\kappa_n^\gamma[\tau] = \kappa_n^\gamma[\mu] + \kappa_n^\gamma[\nu]$.

- Finite free probability:
  additive convolution/cumulants (parameter $d \in \mathbb{N}$)
  rect. convolution/cumulants (parameters $d \in \mathbb{N}$, $n \in \mathbb{Z}_{\geqslant 0}$)
- High temp. β-sums of eigenvalues/singular values:
  Eigenvalues $\Rightarrow$ γ-convolution ($\gamma > 0$)
  Sing. values $\Rightarrow$ (q, γ)-convolution ($\gamma > 0$, $q \in [1, \infty)$)
- In terms of formulas, there is a mysterious match between formulas, upon identification:

$$\boxed{\begin{array}{l} \gamma \leftrightarrow -d \\ q \leftrightarrow 1 + \dfrac{n}{d} \end{array}}$$

Thank you for your attention!