Entry-wise application of non-linear functions on orthogonally invariant random matrices

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Question

How does this change the eigenvalue distribution, asymptotically, if $N \to \infty$?

What assumptions should we make about our random matrices X_N

previous work

- *kernel matrices:*
  El Karou 2010; Cheng and Singer 2013; Fan and Montanari 2019
- *covariance matrices:*
  Pennington and Worah 2017; Louart, Liao and Couillet 2018; Benigni and Péché 2021; Piccolo and Schröder 2021

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Our general setting

orthogonally invariant random matrices

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Typical examples (say, symmetric real random matrices)

- given by density of the form

$$\exp\left(-N\mathsf{Tr}(q(X_N))\right) dX_N$$

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$$p(A_N, B_N, \dots, D_N)$$

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Properties of this class of random matrices

- orthogonal invariance: entries of X_N have the same joint distribution as entries of OX_NO^T for any orthogonal O
- existence of scaled limit of correlations (c_n are classical cumulants)

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  has leading order N^{-3}, and a subleading contribution of order N^{-4}

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Note that there is some freedom in arranging the entries of the cumulants

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- in particular: are there precise statements in the literature for the orthogonal case for the case of several cycles?

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Relation between free and classical cumulants

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Then
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$$\varphi(X^n) = \sum_{\pi\in NC(n)} \kappa_\pi;$$

and the κ_n are the free cumulants of the asymptotic eigenvalue distribution of X_N.

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How does this change the eigenvalue distribution, asymptotically, if $N \to \infty$?

Theorem

1. *The only cumulants which make asymptotically a contribution in the calculation of the moments of* $\mathbb{E}[\mathrm{tr}(Y_N^n)]$ *are those with a cyclic index structure,*

$$\kappa_n^f := \lim_{N\to\infty} N^{n-1} c_n(y_{i_1 i_2}, y_{i_2 i_3}, \dots, y_{i_n i_1}) \qquad \text{all } i_k \text{ distinct.}$$

   Those κ_n^f *are thus the free cumulants of the asymptotic eigenvalue distribution of the random matrix* Y_N*.*

2. *We have the following relation between the free cumulants* κ_n *of the asymptotic eigenvalue distribution of* X_N *and the free cumulants* κ_n^f *of the asymptotic eigenvalue distribution of the random matrix* Y_N*:*
   1. $\kappa_1^f = 0$;
   2. $\kappa_2^f = c_2(f(g), f(g)) = \mathbb{E}[f(g)^2] - \mathbb{E}[f(g)]^2$;
   3. *for* $n \geq 3$,

$$\kappa_n^f = \kappa_n \cdot \mathbb{E}[f'(g)]^n;$$

   where g *is a Gaussian random variable with mean zero and variance* κ_2*.*

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Consider $f(x) = x^3$. Then

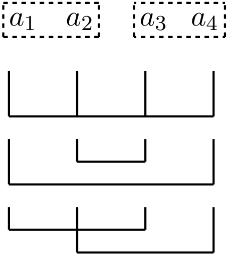
$$c_4(y_{12}, y_{23}, y_{34}, y_{41}) = N^4 c_4(x_{12}x_{12}x_{12}, x_{23}x_{23}x_{23}, x_{34}x_{34}x_{34}, x_{41}x_{41}x_{41})$$

Theorem (Formula of Leonov and Shiryaev)

$$c_m(a_1 \cdots a_{i(1)}, \dots, a_{i(m-1)+1} \cdots a_{i(m)}) = \sum_{\substack{\pi \in P(n) \\ \pi \vee \tau = 1_n}} c_\pi[a_1, \dots, a_n]$$

where

$$\tau = \{(1, \dots, i(1)), \dots, (i(m-1)+1, \dots, i(m))\} \in \mathcal{P}(n).$$



assume $c_1(a_i) = 0$; then

$$c_2(a_1a_2, a_3a_4) = c_4(a_1, a_2, a_3, a_4) + c_2(a_1, a_4)c_2(a_2, a_3) + c_2(a_1, a_3)c_2(a_2, a_4)$$

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Those κ_n^f are thus the free cumulants of the asymptotic eigenvalue distribution of the random matrix Y_N.

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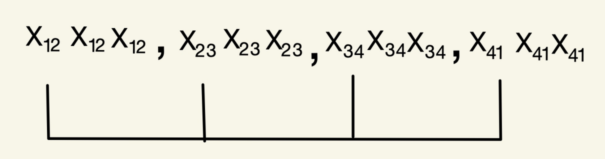

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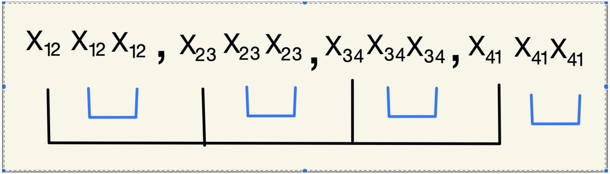

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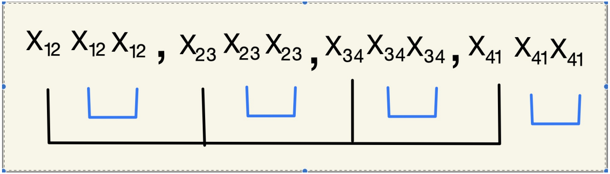

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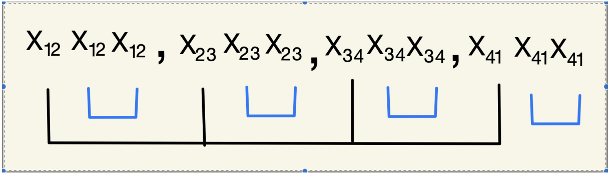

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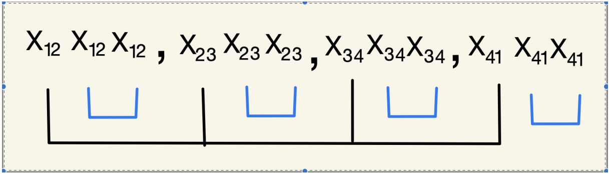

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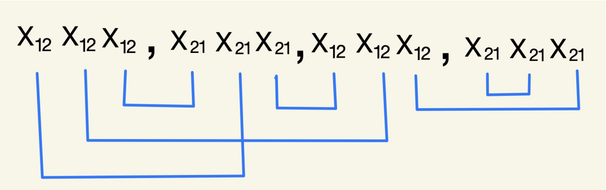

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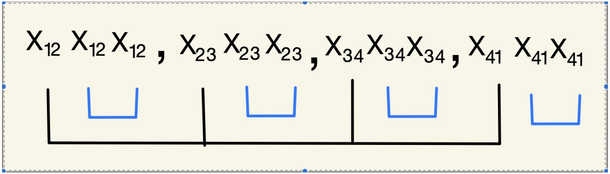

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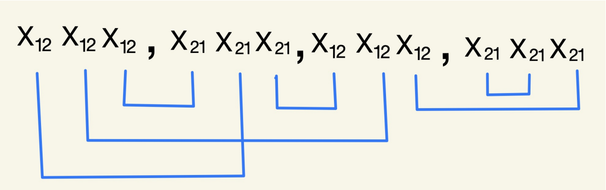

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But this is still small enough to not contribute to $\mathbb{E}[\mathrm{tr}(Y_N^n)]$.

- The only cumulants which make asymptotically a contribution in the calculation of the moments of $\mathbb{E}[\mathrm{tr}(Y_N^n)]$ are those with a cyclic index structure,

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- The κ_n^f are given by
  - for $n = 1$
$$\kappa_1^f = 0$$
  - for $n = 2$
$$\kappa_2^f = c_2(f(g), f(g))$$
  - for $n \geq 3$,
$$\kappa_n^f = \kappa_n \cdot \mathbb{E}[f'(g)]^n;$$

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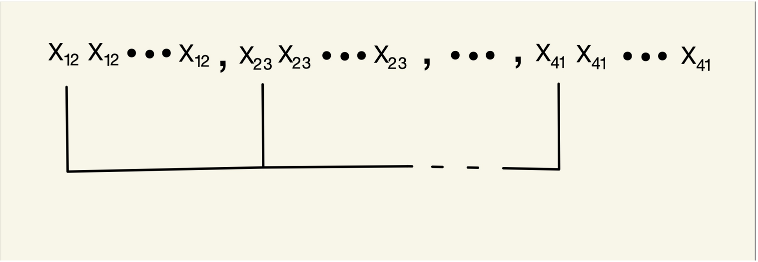

κ_4 m_1 m_2 $\ldots$ m_4

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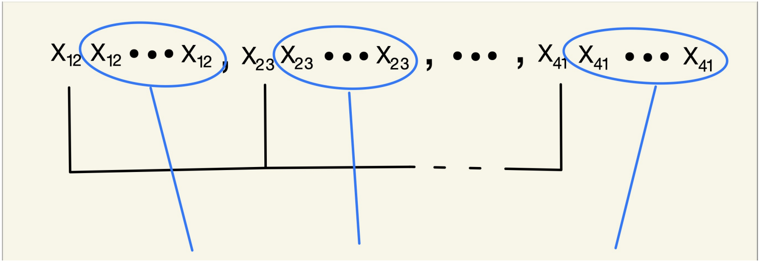

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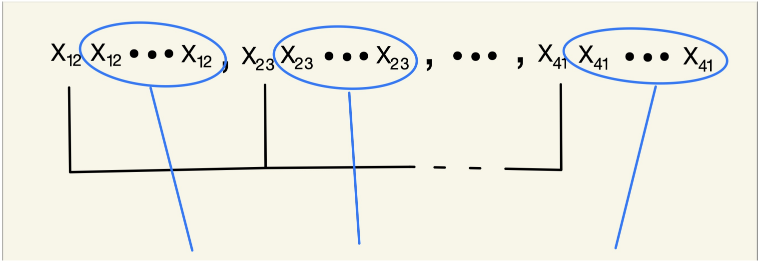

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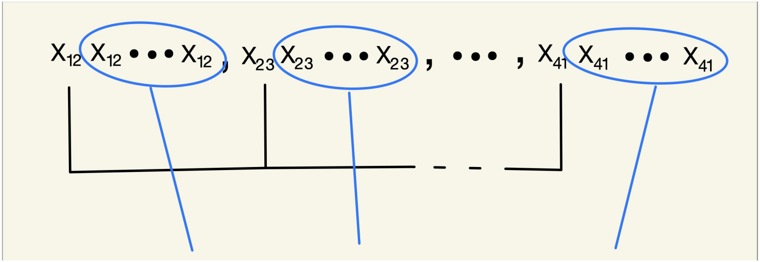

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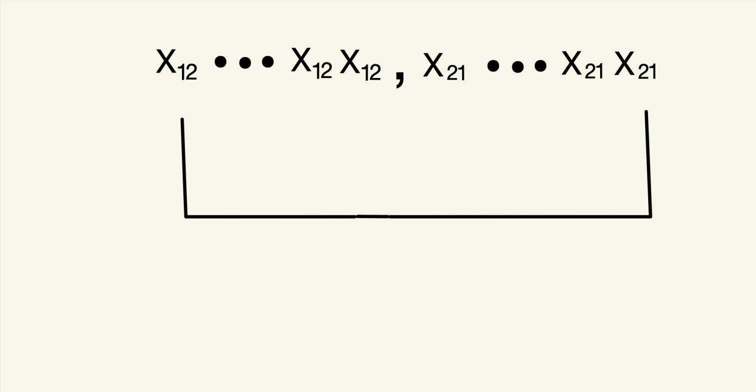

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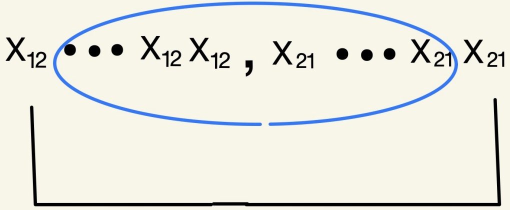

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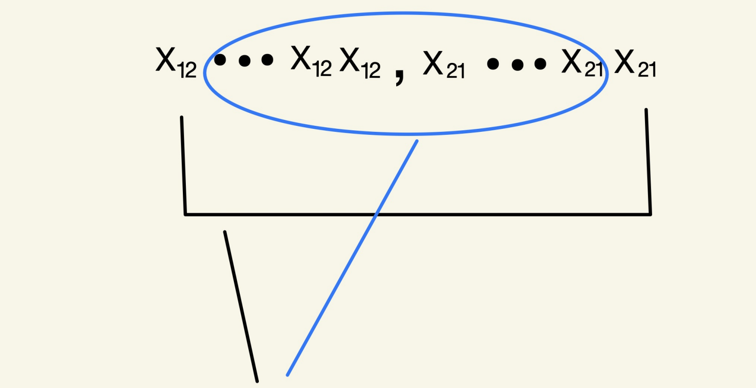

$c_2(f_1(g), f_2(g))$

where g is a Gaussian random variable with mean zero and variance κ_2.

Gaussian equivalence principle

Corollary

Let X_N and Y_N be as above. Then the non-linear random matrix model Y_N has the same asymptotic eigenvalue distribution as the linear model

$$\hat{Y}_N := \theta_1 X_N + \theta_2 Z_N,$$

where Z_N is a symmetric standard GOE random matrix which is independent from X_N and where

$$\theta_1 := \mathbb{E}[f'(g)]$$

and

$$\theta_2 : = \sqrt{\mathbb{E}[f(g)^2] - \mathbb{E}[f(g)]^2 - \kappa_2 \mathbb{E}[f'(g)]^2},$$

where g is a Gaussian random variable with mean zero and variance κ_2.

Example: The ReLU function

The function

$$f(x) = \mathsf{ReLU}(x) := \max(0, x).$$

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$$X_N = A_N^2 + A_N + B_N A_N + A_N B_N + B_N,$$

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We want to compare

non-linear random matrix $\qquad Y_N = \mathsf{ReLU}(X_N)$

versus

linear Gaussian equivalent $\qquad \hat{Y}_N := \frac{1}{2} X_N + \frac{1}{2}\sqrt{5(1 - 2/\pi)} Z_N$

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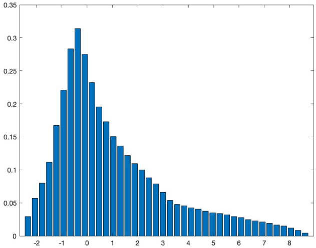

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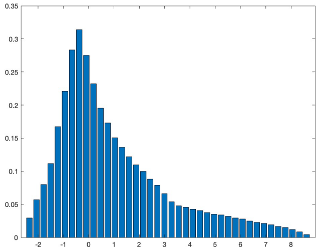
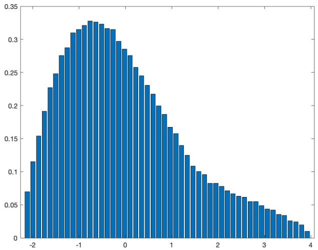


Y_N, after applying ReLU entrywise to X_N



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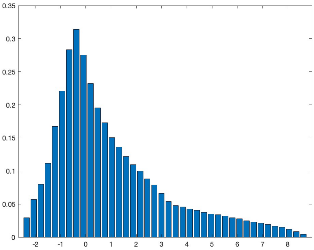

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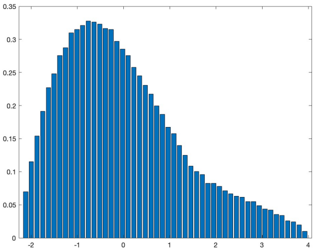
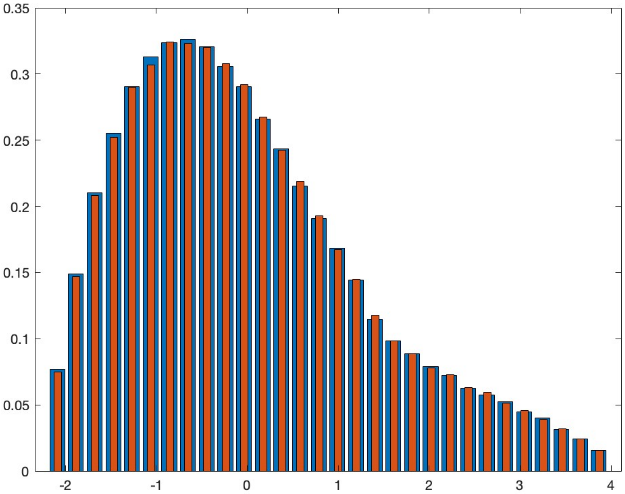


Gaussian equivalent $\hat{Y}_N := \frac{1}{2} X_N + \frac{1}{2}\sqrt{5(1-2/\pi)}\, Z_N$

Superposition of the eigenvalues of the non-linear matrix Y_N and of its Gaussian equivalent $\hat{Y}_N$



Extension to multivariate case

Consider 2 independent matrices

$$X_N^{(1)} = (x_{ij}^{(1)})_{i,j=1}^N \qquad \text{and} \qquad X_N^{(2)} = (x_{ij}^{(2)})_{i,j=1}^N.$$

and a function (for now, a polynomial)

$$f : \mathbb{R}^2 \to \mathbb{R}$$

and define the non-linear random matrix $Y_N = (y_{ij})_{i,j=1}^N$ by

$$y_{ij} = \begin{cases} \frac{1}{\sqrt{N}} f\left(\sqrt{N} x_{ij}^{(1)}, \sqrt{N} x_{ij}^{(2)}\right), & i \neq j \\ 0, & i = j. \end{cases}$$

Corollary

Let $X_N^{(1)}$, $X_N^{(2)}$ *and* Y_N *be as above. Then the non-linear random matrix model* Y_N *has the same asymptotic eigenvalue distribution as the linear model*

$$\hat{Y}_N := \theta_1 X_N^{(1)} + \theta_2 X_N^{(2)} + \theta Z_N,$$

where Z_N *is a symmetric standard* GOE *random matrix which is independent from* $X_N^{(1)}$ *and* $X_N^{(2)}$ *and where*

$$\theta_1 := \mathbb{E}[\partial_1 f(g_1, g_2)], \qquad \theta_2 := \mathbb{E}[\partial_2 f(g_1, g_2)]$$

and

$$\theta := \sqrt{c_2(f(g_1, g_2), f(g_1, g_2)) - \kappa_2^{(1)} \theta_1^2 - \kappa_2^{(2)} \theta_2^2},$$

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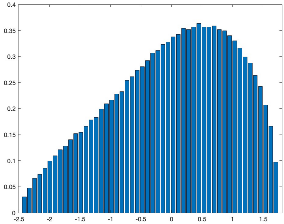

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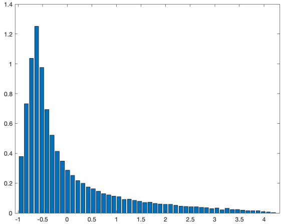

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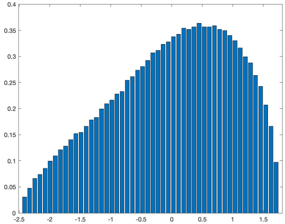

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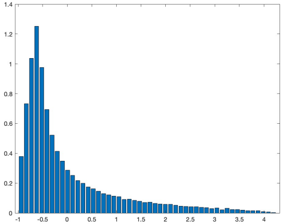

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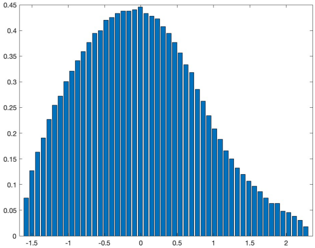

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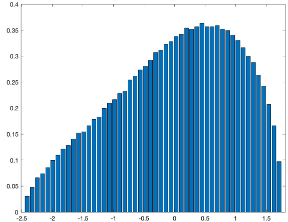

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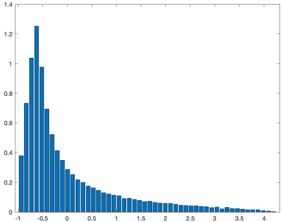

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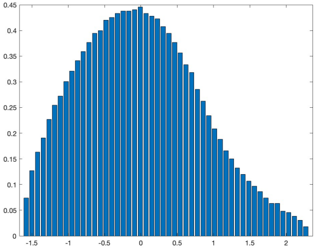

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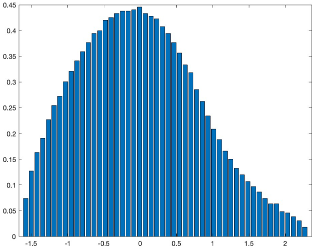
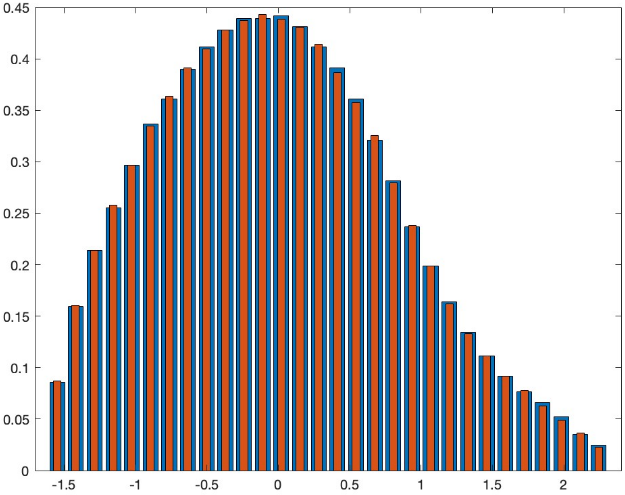


Superposition of the eigenvalues of the non-linear matrix Y_N and of its Gaussian equivalent $\hat{Y}_N$



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Example: something more non-linear

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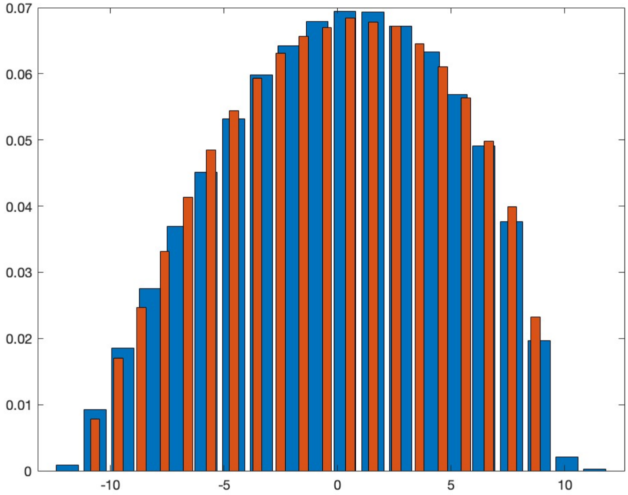

Extension to multivariate case with correlation

Let $X_N^{(1)}$ and $X_N^{(2)}$ be jointly orthogonally invariant random matrices.

$$\kappa_n^{(r_1,\dots,r_n)} := \lim_{N\to\infty} N^{n-1} c_n(x_{i_1 i_2}^{(r_1)}, x_{i_2 i_3}^{(r_2)}, \dots, x_{i_n i_1}^{(r_n)}).$$

and define the non-linear random matrix $Y_N = f(X_N^{(1)}, X_N^{(2)}) = (y_{ij})_{i,j=1}^N$ by

$$y_{ij} = \begin{cases} \frac{1}{\sqrt{N}} f\left(\sqrt{N} x_{ij}^{(1)}, \sqrt{N} x_{ij}^{(2)}\right), & i \neq j \\ 0, & i = j. \end{cases}$$

Corollary

The non-linear random matrix model $Y_N = f(X_N^{(1)}, X^{(2)})$ *has the same asymptotic eigenvalue distribution as the linear model*

$$\hat{Y}_N := \theta_1 X_N^{(1)} + \theta_2 X_N^{(2)} + \theta Z_N,$$

where Z_N *is a symmetric standard* GOE *random matrix which is independent from* $X_N^{(1)}$ *and* $X_N^{(2)}$ *and where*

$$\theta_1 := \mathbb{E}[\partial_1 f(g_1, g_2)], \qquad \theta_2 := \mathbb{E}[\partial_2 f(g_1, g_2)]$$

$$\theta := \sqrt{c_2(f(g_1, g_2), f(g_1, g_2)) - \kappa_2^{(1,1)}\theta_1^2 - \kappa_2^{(2,2)}\theta_2^2 - 2\kappa_2^{(1,2)}\theta_1\theta_2},$$

where g_1 *and* g_2 *are a Gaussian family of random variables with mean zero and covariance*

$$c_2(g_1, g_1) = \kappa_2^{(1,1)}, \qquad c_2(g_2, g_2) = \kappa_2^{(2,2)}, \qquad c_2(g_1, g_2) = \kappa_2^{(1,2)}.$$

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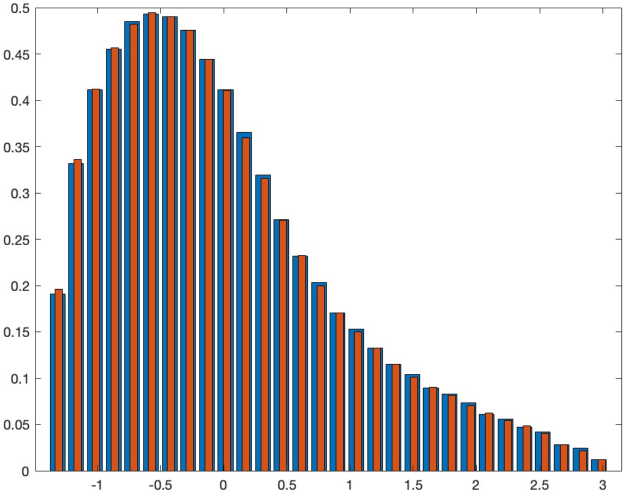

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