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On Conjugate systems with respect to completely positive maps (arXiv:2412.1336)

B-valued Semicircular systems [Shlyakhtenko '99]

Let (B,τ) be a tracial vNa, $\eta_{ij}: B \to B$ linear maps $i,j \in I$. $\{e_{ij} \in B(\ell^2(I)), i,j \in I\}$

Define $\eta: B \to B \otimes B(\ell^2(I))$

$$\eta(b) = \sum_{i,j \in I} \eta_{ij}(b) \otimes e_{ij}$$

Def If $\eta: B \to B \otimes B(\ell^2(I))$ is normal & cp

η is called a covariance matrix.

there exists a vN correspondence over B

$$L^2(B \underset{\eta}{\boxtimes} B, \tau) := \overline{\text{span}}\{B e_i B : i \in I\}$$

$$\langle a e_i b, c e_j d \rangle = \tau(b^* \eta_{ij}(a^* c) d)$$

the full fock space

$$\mathcal{F}_\tau := L^2(B,\tau) \oplus \bigoplus_{d>0} L^2(B \underset{\eta}{\boxtimes} B, \tau)^{\otimes_B^d}$$

the left creation operator on $\mathcal{F}_\tau$

$$\ell(h)b = h \cdot b$$

$$\ell(h)(a_1 \otimes \cdots \otimes a_d) = h \otimes a_1 \otimes \cdots \otimes a_d.$$

Def $S_i := \ell(e_i) + \ell(e_i)^*$ is called B-valued semicircular family with covariance η.

$$\underbrace{\Phi(B,\eta)} := B\langle S_i : i \in I\rangle''$$

$\tau(\eta_{ij}(a)b) = \tau(a\,\eta_{ji}(b))$ $\quad \Phi(B,\eta)$ tracial.

When $B = \mathbb{C}$, it gives us the complex-valued conj. sys.

derivation, op valued conj. system?

Let (M,τ) be a tracial vNa.

$B \leq M$, $X = (x_i)_{i\in I} \subset M_{sa}$

Assume $M = B\langle X \rangle''$

a unique normal faithful

$E_B : M \to B$ $\quad \tau = \tau \circ E_B$.

Covariance matrix $\eta: B \to B \otimes B(\ell^2(I))$

$\eta := \eta \circ E_B : M \to B \otimes B(\ell^2(I))$.

$$L^2(M \underset{\eta}{\boxtimes} M, \tau) := \overline{\text{span}}\{M e_i M : i \in I\}$$

$$\langle a e_i b, c e_j d \rangle = \tau(b^* \eta_{ij}(a^* c) d).$$

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Def Define η- partial derivative $\partial_\eta : B\langle X\rangle \to L^2(M \boxtimes_\eta M, \tau)$
as a linear mapping

$$\partial_\eta(b) = 0$$
$$\partial_\eta(X_i) = e_i$$
$$\partial_\eta(pq) = \partial_\eta(p)\cdot q + p\cdot \partial_\eta(q).$$

When $\eta = \delta_{ij}\tau$, $B = \mathbb{C}$,

Def $\xi_i \in L^2(M,\tau)$ is called a (B,η)-conjugate variable if $\langle \xi_i, p\rangle_\tau = \langle e_i, \partial_\eta(p)\rangle$

If ξ_i exists for $\forall i \in I$,
$(\xi_i)_{i\in I}$ is a (B,η)-conj. sys.

when $p = b_0 X_{i_1} b_1 \cdots X_{i_d} b_d$

$$\tau(\xi_i p) = \langle e_i, \sum_{k=1}^{d} b_0 X_{i_1} b_1 \cdots b_{k-1} \underline{\underline{e_{i_k}}} b_k \cdots X_{i_d} b_d\rangle$$
$$= \sum_{k=1}^{d} \tau(\eta_{i i_k}(b_0 X_{i_1} b_1 \cdots b_{k-1}) b_k \cdots X_{i_d} b_d)$$

If $\eta = \delta_{ij}\tau$, $B = \mathbb{C}$ $\quad \tau(\quad)\tau(\quad)$

. ∂_η is well-defined if ξ_i's exist

thm [L'24]
ξ_i is a (B,η)-conj. var. if and only if

$$\begin{cases} K_B^{(2)}(\xi_i \otimes bX_j) = \eta_{ij}(b) \\ K_B^{(d+1)}(\xi_i \otimes b_1 X_{i_1} \otimes \cdots \otimes b_d X_{i_d}) = 0 \end{cases}$$

Ex B-valued semicircular operators (S_i) have
a (B,η)-conj. sys. $\xi_i = S_i$.

Ex Let $N = M *_B \Phi(B,\eta) = B\langle X\rangle'' *_B B\langle S\rangle''$
Let $X_i(t) = X_i + \sqrt{t}\, S_i$, $\quad N_t = B\langle X_i(t) : i \in I\rangle''$
$E_t : N \to N_t$

$\left(\frac{1}{\sqrt{t}} E_t(S_i)\right)_{i\in I}$ is a (B,η)-conj. sys

for $(X_i(t))_{i\in I}$.

prop [L'24]
Assume there exists a (B,η)-conj. sys.
then ∂_η^* is densely defined, ∂_η is closable.

$$\partial_\eta^*(p e_i q) = p\,\partial_\eta^*(e_i) q - \langle \partial_\eta(p^*) \mid e_i\rangle_M q$$
$$- p\langle e_i \mid \partial_\eta(q)\rangle_M$$

$$\langle a e_i b \mid c e_j d\rangle_M = b^* \eta_{ij}(a^* c) d.$$

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prop [L' 24]

For $p \in M \cap \mathrm{dom}(\bar{\partial}_\eta)$, $u, v \in M_{s.a}$

If $pu = 0$, $p^* v = 0$, $v \bar{\partial}_\eta (p) u = 0$.

Def For a self-adjoint X,

$b \in B_{s.a}$ is called an atom of X in B

if $\ker(X - b) \neq 0$.

thm [L' 24] · $\eta_{ii} = E_B$

a self-adjoint polynomial $P = \sum_j a_j X_i b_j + b_0$

has no atoms in B when $\exists c > 0$ s.t. $\sum a_j b_j \geq c \cdot 1$

· $\eta = (\delta_{ij} E_B)_{ij}$

a self-adjoint polynomial $p = \sum_i \sum_j a_j^{(i)} X_i b_j^{(i)} + b_0$

has no atoms in B when $\exists c > 0$ s.t. $\sum_j a_j^{(i)} b_j^{(i)} \geq c \cdot 1$

ex) $P = X_i$, $Pw = 0$, $P^* w = 0$

$w \partial_\eta (P) w = 0$

$= w e_i w = 0$

$0 = \langle e_i, w e_i w \rangle$

$= \tau(\underline{\eta_{ii}}(w) w) = \| E_B(w) \|_2^2$

$\Rightarrow w = 0$.

Prop [L' 24]

Assume there exists a (B, η)-conj. sys.

Then $E_{B' \cap M}(B\langle X \rangle) \subset \mathrm{dom}(\bar{\partial}_\eta) \cap M$

and $\bar{\partial}_\eta$ is densely defined on $B \vee (B' \cap M)$

Def [popa '06] $A, B \leq M$ a tracial vNa.

We say a corner of A embeds into B inside M, write $A \prec_M B$

if one of the followings holds:

1. there exists projections $p \in P(A)$, $q \in P(B)$
   a nonzero partial isometry $v \in qMp$
   a unital normal $*$-homomorphism
   $\theta : pAp \to qBq$ s.t. $\theta(pap) v = v(pap)$ $\forall a \in A$.

2. there does not exist a net of unitaries $(u_i)_{i \in I} \subset A$ s.t. $\lim_{i \to \infty} \| E_B(x u_i y) \|_2 = 0$ $\forall x, y \in M$.

3. there exists a A, B-correspondence $K \leq L^2(M, \tau)$ $\dim_B(K) < \infty$.

if $M \not\prec_M B$, M is diffuse relative to B.

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prop [L'24]

Assume there exists a (B,η)-conj. sys. for $\eta_{ii} = E_B$.

then (a) $\tilde{x}_i := E_{B'\cap M}(x_i)$ has no atoms in B.

(b) $B\langle \tilde{x}_i\rangle''$ is diffuse relative to B.

(c) $B \vee (B'\cap M)$ is diffuse relative to B.

$\bar{\partial}_\eta(x_i) = e_i$

thm [L'24]

Assume that there exists a (B,η)-conj. sys. for $\eta = (\delta_{ij}E_B)_{ij}$. Then

$$Z(B\vee(B'\cap M)) = Z(B).$$

pf). Let $\tilde{x}_i = E_{B'\cap M}(x_i)$, $A_i = B\langle x_i\rangle''$

$A = B\vee(B'\cap M)$.

step 1 $A_i \not\prec_A B$

Assume $A_i \prec_A B$.

there exists an $A_i - B$ correspondence $K \le L^2(A,\tau)$ with $\dim_B K < \infty$.

Let $N = (A_i)' \cap (JBJ)'$, e_{A_i} be the projection from $L^2(A,\tau)$ onto $L^2(A_i,\tau)$.

Since $B \subset A_i$, $J(B'\cap M)J \subset N$.

$$\left\{\begin{aligned} N e_{A_i} L^2(A,\tau) &= N L^2(A_i,\tau) \supset N L^2(B,\tau) \\ &\supset \underbrace{J(B'\cap M)J\cdot L^2(B,\tau)}_{\text{dense in } L^2(A,\tau)} \end{aligned}\right.$$

$\Rightarrow e_{A_i}$ has full central support in N.

$[K]$ and e_{A_i} are not centrally orthogonal in N.

$\exists$ a non-zero partial isometry $v \in N$ s.t.

$v^*v \le [K]$, $vv^* \le e_{A_i}$.

$vK \le vL^2(M,\tau) \le L^2(A_i,\tau)$

is a $A_i - B$ correspondence $A_i \prec_{A_i} B$.

Step 2: there are no A_i-central vectors in $L^2(A \boxtimes_\eta A,\tau)$ $(= \overline{\mathrm{span}}\{Ae_iA : i\in I\})$.

$$\because\ L^2(A\boxtimes_\eta A,\tau) = \bigoplus_{j\in I} L^2(A\boxtimes_{E_B} A,\tau)$$

Any A_i-central vector in $L^2(A\boxtimes_\eta A,\tau)$ would be a direct sum of A_i-central vectors in $L^2(A\boxtimes_{E_B} A,\tau)$.

Assume $\xi \in L^2(A\boxtimes_{E_B} A,\tau)$ is A_i-central vector, $\|\xi\|_2 = 1$.

Given $\varepsilon > 0$, let $\xi_0 = \sum_{k=1}^{d} x_k e_j y_k$ with $x_k, y_k \in A$

$\|\xi - \xi_0\| < \varepsilon$.

By step 1, $(A_i \not\prec_A B)$

there exists a net of unitaries $(u_m) \subset A_i$ s.t. $\lim_{m\to\infty} \|E_B(x u_m y)\|_2 = 0$ $\forall x, y \in A$.

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In particular, $\exists$ a unitary $u \in A$ s.t.

$$\| E_B(xuy) \|_2 < \varepsilon/d \quad \forall x, y \in \{x_k, y_k, x_k^*, y_k^* : k=1..d\}.$$

$$\left(\xi_0 = \sum_{k=1}^{d} x_k e_j y_k \right).$$

$$|\langle u \xi_0 u^*, \xi_0 \rangle| < \varepsilon^2.$$

$$\| \xi \|_2 < \varepsilon. \qquad \therefore \xi = 0$$

<u>step 3</u> Since $\eta = (\delta_{ij} E_B)_{ij}$

$$L^2(M \bar{\otimes} M, \tau) = \bigoplus_{j \in I} L^2(M \bar{\otimes} M, \tau)_{\eta_{jj}}$$

and $\partial_\eta = \sum_j \partial_{\eta,j}$ where $\partial_{\eta,j} : B\langle X \rangle \to L^2(M \bar{\otimes} M, \tau)_{\partial_{\eta,j}}$

$$\partial_{\eta,j}(x_i) = \delta_{ij}(e_i)$$

positive unbounded operator on $L^2(M, \tau)$

$$\Delta_j = \partial_{\eta,j}^* \bar{\partial}_{\eta,j}$$

a bounded operator on $L^2(M, \tau)$

$$\varsigma_{t,j} = \left(\frac{t}{t + \Delta_j} \right)^{1/2}$$

$$\| \varsigma_{t,j}(y) - y \|_2 \to 0.$$

Since $\varsigma_{t,j}(L^2(M)) \subset \mathrm{dom}(\Delta_j) \subset \mathrm{dom}(\bar{\partial}_{\eta,j})$

$\bar{\partial}_{\eta,j} \cdot \varsigma_{t,j}$ is bounded.

For $i \neq j$, $\bar{\partial}_{\eta,j}(\tilde{x}_i) = 0$.

$$\varsigma_{t,j}([\tilde{x}_i, \xi]) = [\tilde{x}_i, \varsigma_{t,j}(\xi)] \checkmark$$

$$\xi \in L^2(M, \tau).$$

Assume $z \in \mathcal{Z}(A)$,

$$[\tilde{x}_i, z] = 0.$$

$$0 = \bar{\partial}_{\eta,j} \circ \varsigma_{t,j}([\tilde{x}_i, z])$$

$$= [\tilde{x}_i, \bar{\partial}_{\eta,j} \cdot \varsigma_{t,j}(z)].$$

for $b \in B$, $\quad 0 = [b, \underline{\bar{\partial}_{\eta,j} \cdot \varsigma_{t,j}(z)}]$

$A_i = B\langle \tilde{x}_i \rangle''$, $\quad \bar{\partial}_{\eta,j} \cdot \varsigma_{t,j}(z)$ is A_i-central.

$$\therefore \bar{\partial}_{\eta,j} \circ \underbrace{\varsigma_{t,j}(z)}_{\downarrow \, z} = 0.$$

$$z \in \mathrm{dom}(\bar{\partial}_{\eta,j}), \quad \bar{\partial}_{\eta,j}(z) = 0.$$

$$[x_j, z] = 0.$$

$$0 = \partial_{\eta,j}([x_j, z]) = [e_j, z].$$

$$0 = \langle [e_j, z], [e_j, z] \rangle$$

$$= \langle e_j z, e_j z \rangle - \langle e_j z, z e_j \rangle$$

$$- \langle z e_j, e_j z \rangle + \langle z e_j, z e_j \rangle$$

$$= \tau\big(z^* \eta_{jj}(1) z - z^* \eta_{jj}(z)$$

$$- \eta_{jj}(z^*) z + \eta_{jj}(z^* z) \big)$$

$$0 = \tau(z^*z - z^*E_B(z) - E_B(z^*)z + E_B(z^*z))$$
$$\geq \| z - E_B(z)\|_2^2 .$$

$z = E_B(z) \in B.$ $\quad Z(A) \subset B \cap B'$

$\blacksquare$

<u>Cor</u> if $X_i \in B'$, $Z(M) = Z(B)$.

<u>Ex</u> $N_t = B\langle X_i(t)\rangle''$ $\quad X_i(t) = X_i + \sqrt{t}S_i$.

Since $S_i \in B'$, if $X_i \in B'$

$$Z(N_t) = Z(B).$$