# Freeness for tensors

## Probabilistic Operator Algebra Seminar

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November 18, 2024

# Introduction

Joint work with Charles Bordenave.

# Main results

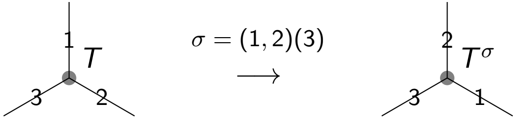
- Definition of freeness associated to a collection of tensors of possibly different orders.
- Associated free cumulants.
- Asymptotic freeness of random tensors.
- Schwinger-Dyson loop equations associated to random tensors.
- Free convolution of tensors (work in progress).

# I. Maps of tensors

# I. Anatomy of a tensor

$$T = (T_{i_1,\ldots,i_p})_{1 \le i_l \le N_l}$$

- $T_{i_1,\ldots,i_p} \in \mathbb{R}$ or $\mathbb{C}$
- $p$ = order $\quad (p = 1$ vector, $p = 2$ matrix,...)
- $l$ = a leg
- $N_l$ = dimension of a leg $\quad = N \quad$ (in this talk)




$$T^{\sigma}_{i_1,\cdots,i_p} = T_{i_{\sigma(1)},\cdots,i_{\sigma(p)}}$$

# I. Combinatorial maps

- $\mathcal{M}_0$ : trace maps $\mathfrak{m} = (\pi, \alpha)$ with
  - $\pi \in S_m$ with $m$ even the number of half-edges and the cycles of $\pi$ are the vertices,
  - $\alpha \in S_m$ an involution without fixed point encoding the edges (matching between half-edges).

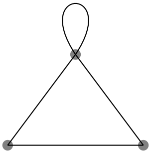


Figure: $\mathfrak{m} = (\pi = (1,2,3,4)(5,6)(7,8), \alpha = (1,2)(3,5)(4,7)(6,8))$.

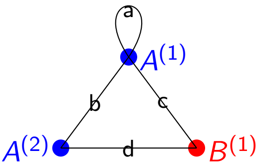
- $\mathcal{M}_q$ : maps with $q$ boundaries ($\alpha$ has $q$ fixed points).

# I. Tensor maps

## Tensor maps

$$\mathfrak{m}((T_v)_{v\in V})_{i_\delta} = \frac{1}{N^\gamma} \sum_{i\in[\![N]\!]^E} \prod_{v\in V} (T_v)_{i_{\partial v}}$$

with $\gamma = \#\{\text{c.c. of } \mathfrak{m}\}$ if $\mathfrak{m} \in \mathcal{M}_0$ and $\gamma = 0$ otherwise.



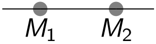

$$\mathfrak{m}(A^{(1)}, A^{(2)}, B^{(1)}) = \frac{1}{N} \sum_{1\leq a,b,c,d\leq N} A^{(1)}_{aabc} A^{(2)}_{bd} B^{(1)}_{cd}.$$

# I. Some examples

- Matrix trace

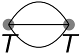

$$\mathfrak{b}_2(M) = \frac{1}{N}\sum M_{ii}$$

- Matrix multiplication



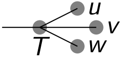

$$\mathfrak{m}(M_1, M_2) = M_1 M_2$$

- Frobenius norm



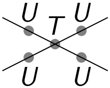

$$\mathfrak{f}_p(T, T) = \frac{1}{N}\sum_{i_1,\dots,i_p} T^2_{i_1,\dots,i_p}$$

- Contraction by vectors




$$T.(u, v, w)$$

- Unitary composition




$$T \cdot U^p$$

# I. Trace invariants

Let $\mathfrak{m}$ be a trace map and consider $\mathfrak{m}((T_v)_{v \in V})$.

It is *orthogonal invariant* : if $U \in O(N)$, then

$$\mathfrak{m}((T_v \cdot U^{p_v})_{v \in V}) = \mathfrak{m}((T_v)_{v \in V}),$$

$$\text{where} \quad (T \cdot U^p)_j = \sum_{i \in [\![N]\!]^p} T_i \prod_{k=1}^{p} U_{j_k i_k}.$$

$\rightarrow$ We call them the *trace invariants*. They are the building blocks of a spectral theory for tensors.

# I. Distribution of tensors

$\mathcal{A} = \{a_i : i \in \mathcal{I}\}$ a finite collection of tensors.

**Distribution of $\mathcal{A}$**

Collection of trace invariants $\mathfrak{m}(T)$ for $\mathfrak{m} \in \mathcal{M}_0$ and $T = (T_v)_{v \in V}$ with $T_v \in \mathcal{A}$ with consistent order.

Example : if $\mathcal{A} = \{a\}, a \in M_N(\mathbb{R})$, we get

$$\frac{1}{N}\mathrm{Tr}(a^{\epsilon_1} \cdots a^{\epsilon_k}), \quad a^{\epsilon_i} \in \{a, a^T\}.$$

# I. Wigner tensors

$(X_i)_{i=(i_1,\cdots,i_p)\in[\![N]\!]^p/\mathrm{S}_p} \in (\mathbb{R}^N)^{\otimes p}$ i.i.d. random variables s.t.

$$\mathbb{E}X_i = 0 \quad \text{and} \quad \mathbb{E}X_i^2 = \frac{1}{(p-1)!} \quad \text{and} \quad \forall k, \mathbb{E}|X_i|^k \leq c(k)$$

**Convergence in distribution (Gurau 20', B. 24')**

$W^N = \frac{(X_i)_{i\in[\![N]\!]^p}}{N^{\frac{p-1}{2}}}$ converges towards $\mathbf{s}_p$ in distribution (in proba).

$$m(W^N) \to \mathfrak{m}(\mathbf{s}_p) = \frac{1}{(p-1)!^{\#\mathfrak{m}/2}} \mathbf{1}_{\mathfrak{m} \text{ melonic}}.$$

# II. Tensor freeness

# II. Action of combinatorial maps

$\mathcal{E}_0 = \mathbb{C}$, $\mathcal{E}_1$ complex vector space, $\mathcal{E}_p = \mathcal{E}_1^{\otimes p}$, $\mathcal{E} = \sqcup \mathcal{E}_p$.
$\mathfrak{m} \in \mathcal{M}_q$, $\mathcal{E}_\mathfrak{m} = \{(x_1, \ldots, x_V) : x_v \in \mathcal{E}_{\deg(v)}\}$.

$\mathcal{M}$ acts on $\mathcal{E}$ via $\mathfrak{m} : \mathcal{E}_\mathfrak{m} \to \mathcal{E}_q$ with properties :

(CI) *Class invariance*

(M) *Morphism*

(L) *Linearity*

(S) *Substitution*

(Id) *Identity for even p*

ex : $\mathcal{E}_1 = \mathbb{C}^N$, $\mathfrak{m}(x) =$ tensor map.

## II. Map vector bundle

**$\mathcal{M}$-bundle**

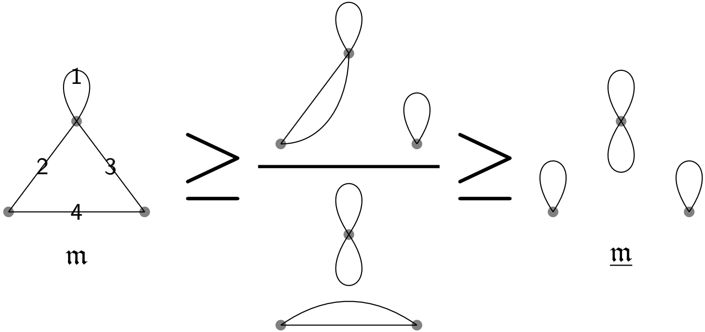
Let $\mathcal{A} \subset \mathcal{E} = \sqcup \mathcal{E}_p$.
$\langle \mathcal{A} \rangle$ is the union of vector spaces spanned by $\mathfrak{m}(x)$, $\mathfrak{m} \in \cup_{q \geq 0} \mathcal{M}_q$, $x_v \in \mathcal{A} \cup 1_{2p}$.

For matrices, we recover the algebra generated by $\mathcal{A}$.

We say that an $\langle \mathcal{A} \rangle$-map $(\mathfrak{m}, x) \in \mathcal{M}_0(\langle \mathcal{A} \rangle)$ is *centered* if $\mathfrak{m}(x) = 0$.

# II. The non-crossing poset

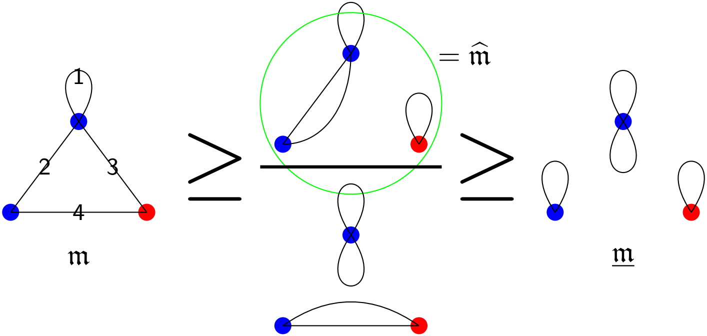
$\mathcal{P}_\pi := \{\mathfrak{m} = (\pi, \alpha) \in \mathcal{M}_0\}$.
$\mathfrak{m}' < \mathfrak{m}$ if $\alpha\alpha'$ is a product of transpositions and $\gamma(\mathfrak{m}') = \gamma(\mathfrak{m}) + 1$.



## II. Definition of freeness - Preliminaries

(P1) $\widehat{\mathfrak{m}}$ has c.c. monochromatic or minimal non monochromatic,

(P2) there is a path from $\mathfrak{m}$ to $\widehat{\mathfrak{m}}$ with only chromatic switches ($\mathcal{A} \neq \mathcal{B}$, switch $(a_1, b_1)$, $(a_2, b_2)$).



# II. Tensor freeness

## Definition of freeness

The sets $(\mathcal{A}_c)_{c\in\mathcal{C}}$ are *free* if for all $\langle\mathcal{A}\rangle$-maps $(\mathfrak{m}, x) \in \mathcal{M}_0(\langle\mathcal{A}\rangle)$, we have $\mathfrak{m}(x) = 0$ as soon as

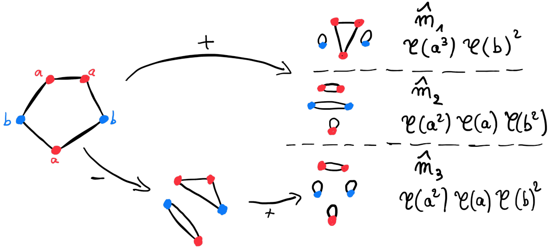
all $\widehat{\mathfrak{m}} \leq \mathfrak{m}$ satisfying (P1)-(P2) has a connected component minimal non monochromatic or centered monochromatic.

## Individual distributions

If $(\mathcal{A}_c)_{c\in\mathcal{C}}$ are free, then the distribution of $\mathcal{A} = \sqcup_{c\in\mathcal{C}}\mathcal{A}_c$ is characterized by the individual distributions of $\mathcal{A}_c$, $c \in \mathcal{C}$.

# II. A matrix example

The connected 2-valent maps are cycles. The $\widehat{\mathfrak{m}}$ are the NC partitions where consecutive elements of the same family are in the same block.




Usual freeness: $\varphi(a^2bab) = \varphi(a^3b)\varphi(b) + \varphi(a^2\mathring{b}ab)$

$$= \varphi(a^3)\varphi(b)^2 + \underbrace{\varphi(a^3\mathring{b})}_{=0}\varphi(b) + \varphi(a^2)\varphi(\mathring{b}ab) + \varphi(\mathring{\overline{a^2}}\mathring{b}ab)$$

$$= \varphi(a^3)\varphi(b)^2 + \varphi(a^2)\varphi(a)\varphi(\mathring{b}b) + 0$$

$$= \varphi(a^3)\varphi(b)^2 + \varphi(a^2)\varphi(a)\varphi(b^2) - \varphi(a^2)\varphi(a)\varphi(b)^2$$

# II. Free cumulants

$\mathcal{P}_{\mathfrak{m}} = \{\mathfrak{n} : \mathfrak{n} \leq \mathfrak{m}\}$.

**Existence of free cumulants**

$\forall \mathfrak{m}$, $\exists ! \kappa_{\mathfrak{m}} : \mathcal{E}_{\mathfrak{m}} \to \mathbb{C}$ s.t. $\forall x \in \mathcal{E}_{\mathfrak{m}}$,

$$\mathfrak{m}(x) = \sum_{\mathfrak{n} \leq \mathfrak{m}} \kappa_{\mathfrak{n}}(x).$$

$\kappa_{\mathfrak{m}}$ satisfy (CI), (M), (L) and a weak form of (S).

**Freeness and free cumulants**

The even families $(\mathcal{A}_c)_{c \in \mathcal{C}}$ are free if and only if for all $\langle \mathcal{A} \rangle$-map $(\mathfrak{m}, x)$ connected non-monochromatic, we have $\kappa_{\mathfrak{m}}(x) = 0$.

# II. Moments and free cumulants of a tensor

$\mathcal{B}_n := \{$ connected rooted $p$-valent maps with $n$ vertices $\}$.
$T$ a real symmetric tensor of order $p$.
Denote $\mathfrak{m}(T) := \mathfrak{m}(T, \dots, T)$ and $\kappa_{\mathfrak{m}}(T) := \kappa_{\mathfrak{m}}(T, \dots, T)$.

**Moments and free cumulants of a tensor**

For $n \geq 0$,

$$m_n(T) = \sum_{\mathfrak{m} \in \mathcal{B}_n} \mathfrak{m}(T) \qquad \text{and} \qquad \kappa_n(T) = \sum_{\mathfrak{m} \in \mathcal{B}_n} \kappa_{\mathfrak{m}(T)}.$$

**Gurau's measure**

$\exists \mu_T$ probability measure on $\mathbb{R}$ s.t. $\forall n, m_n(T) = \int \lambda^n d\mu_T(\lambda)$.

$p = 2$: $\mu_T = \text{ESD}$; $p \geq 3$: $\mu_T$ has unbounded support.

# II. High order semi-circular

## High order semi-circular

Let $\mathbf{s}_p$ the high order semi-circular element given by

$$\mathfrak{m}(\mathbf{s}_p) = \frac{1}{(p-1)!^{\#\mathfrak{m}/2}} \mathbf{1}_{\mathfrak{m} \text{ melonic}} \quad \text{or} \quad \kappa_{\mathfrak{m}}(\mathbf{s}_p) = \frac{1}{(p-1)!} \mathbf{1}_{\mathfrak{m} \text{ a melon}}.$$

Then, for $n \geq 1$,

$$m_n(\mathbf{s}_p) = \mathbf{1}_{n \text{ even}} \text{ Fuss-Catalan}_p(n/2) \qquad \text{and} \qquad \kappa_n(\mathbf{s}_p) = \mathbf{1}_{n=2}.$$

## Convergence of Wigner tensors (Gurau 20', B. 24')

Let $\mu_p$ the measure associated to $\mathbf{s}_p$, then $\mu_{W^N} \to \mu_p$.

# II. Free CLT

$a \in \mathcal{E}_{2p}$ is *standard* if it satisfy

(i) $\mathfrak{b}^{\sigma}_{2p}(a) = 0$ for all $\sigma$ (a is *centered*).

(ii) $\mathfrak{f}^{\sigma}_{2p}(a) = \frac{1}{(2p-1)!}$ for all $\sigma$.

**Free CLT for tensors**

Let $p$ even, $(a_i)_{i\geq 1} \in \mathcal{E}_p$ be a collection of standard free elements. Assume that $\forall \mathfrak{m} \in \mathcal{M}_0$, $\exists C(\mathfrak{m})$ s.t. $\forall i$: $|\mathfrak{m}(a_i)| \leq C(\mathfrak{m})$. Then,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} a_i$$

converges toward $\mathbf{s}_p$.

# III. Asymptotic freenes

# III. Asymptotic freeness

$(\mathcal{A}_c^N)_{c\in\mathcal{C}}$ finite collection of disjoint subsets in $\mathcal{E}^N$ and $\mathcal{A}^N = \sqcup \mathcal{A}_c = \{a_i^N : i \in \mathcal{I}\}$.

**Asymptotic freeness**

$(\mathcal{A}^N)_{N\geq 1}$ *is asymptotically free* if $\forall \mathfrak{m} \in \mathcal{M}_0(\mathcal{I})$ satisfying conditions in the definition of freeness,

$$\lim_{N\to\infty} \mathfrak{m}(\mathcal{A}_N) = 0.$$

If $\mathcal{A}^N = \{\text{random variables}\}$, we can speak of asymptotic freeness *in probability* or *in expectation*

## III. Assumptions

$\mathcal{A}_0^N$ a finite and deterministic collection of tensors.

**(A1)**

$\forall \mathfrak{m} \in \mathcal{M}_0$, $\forall (T_v^N)_{v\in V} \in (\mathcal{A}_0^N)^V$, $\exists C(\mathfrak{m})$ s.t. $\forall N \geq 1$

$$\left|\mathfrak{m}((T_v^N)_{v\in V})\right| \leq C(\mathfrak{m}),$$

**(A2)**

$\forall \mathfrak{m}$ hyper-map, $\forall (T_v^N)_{v\in V} \in (\mathcal{A}_0^N)^V$, $\exists C(\mathfrak{m})$ s.t. $\forall N \geq 1$

$$\left|\mathfrak{m}((T_v^N)_{v\in V})\right| \leq C(\mathfrak{m}),$$

## III. Wigner tensors case

**Theorem 1 - Gaussian case**

If (A1) holds, $\mathcal{A}_0^N$ and $\{W^N\}$ are asymptotically free in probability.

**Theorem 2**

If (A2) holds, $\mathcal{A}_0^N$ and $\{W^N\}$ are asymptotically free in probability.

**Corollary 1**

$(W_1^N, \ldots, W_n^N)$ independent Wigner tensors of possibly different orders. They are asymptotically free in probability.

## III. Haar orthogonal case

$U^N$ Haar distributed on $O(N)$.

**Theorem 3**

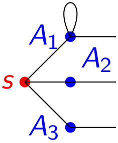
If (A1) holds, $\mathcal{A}_0^N$ and $\{U_N, U_N^*\}$ are asymptotically free in probability.

$$\mathcal{A} \cdot U^{\#} := \{B : \exists p, \exists A \in \mathcal{A} \cap \mathcal{E}_p^N, B = A \cdot U^p\}.$$

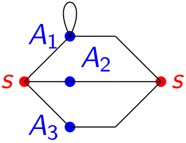
**Theorem 4**

$\mathcal{A}_1^N$ and $\mathcal{A}_2^N$ two finite families of tensors satisfying (A1).
The families $\mathcal{A}_1^N$ and $\mathcal{A}_2^N \cdot U_N^{\#}$ are asymptotically free in probability.

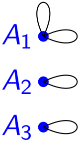
# III. Operations on maps



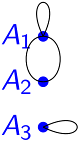

$\mathfrak{m} \in \mathcal{M}_3(\mathcal{A})$




$\mathfrak{m}^{+\mathbf{s}} \in \mathcal{M}_0(\mathcal{A})$




$\mathfrak{m}^{\setminus v} \in \mathcal{M}_0(\mathcal{A})$




$(\mathfrak{m}.\sigma)^{\setminus v}, \sigma = (1,2)(3)$

# III. Schwinger-Dyson equations - Gaussian case

**Proposition 1 - Schwinger-Dyson equations**

$\forall \mathfrak{m} \in \mathcal{M}_p(\mathcal{I})$ connected,

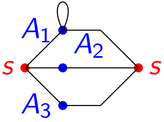
$$\mathbb{E}_N[\mathfrak{m}^{+\mathbf{s}}] = \frac{1}{(p-1)!} \sum_{v,\sigma} \mathbb{E}_N[(\mathfrak{m}.\sigma)^{\setminus v}] + O(\frac{1}{N}),$$

where the sum is over all $v \in V(\mathfrak{m})$ s.t. $w_v = \mathbf{s}$, all permutations in $\mathrm{S}_p$ s.t. $(\mathfrak{m}.\sigma)^{\setminus v}$ has $p$ connected components (this sum might be empty).

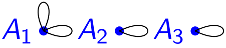
# III. Schwinger-Dyson equations - Sketch of proof

By Gaussian integration by parts,

$$\mathbb{E}\left[\frac{1}{N}\right.$$




$$\left.\right] \simeq \frac{1}{N}$$




$$\underbrace{\qquad}_{\simeq N^p} \quad \underbrace{\mathbb{E}(s^2)}_{= \frac{1}{N^{p-1}}}$$

# III. Variance

## Proposition 2

$\forall \mathfrak{m} \in \mathcal{M}_p(\mathcal{I})$ connected,

$$\mathbb{E}_N[|\mathfrak{m} - \mathbb{E}_N \mathfrak{m}|^2] = O(\frac{1}{N}).$$

Moreover, $\forall \mathfrak{m} \in \mathcal{M}_0(\mathcal{I})$, with connected components $(\mathfrak{m}_1, \ldots, \mathfrak{m}_\gamma)$ we have

$$\mathbb{E}_N[\mathfrak{m}] = \prod_{i=1}^{\gamma} \mathbb{E}_N[\mathfrak{m}_i] + O(\frac{1}{N}).$$

## III. Proof of asymptotic freeness

Proof of Theorem 1 :

- Prop. 2 + Markov inequality $\Rightarrow$ asymptotic freeness in expectation is sufficient.
- Fix $\mathfrak{m}$ and $\widehat{\mathfrak{m}}$ satisfying (P1)-(P2) with a centered component. Prop. 2 $\Rightarrow$ we may assume $\mathfrak{m}$ connected.
- Recurrence on $t := \#\{v \in V(\mathfrak{m}) : w_v = \mathbf{s}\}$.

⋆ $t = 0 \Rightarrow \mathfrak{m} = \widehat{\mathfrak{m}}$ monochromatic $\Rightarrow \mathfrak{m}(\mathcal{A}_N) = 0$ by (i).

⋆ $t = 1 \Rightarrow \mathfrak{m} = \widetilde{\mathfrak{m}}^{+\mathbf{s}}$ and Prop. 1 $\Rightarrow \mathfrak{m}(\mathcal{A}_N) = O(\frac{1}{N})$ as the sum is empty.

⋆ $t \geq 2$. "Delete" 2 vertices of type $\mathbf{s}$ by applying Prop. 1 to $\widetilde{\mathfrak{m}}$ where $\mathfrak{m} = \widetilde{\mathfrak{m}}^{+\mathbf{s}}$ and you get a sum on maps $\widetilde{\mathfrak{m}}^{\setminus v}$ with $t - 2$ vertices of type $\mathbf{s}$.

# III. Non-Gaussian case

*Combinatorial hyper-map* : $\mathfrak{m} = (\pi, \alpha) \in S_m$ where $\alpha$ has cycles of length at least two which are the hyper-edges $E(\mathfrak{m})$.

Proof by comparison,

$$\left|\mathbb{E}_N[\mathfrak{m}] - \mathbb{E}_N^{\mathrm{gauss}}[\mathfrak{m}]\right| = o(1).$$

# IV. Free convolution and openings

# Moment-cumulant formula

Define

$$M_T(z) := \sum_{n\geq 0} m_n(T) z^n \qquad \text{and} \qquad C_T(z) := \sum_{n\geq 0} \kappa_n(T) z^n.$$

We can derive a relation between $M$ and $C$ from the computation

$$m_n(T) = \sum_{s=1}^{n} \kappa_s(T) \sum_{\substack{i_1,\ldots,i_{sp/2} \in [\![ n-s ]\!] \\ s+i_1+\ldots+i_{sp/2}=n}} m_{i_1}(T)\ldots m_{i_{sp/2}}(T).$$

## Moment-cumulant formula

$$M_T(z) = C_T(z M_T(z)^{p/2})$$

# Free convolution

**Example : stable law**

If $T_1 \sim \frac{1}{\sqrt{2}}\mathbf{s}_p$ and $T_2 \sim \frac{1}{\sqrt{2}}\mathbf{s}_p$ are freely independent, then $T_1 + T_2 \sim \mathbf{s}_p$.

Proof : 
$$\begin{aligned} M_{T_1+T_2}(z) &= C_{T_1}(zM_{T_1+T_2}(z)^{p/2}) + C_{T_1}(zM_{T_1+T_2}(z)^{p/2}) - 1 \\ &= 1 + z^2 M_{T_1+T_2}(z)^p \end{aligned}$$

$\rightarrow$ Possible to define $R$-transform, subordination functions, etc. (w.i.p.)

# Variants and perspectives

- We may consider extra symmetries, for example tensors with inputs and outputs as in recent works about *local* invariance.
- To consider tensors with legs of various dimensions, we may decorate the edges of a map and only consider switches between legs of the same color.

- Concentration inequalities for $\mathfrak{m}(T)$ ?
- Connections to $z$-eigenvalues / eigenvectors ($T.v^{p-1} = \lambda v$) ?
- Multiplicative convolution ?
- Free entropy ?

# Conclusion

Thanks for listening !