Quaternionic linear operators, S-spectrum and Grothendieck-Lidskii formula

Irene Sabadini, joint work with P. Cerejeiras, F. Colombo, A. Debernardi Pinos, U. Kähler

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Quaternions I

Let $\mathbb{H}$ denote the algebra of quaternions. An element $q \in \mathbb{H}$ is of the form

$$q = q_0 + q_1 i + q_2 j + q_3 ij \qquad q_\ell \in \mathbb{R},$$

where $ij = -ji = k$, $i^2 = j^2 = k^2 = -1$. They generalize the complex numbers.

- **Sum**: Similar as in the complex case (4-dim. real vector space).
- **Product**: Use the associative law.
- **Product is non-commutative**: In general, $pq \neq qp$, for $p, q \in \mathbb{H}$.
- **Conjugation**: $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$.
- **Existence of inverses**: For every $q \in \mathbb{H}$, $q \neq 0$, there exists a unique $q^{-1} \in \mathbb{H}$ with $q^{-1}q = qq^{-1} = 1$.
- **Modulus**: $|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$.

Quaternions II

Similarly as for complex numbers, we can distinguish between the real and imaginary (vectorial) parts of a quaternion. For

$$q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H},$$

we define

$$\text{Re } q = q_0, \qquad \mathrm{Im}\, q = q_1 i + q_2 j + q_3 k.$$

Remark

Quaternion multiplication may be used to represent rotations in $\mathbb{R}^3$. More precisely, if $q \in \mathbb{H}$ and $s \in \mathbb{H} \setminus \{0\}$, then $s^{-1}qs$ has a rotated imaginary part with respect to q.

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Quaternions II

Note that if $\lambda = \lambda_0 + i\lambda_1 + j\lambda_2 + k\lambda_3 = \lambda_0 + \mathrm{Im}(\lambda) \notin \mathbb{R}$, we can set

$$I = \mathrm{Im}(\lambda)/|\mathrm{Im}(\lambda)|, \qquad \text{and} \qquad \lambda_v = |\mathrm{Im}(\lambda)|$$

so that $\lambda = \lambda_0 + I\lambda_v$.

Set $\mathbb{S} = \{I = \mathrm{Im}(q) \ : \ |\mathrm{Im}(q)| = 1\}$, then $I \in \mathbb{S}$ if and only if $I^2 = -1$.

Then $[\lambda] = \{\lambda_0 + J\lambda_v, \ \ J \in \mathbb{S}\} = \{\lambda' = s^{-1}\lambda s, s \neq 0\}$.

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Quaternionic matrices (linear maps)

We now consider finite-dimensional vector spaces over $\mathbb{H}$ (denoted $\mathbb{H}^n$), and (right-) linear maps between these.

Let $A \in M_n(\mathbb{H})$ and a (right) eigenvalue $\lambda \in \mathbb{H}$ of A, with eigenvector $x \in \mathbb{H}^n$, $x \neq 0$, i.e.,

$$Ax = x\lambda.$$

For any nonzero quaternion s, $\lambda' = s^{-1}\lambda s$ is an eigenvalue of A

$$A(xs) = (Ax)s = x\lambda s = (xs)(s^{-1}\lambda s) = (xs)\lambda'.$$

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Remark

Although being different, λ and λ' satisfy the relations Re $\lambda =$ Re λ' and $|\lambda| = |\lambda'|$. This allows to define equivalence classes $[\lambda]$ of eigenvalues.

Note

The operator $A - I\lambda$ is not right linear: $A(xs) - (xs)\lambda \neq (Ax - x\lambda)s$

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Problems with quaternionic matrices

- How can we define the trace of A and expect it to be equal to the sum of all eigenvalues of A? (in general, there are uncountably many!)
- What about its determinant and relate it to the eigenvalues?
- Can we construct a characteristic polynomial?
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Let us set $\mathbb{C} = \mathbb{C}(i)$. Any quaternion q may be written as $q = z_1 + z_2 j$, where $z_1, z_2 \in \mathbb{C}$. For $A \in M_n(\mathbb{H})$, write $A = A_1 + A_2 j$, with $A_1, A_2 \in M_n(\mathbb{C})$. The companion matrix of A is defined as the matrix

$$\chi_A = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}.$$

This matrix captures the noncommutative nature of A (Lee, '49). It also "solves" some of the problems of working directly with quaternion matrices.

Theorem

Let $A \in M_n(\mathbb{H})$. Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if both λ and $\overline{\lambda}$ are eigenvalues of χ_A.

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Companion matrix and eigenvalues

To each eigenvalue class $[\lambda]$ of A correspond two eigenvalues of χ_A.

Definition

For a matrix $A \in M_n(\mathbb{H})$, we define the *standard eigenvalues* of A as the purely complex eigenvalues of A with positive imaginary part.

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Invariants of quaternionic matrices

Let us define invariants associated to $A = (a_{k\ell})_{k,\ell=1}^n \in M_n(\mathbb{H})$ in terms of

$$\chi_A = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}.$$

- Trace: $T_{\mathbb{H},1}(A) := \operatorname{Tr} \chi_A = 2 \operatorname{Re} \left(\sum_{k=1}^n a_{kk} \right)$. If $\lambda_1, \ldots, \lambda_n$ are the standard eigenvalues of A, then

$$T_{\mathbb{H},1}(A) = 2 \operatorname{Re} \left(\sum_{k=1}^n \lambda_k \right) = \sum_{k=1}^n (\lambda_k + \overline{\lambda_k}).$$

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The (quaternionic) characteristic polynomial

We can also define the "characteristic polynomial" of A as

$$\det{}_{\mathbb{H}}(I - zA) := \det(I - z\chi_A) = \prod_{k=1}^{n}(1 - z\lambda_k)(1 - z\overline{\lambda_k}).$$

It is known that

$$\det(I - z\chi_A) = \sum_{k=0}^{2n}(-1)^k z^k \operatorname{Tr}\left(\bigwedge^k \chi_A\right),$$

where

$$\operatorname{Tr}\left(\bigwedge^k \chi_A\right) = \frac{1}{k!}\begin{vmatrix} \operatorname{Tr}\chi_A & k-1 & 0 & \cdots & 0 \\ \operatorname{Tr}\chi_{A^2} & \operatorname{Tr}\chi_A & k-2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \operatorname{Tr}\chi_{A^{k-1}} & \operatorname{Tr}\chi_{A^{k-2}} & \cdots & \operatorname{Tr}\chi_A & 1 \\ \operatorname{Tr}\chi_{A^k} & \operatorname{Tr}\chi_{A^{k-1}} & \cdots & \operatorname{Tr}\chi_{A^2} & \operatorname{Tr}\chi_A \end{vmatrix}.$$

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A remark on the second-order trace

The previously defined first-order trace satisfies

$$T_{\mathbb{H},1}(A) = 2\,\mathrm{Re}\left(\sum_{k=1}^{n} \lambda_k\right),$$

but still misses some information regarding the eigenvalues of A.
Such an information is found in the second-order trace

$$T_{\mathbb{H},2}(A) = \frac{1}{2}\begin{vmatrix} T_{\mathbb{H},1}(A) & 1 \\ T_{\mathbb{H},1}(A^2) & T_{\mathbb{H},1}(A) \end{vmatrix}$$
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The quaternionic Fredholm determinant $\widetilde{P}_{\chi_A}(z) = \det{}_{\mathbb{H}}(I - zA)$ satisfies the identity

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Example

Consider the matrix

$$A = \begin{pmatrix} 3+i & 0 \\ 0 & ij \end{pmatrix}.$$

The standard eigenvalues of A are $3+i$ and i, the characteristic polynomial is

$$\widetilde{P}_{\chi_A}(z) = \det(I - z\chi_A) = 1 - 6z + 11z^2 - 6z^3 + 10z^4.$$

Observe that the linear term has coefficient $-2\,\mathrm{Re}(3+i+ij) = -6$ and $T_{\mathbb{H},1}(A) = 6$, whilst for the quadratic term the coefficient is

$$T_{\mathbb{H},2}(A) = |3+i|^2 + |ij|^2 + 4\,\mathrm{Re}(3+i)\,\mathrm{Re}(ij) - 2\,\mathrm{Re}((3+i)(ij)) = 11.$$

- The fact that eigenvalues are equivalence classes of elements, explains the need of more information to be encoded in the notion of trace.
- For matrices one may refer to the companion matrix but what to do in the case of linear operators?
- For quaternionic linear operators the appropriate notion of spectrum is the S-spectrum (F. Colombo and I.S., 2006).

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Definition of the S-spectrum

Let $A \in \mathcal{B}(V)$. We define the S-spectrum $\sigma_S(A)$ of A as:

$$\sigma_S(A) = \{\lambda \in \mathbb{H} \quad : \quad A^2 - 2\mathrm{Re}(\lambda)A + |\lambda|^2\mathcal{I} \quad \text{is not invertible in } \mathcal{B}(V)\}.$$

and the S-resolvent set

$$\rho_S(A) = \mathbb{H} \setminus \sigma_S(A).$$

Properties of the S-spectrum

- The S-spectrum is axially symmetric: if $\lambda_0 = x_0 + jy_0 \in \sigma_S(A)$, then $x_0 + Iy_0 \in \sigma_S(A)$, for all $I \in \mathbb{S}$, i.e. $[\lambda_0] \in \sigma_S(A)$
- The point S-spectrum of A defined as

$$\{\lambda \in \mathbb{H} \ : \ \ker(A^2 - 2\,\mathrm{Re}(\lambda)A + |\lambda|^2\mathcal{I}) \neq \{0\}\}.$$

  coincides with the set of right eigenvalues.
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The second order operator

$$Q_\lambda(A) := (A^2 - 2\,\mathrm{Re}(\lambda)A + |\lambda|^2 I)^{-1}, \quad \lambda \in \rho_S(A),$$

is called the **pseudo S-resolvent operator**.

It is a quaternionic right linear operator and it is the linear operator associated with the eigenvalue problem $Av = v\lambda$ (which is not linear: $A - I\lambda$ is not a quaternionic right linear operator), i.e. $v \neq 0$ is such that

$$(A^2 - 2\mathrm{Re}(\lambda)A + |\lambda|^2 I)v = 0$$

and it is eigenvector related with $[\lambda]$

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It is a quaternionic right linear operator and it is the linear operator associated with the eigenvalue problem $Av = v\lambda$ (which is not linear: $A - I\lambda$ is not a quaternionic right linear operator), i.e. $v \neq 0$ is such that

$$(A^2 - 2\mathrm{Re}(\lambda)A + |\lambda|^2 I)v = 0$$

and it is eigenvector related with $[\lambda]$

Towards the infinite-dimensional (Hilbert) case

We can express a linear map A in a finite-dimensional space with respect to an arbitrary orthonormal basis $\{x_k\}_{k=1}^n$ as

$$A = \begin{pmatrix} \langle x_1, x_1' \rangle & \langle x_1, x_2' \rangle & \cdots & \langle x_1, x_n' \rangle \\ \langle x_2, x_1' \rangle & \langle x_2, x_2' \rangle & \cdots & \langle x_2, x_n' \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1' \rangle & \langle x_n, x_2' \rangle & \cdots & \langle x_n, x_n' \rangle \end{pmatrix},$$

with $x_k' = A^* x_k$ (here $A^* = (\overline{A})^T$ is the adjoint matrix of A).

We aim to study operators in Hilbert spaces represented by (infinite) matrices of a similar form.

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We aim to study operators in Hilbert spaces represented by (infinite) matrices of a similar form.

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For finite rank T operators we proved: $T_{\mathbb{H},1}(T) = 2\,\mathrm{Re}\left(\sum_{k=1}^{n}\langle x_k, x_k'\rangle\right)$
for $k \geq 2$,

$$T_{\mathbb{H},1}(T^k) = 2\sum_{m_1=1}^{n}\cdots\sum_{m_k=1}^{n}\mathrm{Re}(\langle x'_{m_2}, x_{m_1}\rangle\langle x'_{m_3}, x_{m_2}\rangle\langle x'_{m_4}, x_{m_3}\rangle\cdots$$

$$\langle x'_{m_k}, x_{m_{k-1}}\rangle\langle x'_{m_1}, x_{m_k}\rangle).$$

In particular, for $k \geq 1$, $T_{\mathbb{H},1}(T^k)$ does not depend on the choice of the vectors x_m, x'_m (since they can be written in terms of $T_{\mathbb{H},1}(T^k)$, $k = 1, \ldots, 2n$).

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$$T_{\mathbb{H},1}(T) = 2\,\mathrm{Re}\left(\sum_{k=1}^{n} \lambda_k\right),$$

$$T_{\mathbb{H},2}(T) = \sum_{k=1}^{n} |\lambda_k|^2 + 4\sum_{k=1}^{n-1}\sum_{m=k+1}^{n} \mathrm{Re}(\lambda_k)\,\mathrm{Re}(\lambda_m)$$

and

$$\det{}_{\mathbb{H}}(I - zT) = \sum_{k=0}^{2n} (-1)^k\, T_{\mathbb{H},k}(T) z^k = \prod_{k=1}^{n} \left(1 - 2\,\mathrm{Re}(\lambda_k) z + |\lambda_k|^2 z^2\right).$$

Trace-class operators

In a Hilbert space H, a linear operator $T : H \to H$ of the form

$$T = \begin{pmatrix} \langle x_1, x_1' \rangle & \langle x_1, x_2' \rangle & \cdots & \langle x_1, x_n' \rangle & \cdots \\ \langle x_2, x_1' \rangle & \langle x_2, x_2' \rangle & \cdots & \langle x_2, x_n' \rangle & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ \langle x_n, x_1' \rangle & \langle x_n, x_2' \rangle & \cdots & \langle x_n, x_n' \rangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is of trace-class if $\sum_{k=1}^{\infty} |\langle x_k, x_k' \rangle| < \infty$ (here $\{x_k\}$ is an orthonormal basis of H). This allows to define its trace. In the classical (complex) case,

$$\text{Tr } T := \sum_{k=1}^{\infty} \langle x_k, x_k' \rangle.$$

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$$\mathrm{Tr}\, T := \sum_{k=1}^{\infty} \langle x_k, x_k' \rangle.$$

Grothendieck-Lidskii formula (classical case)

The Fredholm determinant of T is defined as

$$\det(I - zT) = \sum_{k=0}^{\infty} (-1)^k z^k \operatorname{Tr}\left(\bigwedge^k T\right).$$

Recall:

$$\operatorname{Tr}\left(\bigwedge^k T\right) = \frac{1}{k!} \begin{vmatrix} \operatorname{Tr} T & k-1 & 0 & \cdots & 0 \\ \operatorname{Tr} T^2 & \operatorname{Tr} T & k-2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \operatorname{Tr} T^{k-1} & \operatorname{Tr} T^{k-2} & \cdots & \operatorname{Tr} T & 1 \\ \operatorname{Tr} T^k & \operatorname{Tr} T^{k-1} & \cdots & \operatorname{Tr} T^2 & \operatorname{Tr} T \end{vmatrix}.$$

Question

Can we relate the trace and Fredholm determinant of T with its eigenvalues?

Theorem (Grothendieck-Lidskii formula)

Let T be a trace-class operator (on a complex Hilbert space H), and let $\{\lambda_k\}$ be the sequence of eigenvalues of T. Then, the Fredholm determinant $\det(I - zT)$ is an entire function of order 1 and genus 0, and, moreover,

$$\mathrm{Tr}\, T = \sum_{k=1}^{\infty} \lambda_k,$$

$$\det(I - zT) = \prod_{k=1}^{\infty} (1 - z\lambda_k).$$

Invariants of operators in quaternionic Hilbert spaces

Let T be a linear operator acting on a quaternionic Hilbert space H.

$$T = \begin{pmatrix} \langle x_1, x_1' \rangle & \langle x_1, x_2' \rangle & \cdots & \langle x_1, x_n' \rangle & \cdots \\ \langle x_2, x_1' \rangle & \langle x_2, x_2' \rangle & \cdots & \langle x_2, x_n' \rangle & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ \langle x_n, x_1' \rangle & \langle x_n, x_2' \rangle & \cdots & \langle x_n, x_n' \rangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For a quaternionic matrix $A \in M_n(\mathbb{H})$, we have

$$T_{\mathbb{H},1}(A) = 2\,\mathrm{Re}\left(\sum_{k=1}^{n} \langle x_k, x_k' \rangle\right),$$

$$\det{}_{\mathbb{H}}(I - zA) = \sum_{k=0}^{2n} (-1)^k z^k T_{\mathbb{H},k}(A).$$

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For the operator T, we define

$$T_{\mathbb{H},1}(T) = 2\,\mathrm{Re}\left(\sum_{k=1}^{\infty} \langle x_k, x_k' \rangle\right),$$

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The definitions do not depend on the choice of the basis $\{x_k\}$

Quaternionic Grothendieck-Lidskii formula

Theorem (Cerejeiras, Colombo, Debernardi Pinos, Kähler, I. S.)

Let T be a trace-class operator on a quaternionic Hilbert space H, and let $\{\lambda_k\}$ be the sequence of **standard** eigenvalues of T. Then, the quaternionic Fredholm determinant $\det_{\mathbb{H}}(I - zT)$ is an entire function of order 1 and genus 0, and, moreover,

$$T_{\mathbb{H},1}(T) = 2\,\mathrm{Re}\left(\sum_{k=1}^{\infty} \lambda_k\right)$$

$$T_{\mathbb{H},2}(T) = \sum_{k=1}^{\infty} |\lambda_k|^2 + 4 \sum_{k=1}^{\infty} \sum_{m=k+1}^{\infty} \mathrm{Re}(\lambda_k)\,\mathrm{Re}(\lambda_m)$$

$$\det{}_{\mathbb{H}}(I - zT) = \prod_{k=1}^{\infty}(1 - z\lambda_k)(1 - z\overline{\lambda_k}) = \prod_{k=1}^{\infty}(1 - 2\,\mathrm{Re}(\lambda_k)z + |\lambda_k|^2 z^2).$$

Nuclear operators in quaternionic Banach spaces

Let X be a quaternionic Banach space and $0 < r \leq 1$. A linear operator $T : X \to X$ is said to be r-nuclear if it admits a representation

$$Tx = \sum_{k=1}^{\infty} x_k \cdot x_k'(x), \qquad x_k \in X, \quad x_k' \in X',$$

and

$$\|T\|_r := \inf \left(\sum_{k=1}^{\infty} \|x_k\|_X^r \|x_k'\|_{X'}^r \right)^{\frac{1}{r}} < \infty.$$

If $r = 1$ we say that T is nuclear.

Remark

If X is a Hilbert space, then nuclear operators are precisely trace-class operators.

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Classical case: nuclear operators and well-definiteness of the trace

Question

For nuclear operators, is it true that the quantity

$$\sum_{k=1}^{\infty} \langle x'_k, x_k \rangle$$

is well defined? (independently of the "basis" $\{x_k\}$, $\{x'_k\}$).

Answer: it depends.

- True if X possesses the approximation property.
- False in general.

Grothendieck showed that the answer is always positive for 2/3-nuclear operators

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Theorem (Quaternionic Grothendieck-Lidskii formula, 2024)

Let X be a quaternionic Banach space having the approximation property, $T \in L(X)$ be a $\frac{2}{3}$-nuclear operator, and let $\{\lambda_k(T)\}$ be the sequence of standard eigenvalues of T. Then:

$$T_{\mathbb{H},1}(T) = 2\,\mathrm{Re}\left(\sum_{k=1}^{\infty}\lambda_k(T)\right),$$

$$T_{\mathbb{H},2}(T) = \sum_{k=1}^{\infty}|\lambda_k(T)|^2 + 4\sum_{k=1}^{\infty}\sum_{m=k+1}^{\infty}\mathrm{Re}(\lambda_k(T))\,\mathrm{Re}(\lambda_m(T)),$$

and

$$\det{}_{\mathbb{H}}(I+T) = \prod_{k=1}^{\infty}(1+2\,\mathrm{Re}(\lambda_k(T))+|\lambda_k(T)|^2) = \sum_{k=0}^{\infty}T_{\mathbb{H},k}(T).$$

Note: Independence of x_n, x_n'.

Trace and tensor products

More generally, Grothendieck considered the general context of locally convex spaces E, and their tensor products, with operators represented as

$$T \sim \sum_{k=1}^{\infty} \mu_k (x_k \otimes x_k'), \qquad \mu_k \in \mathbb{R},\ x_k \in E,\ x_k' \in E'.$$

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Depending on the topological properties of E, and assumptions on T, the trace may be properly defined.

A few words on the trace functional

For a (real) vector space E and its dual E', the tensor product $E \otimes E'$ is defined as the free Abelian group with generators $x \otimes x'$, $x \in E$, $x' \in E'$, modulo the relations

1. $(x + y) \otimes x' = x \otimes x' + y \otimes x'$,
2. $x \otimes (x' + y') = x \otimes x' + x \otimes y'$,
3. $a(x \otimes x') = (ax) \otimes x' = x \otimes (ax') = (x \otimes x')a$, $a \in \mathbb{R}$.

Linear operators may be described in terms of tensor products as follows: to any tensor $x \otimes x'$, we associate the (rank 1) operator

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Tensor products of quaternionic spaces

Given a right $\mathbb{H}$-vector space E, with dual (left) $\mathbb{H}$-vector space E' (both $\mathbb{R}$-linear), $E \otimes E'$ is defined similarly as before, with the relations

1. $(x + y) \otimes x' = x \otimes x' + y \otimes x'$,
2. $x \otimes (x' + y') = x \otimes x' + x \otimes y'$,
3. $(xq) \otimes x' = x \otimes (qx')$, $q \in \mathbb{H}$ (balance property).

Remarks

1. $E \otimes E'$ can be given a left (or right) $\mathbb{H}$-linear structure via the relations $q(x \otimes x') = (xq) \otimes x'$.
2. In order for $x \otimes x'$ to properly represent a right $\mathbb{H}$-linear operator, we can only have $\mathbb{R}$-linear structure!
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A map $\Phi : E \times E' \to M$ is said to be balanced if

$$\Phi(xq, x') = \Phi(x, qx').$$

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For instance, the canonical map $(x, x') \mapsto x \otimes x'$ is balanced.

The trace functional: from real to quaternionic

If we wish to define a quaternionic trace for operators in $E \otimes E'$, it should respect the linear structure of such a space. Thus, we have to consider a trace that takes values on $\mathbb{R}$.

The quaternionic trace as the canonical form

If we consider $\Phi(x, x') = \mathrm{Re}(\langle x', x \rangle)$, we obtain an additive balanced form on $E \times E'$ (thus, it can be extended to $E \otimes E'$), which is a candidate for a canonical form on $E \otimes E'$:

Theorem

Any balanced $\mathbb{R}$-bilinear form $\Phi : E \times E' \to \mathbb{R}$ is the real part of a left and right $\mathbb{H}$-linear form $\Psi : E' \times E \to \mathbb{H}$.

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Proof

Theorem

Any additive balanced $\mathbb{R}$-bilinear form $\Phi : E \times E' \to \mathbb{R}$ is the real part of a and right/left $\mathbb{H}$-linear form $\Psi : E' \times E \to \mathbb{H}$.

Given Φ, define

$$\Psi(x, x') = \Phi(x, x') - \Phi(xi, x')i - \Phi(xj, x')j - \Phi(xij, x')ij.$$

It is clear that $\Phi = \operatorname{Re}\Psi$, and that Ψ is additive. Then one may directly check the linearity properties in the first and second entries.

From Banach spaces to locally convex spaces

In order to derive a trace formula in locally convex spaces, we first have to introduce the relevant operators.

Remark

In Banach spaces X, we defined r-nuclear operators

$$Tx = \sum_{k=1}^{\infty} x_k \cdot x_k'(x), \qquad x_k \in X, \quad x_k' \in X',$$

by the condition (which defines a semi-norm on the linear space of such operators)

$$\|T\|_r := \inf \left(\sum_{k=1}^{\infty} \|x_k\|_X^r \|x_k'\|_{X'}^r \right)^{\frac{1}{r}} < \infty.$$

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From Banach spaces to locally convex spaces

Let E be a locally convex right linear space on $\mathbb{H}$. A right-linear operator $T : E \to E$ is called a *Fredholm operator* if it is of the form

$$Tx = \sum_{k=1}^{\infty} \mu_k x_k \langle x'_k, x \rangle, \qquad x \in E, \tag{1}$$

where $\{\mu_k\} \in \ell^1$ is a real sequence, and $\{x_k\}$ (resp. $\{x'_k\}$) is contained in a suitable absolutely convex set $B \subset E$ (resp. $B' \subset E'$).

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We denote by $E\overline{\overline{\otimes}}E'$ the subspace of $E\overline{\otimes}E'$ consisting of all elements of the form

$$u = \sum_{k=1}^{\infty} \mu_k x_k \otimes x'_k, \tag{2}$$

with $\{\mu_k\}$, $\{x_k\}$, and $\{x'_k\}$ as before. We say that $u \in E\overline{\overline{\otimes}}E'$ is a *Fredholm kernel* on E.

Let $0 < p \leq 1$. We say that $u \in E\overline{\overline{\otimes}}E'$ is a *p-summable Fredholm kernel* on E if u is a Fredholm kernel on E, with $\{\mu_k\} \in \ell^p$.

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From Banach spaces to locally convex spaces

Let us denote by $\mathcal{L}(E)$ the $\mathbb{R}$-linear space of all weakly continuous right-linear operators in E endowed with the weak operator topology, given by the semi-norms $A \to |\langle x', Ax\rangle|$, with $A \in \mathcal{L}(E)$, $x \in E$, and $x' \in E'$. Let us denote by Γ the $\mathbb{R}$-linear mapping $E \otimes E' \to \mathcal{L}(E)$, under which the tensor

$$u = \sum_{k=1}^{n} \mu_k x_k \otimes x'_k$$

is mapped into the finite-rank operator

$$\Gamma(u) : \ x \mapsto \sum_{k=1}^{n} \mu_k x_k \langle x'_k, x\rangle,$$

From Banach spaces to locally convex spaces

Let E be a locally convex right linear space on $\mathbb{H}$ and let $0 < p \leq 1$ and let $u \in E\overline{\overline{\otimes}}E'$ be a *p-summable Fredholm kernel* on E. The image $\Gamma(u)$ of such kernel is called *p-summable Fredholm operator*. If the sequence $\{x'_k\}$ defining u is equicontinuous, $\Gamma(u)$ is called a *p-nuclear operator*.

The problem of well-defined traces

Question

How do we understand the trace of the operator $T = \Gamma(u)$?

Simple answer: set $T_{\mathbb{H},1}(\Gamma(u)) := T_{\mathbb{H},1}(u)$. Given $u \in E\overline{\otimes}_r E'$, one may define the (quaternionic) trace of u as

$$T_{\mathbb{H},1}(u) = \sum_{k=1}^{\infty} \mu_k \langle x'_k, x_k \rangle.$$

Problem: Γ need not be bijective. As in the Banach space case, this is related to approximation properties of E. This is called the **uniqueness problem**.

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The uniqueness problem and 2/3-nuclear operators

For 2/3-nuclear operators we show that the uniqueness problem has a positive solution, i.e.,

$$T_{\mathbb{H},1}(\Gamma(u)) = T_{\mathbb{H},1}(u)$$

is well defined (no approximation property on E is required due to the "good enough" behavior of 2/3-nuclear operators). In fact, we show that

$$T_{\mathbb{H},1}(u) = 2\,\mathrm{Re}\left(\sum_{k=1}^{\infty} \lambda_k(\Gamma(u))\right),$$

where $\{\lambda_k(\Gamma(u))\}$ are the standard eigenvalues of $\Gamma(u)$.

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Further comments

We introduce (formally) the Fredholm determinant of u:

$$\det{}_{\mathbb{H}}(I - zu) = \sum_{k=0}^{\infty} (-1)^k T_{\mathbb{H},k}(u) z^k,$$

and show that it has the expected properties.

Remark

- The trace formula follows from the fact that $\det{}_{\mathbb{H}}(I - zu)$ is of order one and genus zero.
- We also show that for 2/3-nuclear operators $T = \Gamma(u)$, we have $\{\lambda_k(T)\} \in \ell^1$.

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Thank you for your attention!