PROBABILISTIC OPERATOR ALGEBRA SEMINAR. 10/21/24

STRONG ASYMPTOTIC FREENESS OF HAAR UNITARIES IN QUASIEXPONENTIAL-DIMENSIONAL REPRESENTATIONS.

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STATEMENT OF RESULT.

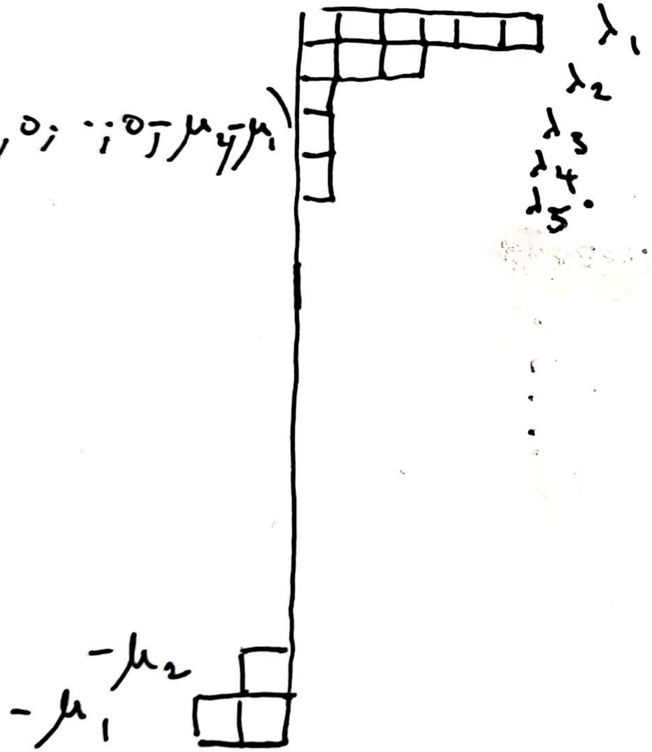
IRREDUCIBLE REPRESENTATIONS OF $\mathcal{U}(n) \overset{1:1}{\longleftrightarrow}$ DOMINANT WEIGHTS.

= NON INCREASING SEQUENCES IN $\mathbb{Z}^n$.

IN THIS TALK: ONLY CONSIDER SEQUENCES WITH ZEROS.

SEQUENCE

$(\lambda_1, \lambda_2, \ldots, \lambda_5, 0, 0, \ldots, 0, -\mu_2, -\mu_1)$



THESE ARISE FROM PAIRS OF YOUNG DIAGRAMS.

λ, μ WITH $|\lambda| + |\mu| < n$

CORRESPONDING REP. $(V^{\lambda,\mu}, \rho_{\lambda,\mu})$.

$\rho_{\lambda,\mu} : \mathcal{U}(n) \to V^{\lambda,\mu}$.

$n > \ell(\lambda) + \ell(\mu)$.

STRONG CONVERGENCE / STRONG ASYMPTOTIC FREENESS.

If $U_1^{(n)}, \dots, U_r^{(n)}$ are (random) unitaries of dimension $d(n)$, $n = 1, \dots, \infty$, say they are (almost surely) <u>strongly asymptotically free</u> if for any $*$-polynomial p,

$$\| p(U_1^{(n)}, \dots, U_r^{(n)}) \|_{\mathrm{Mat}_{d \times d}} \longrightarrow \| p(X_1, \dots, X_r) \|_{C^*_{\mathrm{red}}(F_r)}$$

(almost surely) where $X_1, \dots, X_r$ generate free group F_r.

Also say the representations $\rho_n : F_r \to U(d(n))$, $X_i \mapsto U_i^{(n)}$ strongly <u>converge</u> to the regular representation.

MAIN RESULT. (M - de la Salle).

Let $U_1^{(n)}, \dots, U_r^{(n)}$ be i.i.d. Haar in $U(n)$.

Let $\epsilon > 0$. If. $|\lambda(n)| + |\mu(n)| < n^{\frac{1}{24} - \epsilon}$

Then $\rho_{\lambda(n),\mu(n)}(U_1^{(n)}), \dots, \rho_{\lambda(n),\mu(n)}(U_r^{(n)})$

are a.s. strongly asymptotically free.

In fact this is uniform over all $\lambda(n), \mu(n)$ in this regime.

MOTIVATION 1. $(n \geq 2)$. THE RUZIEWICZ PROBLEM.

RUZIEWICZ: (1921) IS LEBESGUE MEASURE ON n-SPHERE THE UNIQUE FINITELY ADDITIVE ROTATION-INVARIANT MEASURE ON THE LEBESGUE SETS?

SUFFICES TO FIND $\mathcal{O}_1^{(n)}, \dots, \mathcal{O}_r^{(n)} \in SO(n)$, $r \geq 2$.

S.T. $\| \underbrace{\lambda_{SO(n)/SO(n-1)}} (\mathcal{O}_1^{(n)} + \dots + \mathcal{O}_r^{(n)} + \mathcal{O}_1^{(n)-1} + \dots + \mathcal{O}_r^{(n)-1}) \| < 2r.$

rep by translations on mean zero L^2 functions on $SO(n)/SO(n-1)$.

STP: $\sup_{\substack{\pi < \lambda \\ \pi \text{ irreducible}, \pi \neq \text{triv.}}} \| \pi (\mathcal{O}_1^{(n)} + \dots + \mathcal{O}_r^{(n)} + \mathcal{O}_1^{(n)-1} + \dots + \mathcal{O}_r^{(n)-1}) \| < 2r.$

INFINITELY MANY IRREPS HERE.

RESOLVED 80's MARGULIS, SULLIVAN $n \geq 4$.

DRINFELD. $n = 3$ USING DELIGNE + JACQUET-LANGLANDS 1984.

NUMBER THEORY REMOVED BY BOURGAIN-GAMBURD 2008

– SUFFICES $D_i^{(2)}$ HAVE ALG. ENTRIES AND GEN. DENSE SUBGROUP.

STILL OPEN: DO GENERIC IN MEASURE (HAAR) $(D_i^{(2)}) \in SO(3)$ HAVE A SPECTRAL GAP.? (IN THIS CASE ALL $\pi \in \widehat{SO(3)}$ – triv APPEAR).

GAMBURD-JAKOBSON-SARNAK).

D. FISHER: IT IS ZERO OR ONE!

MOTIVATION 2. CAYLEY GRAPHS.

cf. Kassabov

OPEN QUESTION IS A UNIFORMLY RANDOM $d = 2r$ - REGULAR CAYLEY GRAPH OF S_n A UNIFORM EXPANDER WITH $\mathbb{P} \to 1$ AS $n \to \infty$?

A RELAXATION: PICK $\sigma_1, \ldots, \sigma_r \in S_n$ i.i.d. UNIFORM. AND FOR $1 \leq k \leq n$. FORM THE SCHREIER GRAPH.

OF $\sigma_1, \ldots, \sigma_r$ $\curvearrowright$. DISTINCT K-TUPLES IN $\{1, \ldots, n\}$. ORDERED.

def.

$= G_{k,n}(\sigma_1, \ldots, \sigma_r)$.

EG $k=1$ EDGE FROM i TO j FOR EACH

$2r$ - reg. graph on n vertices.

$k=1$ FRIEDMAN. $\sigma_1, \ldots, \sigma_r$ AS BEFORE, STD $n-1$ DIM IRREP OF S_n

WITH $\mathbb{P} \to 1$ AS $n \to \infty$

$$\| \mathrm{STD}(\sigma_1 + \sigma_1^{-1} + \cdots + \sigma_n + \sigma_n^{-1}) \| \to \overbrace{2\sqrt{2r-1}}^{\text{RAMANUJAN.}} = \| \lambda_{F_r}(x_1 + x_1^{-1} + \cdots + x_r + x_r^{-1}) \|$$

EXTENDED TO ALL NC $*$-POLYS OF $\sigma_1, \ldots, \sigma_n$

BY BORDENAVE - COLLINS 2019 AND EXTENDED TO

IRREPS $\sim$ $k=2$ (QUANTUM EXPANDERS).

THIS YEAR: CHEN - GARZA - VARGAS - TROPP - VAN HANDEL

- A NEW AND REMARKABLE PROOF OF THE PREVIOUS 2 RESULTS AND EXTENSION TO IRREPS ~ ANY FIXED K AS $n \to \infty$ (STABLE REPRESENTATIONS).

MORE ON THIS SHORTLY...

CASSIDY (IN PREPARATION) (DURHAM PHD THESIS).

$G_{k,n}(\sigma_1, \ldots, \sigma_r)$ HAS SAME PROPERTIES AS IN FRIEDMAN'S THEOREM.

(ALSO STRONG CONVERGENCE)

PROVIDED $k \leq n^{\alpha}$, $\alpha > 0$ ABSOLUTE.

($\alpha = 1$ WOULD RESOLVE PROBLEM ON CAYLEY GRAPHS).

MAIN INGREDIENTS IN MAIN THEOREM PROOF.

(1) WE RELY CRUCIALLY ON THE METHOD OF [C-GV-T-VH].

WE ALSO EXTEND THIS METHOD TO AVOID PISIER'S LINEARIZATION.

(TO PROVE STRONG CONVERGENCE IT SUFFICES TO PROVE FOR LINEAR POLYS WITH MATRIX COEFFICIENTS) .

$$\mathrm{Tr}^{(n)}_{\lambda,\mu}(w) \overset{\text{def.}}{=}$$

(2) WE RELY ON ESTIMATES FOR $\mathbb{E}[\mathrm{Tr}(\rho_{\lambda,\mu}(w(U_1^{(n)},\ldots,U_r^{(n)})))]$ (*).

that improve as $|\lambda|$ and $|\mu|$ get larger (up to a certain point)

First observed in M-Puder 'Matrix group integrals .... I'.

THE ISSUE OF WHETHER THERE EXIST

$\sigma_1^{(n)}, \ldots, \sigma_r^{(n)}$ in S_n.

s.t.

$\sigma_i^{(n)} \curvearrowright \ell_0^2(S_n)$ are strongly asymptotically free

also intervenes in the following problem.

Do there exist $\phi_n : F_2 \times F_2 \to S_n$

such that $\mathrm{std} \circ \phi_n \xrightarrow{\text{STRONG}} \lambda_{F_2 \times F_2}$.

[See forthcoming survey. [M] Proc. E.C.M.]

[Not hard to check ϕ_n has to be of form $\phi_1 \times \phi_2$.

$\phi_1 : F_2 \to H \overset{\text{left}}{\curvearrowright} \ell_0^2(H)$. $\quad H$ a finite group

$\phi_2 : F_2 \to H \overset{\text{right}}{\curvearrowright} \ell_0^2(H)$.]

Let $X^{(n)} = \rho_{\lambda,\mu}\left(U_1^{(n)} + \cdots + U_r^{(n)} + U_1^{(n)-1} + \cdots + U_r^{(n)-1}\right)$.

In fact, suppose $\mu = \phi$ (polynomial representation).

Most naive approach to a.s. bound $\|X^{(n)}\|$

is perhaps to estimate

$$\mathbb{E}[\mathrm{Tr}((X^{(n)})^{2p})] = \sum_{w \in W_{2p}} \underbrace{\mathbb{E}[\mathrm{Tr}(\rho_{\lambda,\mu}(w(-)))]}_{\text{as in } (*)}.$$

$$\vee\!/$$

$$\mathbb{E}[\|X^{(n)}\|^{2p}].$$

And input: if w has certain algebraic properties

$$\mathrm{Tr}^{(n)}_{\lambda,\mu}(w) = \frac{c_w}{n^{K(w)}} + O\left(\frac{1}{n^{K(w)+1}}\right).$$

Then one can try to make the O uniform over $|w| < c \log n$, and count # of $w \in W_{2p}$ with $K(w) = K_0$ etc.

it's a mess

Skipping ahead, Doron and I proved. for λ fixed ($\mu = \phi$).

$$\mathrm{Tr}_{\lambda}^{(a)}(w) = O\left(\frac{1}{n^{2\,\mathrm{scl}(w)|\lambda|}}\right).$$

cf. Voiculescu RMT paper

for fixed w and $n \to \infty$. $(w \neq \mathrm{id})$.

$$\mathrm{cl}(w) = \inf\left(\{ g : w = [u_1, v_1] \cdots [u_g, v_g] \text{ in } F_r\} \cup \{\infty\}\right)$$

$$\mathrm{scl}(w) = \lim_{m \to \infty} \frac{\mathrm{cl}(w^m)}{m}.$$

Thm (Duncan–Howie) If $w \neq \mathrm{id}$, $\mathrm{scl}(w) \geq \frac{1}{2}$.

Hence. $\mathrm{Tr}_{\lambda}^{(a)}(w) = O\left(\frac{1}{n^{|\lambda|}}\right) = O\left(\frac{1}{\dim(\rho_\lambda)}\right).$

Cor. (M-Puder).

(for fixed w, as $n \to \infty$.).

[C-GV-T-VH]. METHOD:

TAKES ESTIMATES LIKE THIS AS INPUT.

DEPENDENCE OF CONSTANTS IN 'O' ON W ETC HANDLED DIFFERENTLY.

SKETCH. $X^{(n)}$ AS BEFORE, λ FIXED. h POLYNOMIAL DEGREE q.
$X^{\infty} = x_1 + x_1^{-1} + \cdots + x_r + x_r^{-1} \in \mathbb{C}[F_r]$. τ CAN. TRACE ON $C^*_{red}(F_r)$.

A PRIORI BOUND: $\mathbb{E}[\mathrm{tr}(X^{(n)})^{2p}]$

$$\mathbb{E}[\mathrm{tr}_\lambda(h(X^{(n)}))] \leq \underbrace{\sup_{[-2r,2r]} |h| \cdot 4r}_{\|h\|_0}.$$

IS PROPAGATED TO

$$|\mathbb{E}[\mathrm{tr}_\lambda(h(X^{(n)}))] - \tau(h(X^{\infty}))| \leq \frac{C q^4 \|h\|_0}{n}. \quad (\dagger)$$

USING FUNDAMENTAL THEOREM OF CALCULUS + MARKOV BROTHERS INEQUALITY.

$\dagger)$ IS NOT ENOUGH FOR C-GV-T-VH, WE'LL RETURN TO THIS.

THIS USES:

$\triangle$ $\mathbb{E}[\operatorname{tr}_\lambda(h(X^{(n)}))]$ is essentially a polynomial in n^{-1} with degree $\ll |\lambda| q$ for $n \gg q|\lambda|$.

$\square$ $\mathbb{E}[\operatorname{tr}_\lambda(h(X^{(n)}))] - \tau(h(X^\infty))$ is $O(1/n)$ as $n \to \infty$.

$[X^\infty = x_1 + x_1^{-1} + \cdots + x_r + x_r^{-1} \in \mathbb{C}[F_r]]$.

i.e. $\tau(h(X^\infty))$ is the zeroth term of Taylor series in n^{-1}.

Since for us, $O(\frac{1}{n})$ can be replaced by $O\left(\frac{1}{n^{|\lambda|+1}}\right)$

we can Taylor expand to this order

and use higher order version of MB inequality.

then one can take λ up to the point where this proof breaks down.

The result is along lines of.

$$\left| \mathbb{E}[\mathrm{tr}_\lambda(h(X^{(n)}))] - \tau(h(X^\infty)) \right| \leq \frac{C_{|\lambda|}\, q^{4|\lambda|}}{n^{|\lambda|}} \|h\|_0 .$$

'Standard' results in FA then $\Rightarrow$. above extends

to $h \in C^k[-2r, 2r]$.

$k = k(|\lambda|) . < \infty$.

$= K|\lambda|$.

$K > 0$.

AND THE LARGE GAIN IN POWERS OF n MEANS

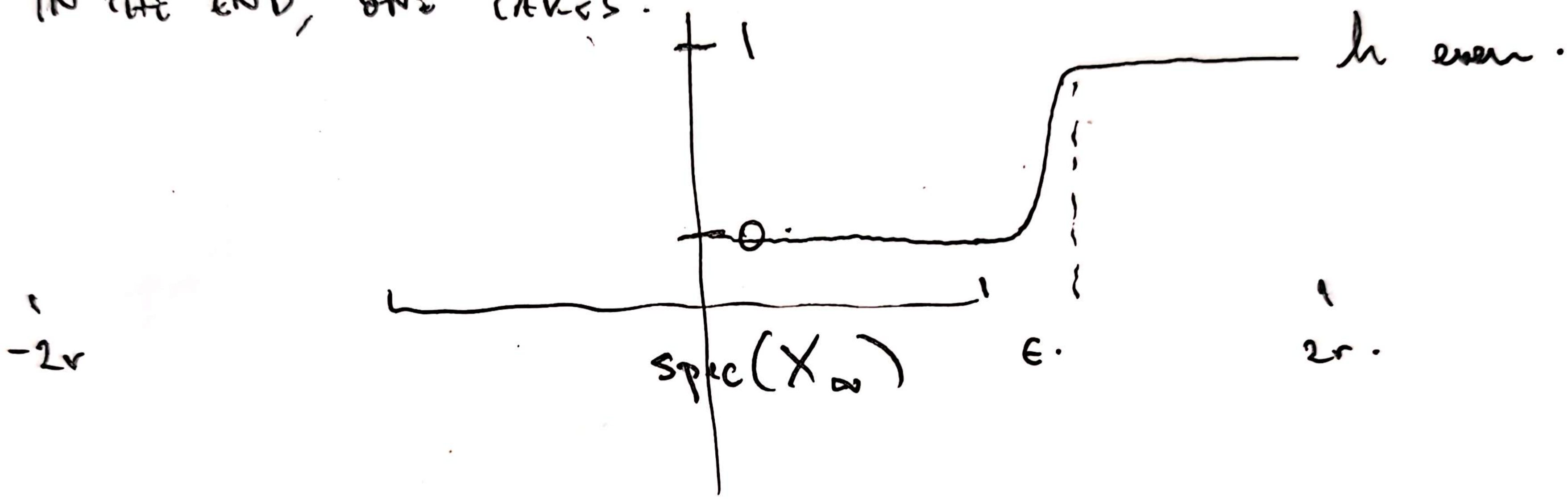
i). RESULTING LINEAR FUNCTIONALS REMAIN UNIFORMLY BOUNDED. ON C^k

2) WE CAN SUM OVER $|\lambda| < n^\alpha$ AND GET UNIFORM BOUNDS.

NOTE: $C_{|\lambda|}$ BEHAVES LIKE $|\lambda|^{\beta|\lambda|}$

WHICH $=$ $n^{|\lambda|}$ WHEN $|\lambda| = n^{1/\beta}$.

IN THE END, ONE TAKES.



$\Rightarrow$ $\mathbb{E}[\mathrm{Tr}_d(h(X^{(n)}))] = o(1)$ as $n \to \infty$.

Conclude using matrix concentration there are no eigenvalues outside $\mathrm{spec}(X_\infty) + [-\epsilon, \epsilon]$.

THIS OVERVIEW PASSED OVER TWO IMPORTANT POINTS.

(A) . THE ESTIMATE $\mathbb{E}[\mathrm{Tr}_\lambda(w)] = O\left(\frac{1}{n^{|\lambda|}}\right)$ $w \neq \mathrm{id}$.

does not extend to general pairs λ, μ

and in fact, $\mathbb{E}[\mathrm{Tr}_{\lambda,\mu}(w)] = \Omega(1)$

if w is a ~~f~~ proper power and λ, μ are certain YDs.

- this is also the case for [C-GV-T-VH]
  and how we deal with this extends their method.

(B) If w is <u>not</u> a proper power we do not know. (before now)

anything like $\mathbb{E}[\mathrm{Tr}_{\lambda,\mu}(w)] = O\left(\frac{1}{n^{|\lambda|+|\mu|}}\right)$ as $n \to \infty$.

(in fact this is still open).

Instead we proved $\mathbb{E}[\mathrm{Tr}_{\lambda,\mu}(w)] = O\left(\frac{1}{n^{\frac{1}{3}(|\lambda|+|\mu|)}}\right)$

which suffices, but proving this is not easy.

ISSUE (A). (PROPER POWERS). ALSO ARISES FOR [C-GV-T-VH].

LEADS TO. (GOING BACK TO (+)) AFTER AN ITERATION OF MB $\leq$

$$\left| \mathbb{E}[\mathrm{tr}(h(X^{(n)}))] - \tau(h(X^{\infty})) - \frac{\nu[h(X^{\infty})]}{n} \right| \leq \frac{C q^{8} \|h\|}{n^{2}}.$$

WHERE ν IS A NON-ZERO FUNCTIONAL ON $\mathbb{C}[F_r]$.

IBID. PROVE 1) ν EXTENDS TO $h(X^{\infty})$: $h \in C^{k}[-2r, 2r]$
AND. (IMMEDIATE FROM PREVIOUS).

2) ν VANISHES ON $h(X^{\infty})$ IF h IS SUPPORTED OFF. $\mathrm{spec}(X_{\infty})$.

— ν IS 'TEMPERED' IN OUR LANGUAGE.

(A) ctd.

BECAUSE OF LINEARIZATION, FOR C(IBID).

PT 2) BOILS DOWN TO GENERALIZED COUNTING OF PROPER POWERS.

WRT WORD LENGTH IN STANDARD GENERATORS.

BECAUSE WE DON'T LINEARIZE, WE DO SOMETHING NEW:

i) WE CLARIFY THE RELATION BETWEEN TEMPEREDNESS

IN SENSE $\limsup_{n\to\infty} |\tau((x^*x)^n)|^{1/n} \leq \|\lambda(x)\|. \quad \forall x \in \mathbb{C}[\Gamma]$

sup

OF 'LAURENT COEFFICIENTS' (LIKE τ ABOVE).

AND STRONG CONVERGENCE.

ii) WE GIVE A CRITERION FOR TEMPEREDNESS OF FUNCTION ON F_r. IN TERMS OF ITS VALUES ON THE LAWS OF RANDOM WALKS. (IN PARTICULAR, POSITIVE FUNCTIONS).

iii) FOR PARTICULAR RANDOM WALKS RELEVANT TO ii), GIVE ESTIMATES FOR HOW LIKELY THEY ARE TO HIT THE PROPER POWERS IN F_r.

FINITE, >0 MASS AT ID, SUPPORT GENERATES F_r SYMMETRIC

(B) $V^{\lambda,\mu}$ APPEARS FOR THE FIRST TIME

{ RAPID DECAY OF EXPECTED TRACES.

IN $(\mathbb{C}^n)^{\otimes k} \otimes ((\mathbb{C}^n)^\vee)^{\otimes \ell} \overset{\text{def.}}{=} V_{k,\ell}, \rho_{k,\ell}$

WITH $k = |\lambda|$, $\ell = |\mu|$ BUT ALSO APPEARS FOR LARGER k, ℓ.

EG: IF $k = \ell$ $V_{k,k}$ CONTAINS $V_{\lambda',\mu'}$

WHEN $|\lambda'| = |\mu'| < k$. (BECAUSE $\mathbb{C}^n \otimes (\mathbb{C}^n)^\vee$ has inv. vector).

IF $tr_{k,\ell}$ IS THE NORMALIZED TRACE OF $\rho_{k,\ell}$

DO NOT EXPECT $\mathbb{E}[tr_{k,\ell}(w)] = O\left(\frac{1}{n^{k+\ell+1}}\right)$

WORD IN HAAR UNITARIES.

IF w NOT A PROPER POWER.

IT <u>IS</u> TRUE THAT. $\rho_{k,\ell}^{(w)}$ IS LIN. COMBO OF.

$$tr(w^{k_1}) \dots tr(w^{k_p})\, tr(w^{-\ell_1}) \dots tr(w^{-\ell_p})$$

WITH $\sum_1 k_i = k$ AND $\sum \ell_i = \ell$.

<u>BUT</u> THE COEFS ARE PRODUCTS OF

LR COEFS. (GOOD LUCK...).

INSTEAD.

M – RANDOM UNITARY REPRESENTATIONS OF SURFACE GROUPS.

GIVES.

i) A FORMULA FOR THE PROJECTION $p_{\lambda,\mu}$

ONTO A COPY OF $V_{\lambda,\mu}$ IN $(\mathbb{C}^n)^{\otimes k} \otimes (\check{\mathbb{C}}^n)^{\otimes \ell}$

AS LIN. COMBINATION OF ELEMENTS OF $\mathbb{C}[S_{k+\ell}]$. VIA.

$$\mathrm{End}\left((\mathbb{C}^n)^{\otimes k} \otimes (\check{\mathbb{C}}^n)^{\otimes \ell}\right) \overset{\text{FIX}}{\cong} \mathrm{End}\left((\mathbb{C}^n)^{\otimes k+\ell}\right).$$

$$\uparrow$$

$$\mathbb{C}[S_{k+\ell}].$$

ii) A PRIORI $p_{\lambda,\mu}$ KILLS THE ORTHOCOMPLEMENT

TO THE INTERSECTION OF ALL CONTRACTIONS.

$$c_{p,q} : (\mathbb{C}^n)^{\otimes k} \otimes (\check{\mathbb{C}}^n)^{\otimes \ell} \longrightarrow (\mathbb{C}^n)^{\otimes k-1} \otimes (\check{\mathbb{C}}^n)^{\otimes \ell-1}.$$

$\mathring{T}^{k,\ell} =$ this intersection.

This technology arises from

Koike: on $\dot{T}^{k,l}$, $U(n)$ and $S_k \times S_l$

are full mutual centralizers.

(and only 'new' $V_{\lambda,\mu}$ appear here).

i) and ii) lead to a diagrammatic expansion (via Wg calculus).
where

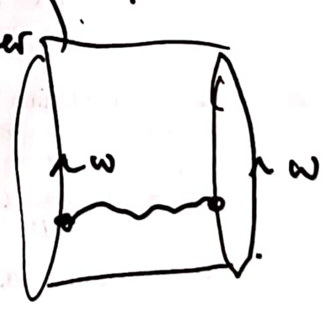
"one does not see diagrams arising from lower k and l"

eg. $k = l = 1$ $\lambda = \square$ $\mu = \square$.

$$\mathrm{Tr}_{\lambda,\mu}(g) = \mathrm{Tr}(g)\mathrm{Tr}(g^{-1}) - 1 = |\mathrm{Tr}(g)|^2 - 1.$$

using Wg to do this: (via M-Puder)

$\mathrm{Tr}(\omega)\mathrm{Tr}(\omega^{-1})$ gives an annulus:



giving a '1'
cancelling the -1.

But this example has bad property that there are
two 'parallel' w: so would work for any w.

In our method, this diagram never appears!

Next, we prove a topological result
(weaker, but more robust version of Duncan-Howie)
that controls orders of terms in diagrammatic
expansion.

Key ideas:

If w not a proper power, and cyclically reduced,

two equal subsegments of w^{∞} of length $> |w|$
begin at the same point of w.

no two nonempty subsegments of $w^{+\infty}$ and $w^{-\infty}$ of length
$> |w|$ can be equal.

The End

Thanks for your attention.