Supports of free convolutions

joint work with C.-W. Ho and S.T. Belinschi

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Connected components

- $\mu_1, \mu_2 \in \mathscr{P}_{\mathbb{R}}$, $\mu_1 \boxplus \mu_2$ their free convolution, $\mu_1^{\boxplus t}$ tth power $(t > 1)$
- If supp(μ) has finitely many components, $\mu^{\boxplus t}$ may have more components for t small. The number decreases along with $1/t$ and reduces to 1 eventually (Huang; earlier related results by Biane and V-B)
- If $\mu_{1,}\mu_2$ have (bounded) connected supports and Jacobi type densities, then $\mu_1 \boxplus \mu_2$ has connected support, good endpoint behavior (Bao, Erdös, Schnelli; multiplicative $\mathbb{R}_+$ analog by Ji)
- If μ_j has n_j (bounded) connected components in supp, Jacobi densities on each component, then $\mu_1 \boxplus \mu_2$ has $< 2n_1 n_2$ components, good endpoint behavior, compatible with observed RM models. (Moreillon-Schnelli; lower estimates sometimes)
  $(\mu_1 * \mu_2$ has at most $n_1 n_2$ components)

Is regularity necessary? bounded support?

- No.

Theorem
If μ_j has n_j (bounded) connected components, then $\mu_1 \boxplus \mu_2$ has $< 2n_1 n_2$ components in supp. When $n_1 = n_2 = 1$, components need not be bounded.

- Bounded version derived from M-S via simple spectral theory.
- Unbounded version (connected spectrum) seems to require a direct argument. The delicate combinatorics of M-S may perhaps be reproducible when $n_1 n_2 > 1$ in order to count the bounded components. Optimal uper bound is probably smaller.

Spectral argument

- $\mathscr{A}$ Banach algebra, $x, y \in \mathscr{A}$ with $\|x - y\| < \varepsilon_x$, then the spectrum of y has at least as many components as the spectrum of y. (Folklore? Maybe, but see Newburgh-1951.)
- $(\mathscr{A}, \tau)$ tracial W^*-probability space, $x_1, x_2 \in \mathscr{A}$ free, selfadjoint, with distribution μ_1, μ_2, so $\mu_{x_1+x_2} = \mu_1 \boxplus \mu_2$
- One can find, perhaps in a larger algebra, y_1, y_2 free, selfadjoint, $\|y_j - x_j\| < \varepsilon_{x_1+x_2}/2$, so y_j satisfy the hypotheses of M-S (same n_1, n_2)
- Thus $\operatorname{supp}(\nu_1 \boxplus \nu_2)$ has $< 2n_1 n_2$ components.
- $\operatorname{supp}(\nu_1 \boxplus \nu_2) = \sigma(y_1 + y_2)$ has at least as many components as $\sigma(x_1 + x_2)$, so $\sigma(x_1 + x_2) = \operatorname{supp}(\mu_1 \boxplus \mu_2)$ also has $< 2n_1 n_2$ components.

Preliminaries

Lemma

(Lehner) X a topological space, $u_1, u_2 : X \to \mathscr{A}$ norm-continuous functions, for every $x \in X$, $u_1(x)$ and $u_2(x)$ are $*$-free, $\tau(u_1(x)) = \tau(u_2(x)) = 0$, and $1 - u_1(x)u_2(x)$ is invertible. Then

$$Y = \{x \in X : \|u_1(x)\|_2 \|u_2(x)\|_2 < 1\}$$

is both open and closed in X. If X is connected, either $Y = X$ or $Y = \varnothing$.

Preliminaries

Example

$x = x^*$ affiliated with $\mathscr{A}$, $G_x(\lambda) = \tau((\lambda - x)^{-1})$, $F_x(\lambda) = 1/G(\lambda)$ $(\lambda \in \mathbb{H} = \{z : \Im z > 0\})$. Then

$$\sphericalangle \lim_{\lambda \to \infty} \|F(\lambda)(\lambda - x)^{-1} - 1\|_2 = 0.$$

Fact

Suppose $a_j = b_j^{-1} - 1_{\mathscr{A}}$, $b_1, b_2 \in \widetilde{\mathscr{A}}$ *boundedly invertible. Then:*

$$1_{\mathscr{A}} - a_1 a_2 = b_1^{-1}(1_{\mathscr{A}} - b_1 - b_2)b_2^{-1}.$$

(Haagerup applies this when $\tau(b_1^{-1}) = \tau(b_2^{-1}) = 1$ *to prove additivity of R. Also helps verify lemma above.)*

Preliminaries

- $\varphi : \mathbb{H} \to \mathbb{H}$ analytic, $\varphi^n = \underbrace{\varphi \circ \cdots \circ \varphi}_{n \text{ times}}$ converges to a constant $\lambda_\varphi \in \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ unless φ is a hyperbolic rotation. (Denjoy-Wolff, 1920s)
- λ_φ depends continuously on φ. (Heins, 1951)
- If $\lambda_\varphi \in \mathbb{H}$, then $|\varphi'(\lambda_\varphi)| < 1$
- If $\lambda_\varphi \in \mathbb{R}$, then $\varphi'(\lambda_\varphi) \leq 1$. (Carathéodory-Julia derivatives exist at such points.) Similar statement for $\lambda_\varphi = \infty$ via conformal map. φ may have many "fixed" points on the real line, but only one of them can satisfy this derivative condition.

Subordination

- x_1, x_2 selfadjoint affiliated with $\mathscr{A}$, $\mu_j = \mu_{x_j}$, $\mu = \mu_1 \boxplus \mu_2$, $G_\mu = G_x$, $F_\mu = F_x$, $h_j(\lambda) = F_j(\lambda) - \lambda$. Suppose x_j is not a scalar multiple of 1.
- $\varphi_\alpha^{(1)}(\lambda) = \alpha + h_2(\alpha + h_1(\lambda)), \varphi_\alpha^{(2)}(\lambda) = \alpha + h_1(\alpha + h_2(\lambda))$
- For $\alpha \in \mathbb{H} \cup \mathbb{R}$ and $j = 1, 2$, denote by $\omega_j(\alpha)$ the Denjoy-Wolff point of $\varphi_\alpha^{(j)}$. Then

Theorem

1. *ω_j is continuous on $\mathbb{H} \cup \mathbb{R}$, analytic on $\mathbb{H}$*
2. *$F_\mu(z) = F_{\mu_1}(\omega_1(z)) = F_{\mu_2}(\omega_2(z)) = \omega_1(z) + \omega_2(z) - z$ for every $z \in \mathbb{H}$.*
3. $\lim_{y \uparrow \infty} \omega_j(iy)/iy = 1$ *for $j = 1, 2$, and*

More preliminaries

Fact

(Lehner) $x \in \widetilde{\mathscr{A}}$ *selfadjoint with distribution* ν,, $t \in \mathbb{R} \setminus \mathrm{supp}(\nu)$, *and* $G_\nu(t) \neq 0$. *Set* $b = G_\nu(t)(t1_{\mathscr{A}} - x)$, $a = b^{-1} - 1_{\mathscr{A}}$. *Then* $\|a\|_2^2 = F'_\nu(t) - 1$.

Spectrum of a sum

- $x_1, x_2 \in \widetilde{\mathscr{A}}$ selfadjoint, free, $x = x_1 + x_2$, μ, μ_1, μ_2 distributions of x, x_1, x_2
- $J \subset \mathbb{R} \backslash \mathrm{supp}(\mu)$ open interval where $G_\mu \neq 0$, then $\omega_k(J) \subset \mathbb{R} \backslash \mathrm{supp}(\mu_k)$
  (use $G_\mu(z) = G_{\mu_k}(\omega_k(z))$ for $z = t + i\varepsilon$, $\varepsilon \downarrow 0$)
- $t \in J, t_k = \omega_k(t)$, then
  $\varphi_t^{(k)}(t_k) = (F'_{\mu_1}(t_1) - 1)(F'_{\mu_2}(t_2) - 1) < 1$
  (use connected set lemma)
- converse: suppose $t_1, t_2 \in \mathbb{R}, F_{\mu_1}(t_1) = F_{\mu_2}(t_2)$, and

$$(F'_{\mu_1}(t_1) - 1)(F'_{\mu_2}(t_2) - 1) < 1.$$

  Then $t = t_1 + t_2 - F_{\mu_k}(t_k) \notin \mathrm{supp}(\mu)$ and $t_k = \omega_k(t)$.

Spectrum of a sum

- To understand $\mathbb{R}\backslash\mathrm{supp}(\mu)$ we must look at

$$\{(t_1,t_2)\in\mathbb{R}^2 : t_k\notin\mathrm{supp}(\mu_k), F_{\mu_1}(t_1)=F_{\mu_2}(t_2)\neq\infty,$$
$$(F'_{\mu_1}(t_1)-1)(F'_{\mu_2}(t_2)-1)<1\}$$

- This is a union of smooth curves whose number generally exceeds the number of components of $\mathrm{supp}(\mu)$. When $\mathrm{supp}(\mu_k)$ is connected, $k=1,2$, there are at most two such curves.
- Say $(s,\infty)\cap\mathrm{supp}(\mu_1)=\varnothing$. Then (Nevanlinna)

$$F_{\mu_1}(z)-z=\alpha+\int_{\tau\leq s}\frac{1+\tau z}{\tau-z}d\rho(\tau),$$
$$F''_{\mu_1}(z)=\int_{\tau\leq s}\frac{1+\tau^2}{(\tau-z)^3}\,d\rho(\tau)<0,\quad z>s,$$

so $F_{\mu_1}>0$ increases, $F'_{\mu_1}-1$ decreases there. Similar for μ_2. Only ≤ 1 component in $\mathbb{R}\backslash\mathrm{supp}(\mu)$ comes from that side, etc.

Spectrum of a sum

- If J is a bounded component of $\mathbb{R}\backslash\mathrm{supp}(\mu_1)$, F_{μ_1} may be ∞ at one point in J, may also change sign and convexity. M-S use additional information about ω_k to find a bound on the number of resulting components.

Multiplicative version

- runs along analogous lines
- (Haagerup identity) Suppose that $y_1, y_2 \in \widetilde{\mathscr{A}}$, $1 - y_1$ and $1 - y_2$ are boundedly invertible, and $\beta \in \mathbb{C} \setminus \{0, 1\}$. Then

$$(1-y_1)\left\{1-\left[(1-y_1)^{-1}-\beta\right]\beta^{-1}(\beta-1)^{-1}\left[(1-y_2)^{-1}-\beta\right]\right\}(1-y_2)$$
$$=\beta^{-1}-y_1(\beta-1)^{-1}y_2$$

- Under this form it applies in a Banach op. valued prob. space (with β in the "scalar" algebra). Take y_1, y_2 free,

$$\mathbb{E}[(1-y_1)^{-1}] = \mathbb{E}[(1-y_2)^{-1}] = \beta$$

to obtain a form of Dykema's "twisted" multiplicativity for S-transforms.

Multiplicative version

- Replace G, F by

$$\varphi_x(\lambda) = \tau(\lambda x(1-\lambda x)^{-1}), \quad \eta_x(\lambda) = \lambda \widetilde{\eta}_x(\lambda) = \frac{\varphi_x(\lambda)}{1+\varphi_x(\lambda)}$$

- For x_1, x_2 free, define

$$\psi_\alpha^{(1)}(\lambda) = \alpha \widetilde{\eta}_{x_2}(\alpha \widetilde{\eta}_{x_1}(\lambda)), \ \psi_\alpha^{(2)}(\lambda) = \alpha \widetilde{\eta}_{x_1}(\alpha \widetilde{\eta}_{x_2}(\lambda))$$

and let $\omega_k(\alpha)$ be the Denjoy-Wolff point of $\psi_\alpha^{(k)}$ on an appropriate domain. With $x = x_1 x_2$, we have

$$\widetilde{\eta}_x(\lambda) = \widetilde{\eta}_{x_k}(\omega_k(\lambda)) = \omega_1(\lambda)\omega_2(\lambda).$$

Here $\lambda \in \mathbb{H} \cup \mathbb{R}$ when $x_k \geq 0$ and $\lambda \in \overline{\mathbb{D}}$ when x_k unitary.

Multiplicative version

- The Julia-Carathéodory derivative is more complicated (but, curiously, the same formula holds for the positive and unitary cases)
- For the positive case with connected supports, there is again a convexity argument that yields connectivity for the free multiplicative convolution. (No convexity was observed in the unitary case.)
- The existence of subordination functions survives in the Banach algebra-valued case.

Thanks for listening!