Thm (Ito-M) $\quad l_u + l_u^*$

$$\left\{ \alpha_{[u]}\, \delta l_u \Omega^{\oplus n} \;\middle|\; [u] \in [n]^{k+1} / \mathfrak{S}_{k+1} \right\}$$

$$\alpha_{[u]}^{-1} = \underbrace{|(\mathfrak{S}_{k+1})_u|}_{\text{Stabilizer subgrp}} \sqrt{\underbrace{|[u]|}_{\#\text{element in }[u]}} = (k+1)! \big/ \sqrt{|[u]|}$$

is an orthonormal basis of $\mathfrak{Z}_k$

Proof Orthogonality

$u, u' \in [n]^{k+1}$ $\quad u = i_0 \cdots i_k \quad u' = i'_0 \cdots i'_k$

$$\delta l_u \Omega^{\oplus n} = \sum_{j=0}^{k} e_{i_{j+1} \cdots i_k i_0 \cdots i_{j-1}} \otimes f_{i_j}$$

$$\left([(\mathbb{C}^n)^{\otimes k}]^n \cong (\mathbb{C}^n)^{\otimes k} \otimes \mathbb{C}^n \right.$$

$$\delta l_{u'} \Omega^{\oplus n} = \sum_{j'=0}^{k} e_{i'_{j'+1} \cdots i'_k i'_0 \cdots i'_{j'-1}} \otimes f_{i'_{j'}}$$

If $\delta l_u \Omega^{\oplus n} \not\perp \delta l_{u'} \Omega^{\oplus n}$

$i_j = i'_{j'}$

Then $\exists\, j, j'$ s.t. $i_{j+1} \cdots i_k i_0 \cdots i_{j-1} = i'_{j'+1} \cdots i'_k i'_0 \cdots i'_{j'-1}$

$$\Rightarrow \exists\, j, j' \text{ s.t. } \underset{\substack{\downarrow \\ i_0 \cdots i_k}}{i_{j+1} \cdots i_k i_0 \cdots i_j} = \underset{\substack{\downarrow \\ i'_0 \cdots i'_k}}{i'_{j'+1} \cdots i'_k i'_0 \cdots i'_{j'}}$$

$$\Rightarrow u' \in [u]$$

This implies $u' \notin [u] \Rightarrow$

$$\begin{gathered} \delta l_u \Omega^{\oplus n} \\ \perp \\ \delta l_{u'} \Omega^{\oplus n} \end{gathered}$$

$\square$

<u>Cor</u> P : projection onto $\overline{\delta^{\ell}\mathbb{C}[\ell_1, \ldots, \ell_n]\Omega^{\oplus n}}$

$$\overline{\delta^{s}\mathbb{C}[s_1, \ldots, s_n]\Omega^{\oplus n}}$$

$$\begin{pmatrix} e_{u_1} \\ \vdots \\ e_{u_n} \end{pmatrix} \in \left(\mathcal{F}(\mathbb{C}^n)\right)^{\oplus n}$$

Then

$$P\begin{pmatrix} e_{u_1} \\ \vdots \\ e_{u_n} \end{pmatrix} = \sum_{j=1}^{n} \frac{\delta \ell_{j u_j} \Omega^{\oplus n}}{(u_j | t)}$$

Free Euler Equation (Voiculescu 2020)

$$\begin{cases} u = u(t) = u(t)^* \in \mathrm{Vect}\,\mathbb{C}^s\langle n \mid \tau\rangle_{sa} \\ \dfrac{du}{dt} + \underbrace{(1-P)}_{\text{projection onto } \overline{\mathrm{Vect}\,\mathbb{C}^s\langle n\mid\tau\rangle}} (D_u u_i)_{i=1}^{n} = 0 \end{cases}$$

Free Leray projection

Classical setting

$$\begin{cases} \dfrac{du}{dt} + u \cdot \nabla u + \nabla P = 0 \\ \operatorname{div} u = 0 \end{cases}$$

Future works

- Find and analyze particular solutions

For example, all vectors in $\bar{\mathcal{X}}_1$ satisfy

$$(1-P)(D_u u_i)_{i=1}^{n} = 0$$

- Asked by Voiculescu

What is the structure of the vector space

$$\mathrm{Out}\left(\mathrm{Vect}\,\mathbb{C}^{s}_{\langle n|t\rangle}\right) = \mathrm{Vect}\,\mathbb{C}^{s}_{\langle n|t\rangle} \Big/ \left\{ \begin{pmatrix} [s_1,p] \\ \vdots \\ [s_n,p] \end{pmatrix} \;\middle|\; p \in \mathbb{C}^{s}\langle s_1,\dots,s_n\rangle \right\}$$