Free semi-circular system

$$X=(X_1,\dots X_n) \longrightarrow S=(S_1,\dots S_n)$$

Free semicircle

$$S_i = l_i + l_i^* \in B(\mathcal{F}(\mathbb{C}^n))$$

$$\mathcal{F}(\mathbb{C}^n) = \mathbb{C}\Omega \oplus \bigoplus_{k=1}^{\infty} (\mathbb{C}^n)^{\otimes k}, \quad \|\Omega\|=1$$

$\{e_i\}_{i=1}^n$ o.n.b of $\mathbb{C}^n$

$l_i(\xi_1 \otimes \cdots \otimes \xi_m) = e_i \otimes \xi_1 \otimes \cdots \otimes \xi_m \quad \forall m \geq 1 \quad l_i\Omega = e_i$

$\tau(\cdot) = \langle \cdot\, \Omega, \Omega \rangle$ is the trace on $W^*(S)$

Isomorphism of full Fock space

$$L^2(W^*(s_1, \ldots, s_d), \tau) \simeq \mathcal{F}(\mathbb{C}^n)$$

$$\cup \qquad\qquad\qquad \cup$$

$$\mathbb{C}\langle s_1, \ldots, s_d\rangle \simeq \underline{\mathcal{F}_{alg}(\mathbb{C}^n)}$$ algebraic Fock space

$$\Psi$$

$$P(s) \longmapsto P(s)\Omega$$

$$U_w(s) = U_{k_1}(s_{i_1}) \cdots U_{k_\ell}(s_{i_\ell}) \longleftarrow\!\shortmid \; e_w = e_{w_1} \otimes e_{w_2} \otimes \cdots \otimes e_{w_m}$$

$$w = i_1^{k_1} \cdots i_\ell^{k_\ell} \qquad w \in [n]^*$$

$$i_1 \neq i_2 \neq i_3 \neq \cdots \neq i_\ell \qquad \overset{\shortparallel}{w_1 w_2 \cdots w_m} \qquad w_i \in \underset{\{1, \ldots, d\}}{\overset{\shortparallel}{[n]}}$$

Thm (Voiculescu 2002)

$$\mathrm{Vect}\,\mathbb{C}^s\langle n|\tau\rangle = \bigoplus_{k=0}^{\infty} \mathcal{H}_k \subset \mathcal{F}(\mathbb{C}^n)^{\oplus n}$$

$$\wedge$$

$$L^2(W^*(S),\tau)^{\oplus n}$$

$$\mathcal{H}_k = \left\{ \left[(l_i^* - r_i^*)\xi \right]_{i=1}^{n} \;\middle|\; \xi \in [\mathbb{C}^n]^{\otimes k+1} \right\} \subset \left([\mathbb{C}^n]^{\otimes k} \right)^{\oplus n}$$

where $r_i\, \xi_1 \otimes \cdots \otimes \xi_m = \xi_1 \otimes \cdots \otimes \xi_m \otimes e_i$

Exercise $f = (S_1^2 + S_2^2)^m \quad m \in \mathbb{N}$

$$\begin{pmatrix} \delta_2 f \\ -\delta_1 f \end{pmatrix} \in \mathrm{Vect}\,\mathbb{C}^s\langle n|\tau\rangle$$

A question asked in his lectures

Q. $\dim \mathcal{H}_k = ?$

Thm (Ito-M)

set of words of length $k+1$

$$\dim \mathcal{H}_k = n^{k+1} - \left| [n]^{k+1} \Big/ \underbrace{\mathbb{Z}_{k+1}}_{\text{cyclic grp}} \right|$$

The set of orbits under $\mathbb{Z}_{k+1}$-action

$$(i_0, i_1, \cdots, i_k) \longmapsto (i_k, i_0, i_1, \cdots, i_{k-1})$$

Proof of our result

We focus on $\delta^s\mathbb{C}\langle s_1,\dots,s_n\rangle$ rather than $\mathrm{Vect}\,\mathbb{C}^s\langle n|t\rangle$

Thm (Voiculescu 2002)

$$\delta^s\mathbb{C}\langle s_1,\dots,s_n\rangle\Omega^{\oplus n} = \delta^\ell\mathbb{C}\langle\overbrace{\ell_1,\dots,\ell_n}^{\text{left creations}}\rangle\Omega^{\oplus n} = \bigoplus_{k=0}^{\infty}\mathcal{H}_k$$

$u = u_1\cdots u_{k+1}$ $\quad \ell_u = \ell_{u_1}\ell_{u_2}\cdots\ell_{u_{k+1}}$

$[e_w]_{|w|=k}$

$$\mathcal{H}_k = \mathrm{span}\{\delta\,\ell_u\,\Omega^{\oplus n} \mid u\in[n]^{k+1}\}$$

$$\mathcal{X}_k = \left[(\mathbb{C}^n)^{\otimes k}\right]^{\oplus n} \ominus \mathcal{Y}_k$$

$$\operatorname{Vect} \mathbb{C}^{S}\langle n|\tau\rangle = \bigoplus_{k=0}^{\infty} \mathcal{X}_k$$

$$\dim \mathcal{X}_k = n^{k+1} - \dim \mathcal{Z}_k$$

Thus we want to show

$$\dim \mathcal{Z}_k = \left| [n]^{k+1} \big/ \mathcal{R}_{k+1} \right| \quad \forall k$$

__Observation__ δ is invariant under cyclic permutation

$$\delta l_{u_0} \cdots l_{u_k} = \delta l_{u_k} l_{u_0} \cdots l_{u_{k-1}}$$

We can define the map

$$[n]^{k+1}/\mathbb{Z}_{k+1} \ni [u] \longmapsto \underline{\delta l_u} \in \mathbb{C}\langle l_1, \dots, l_n \rangle^{n}$$