# Voiculescu's non-commutative divergence free vector fields

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Joint work with

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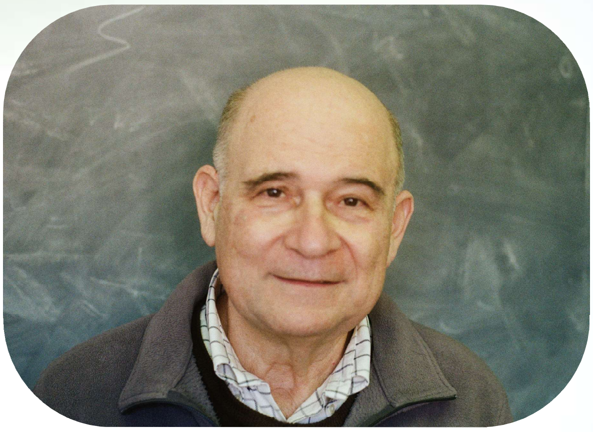
スーパーグローバルコース数学特別講義 1

# Noncommutative hydrodynamic Euler equation in a free group factor



***Dan-Virgil Voiculescu***

**University of California, Berkeley**

**Abstract.**

Combining free probability ideas and the Arnold approach to Euler equations on a Lie algebra, leads to a noncommutative analogue of the Euler equations describing an inviscid flow which preserves a Gaussian measure. The equations deal with infinitesimal automorphisms of free group factors and there are natural noncommutative smoothness spaces to which they extend. There is also a notion of cyclic vorticity, which satisfies corresponding vorticity equations and there are conserved quantities that arise. I will explain the analogy, the results and some of the problems.

【講義情報】

**Date**：January 16 (Mon)－20 (Fri), 2023

**Time**：10:30－12:30（Hybrid lecture）

**Venue**：127 Conference Room, Faculty of Science Bldg. #3, Kyoto University

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要申込：受講希望者は、Google フォームにて申込みを行って下さい。
右記 QR コードまたは下記 URL からアクセスしてください。
オンライン視聴を希望する場合でも参加登録が必要です.

**URL：https://forms.gle/tbdCKRZjLW1iJFJh7**

締切日：１月１３日（金）１７時厳守

本講義はスーパーグローバルコース登録学生のコース修了要件の１単位となります。
ただし、大学院科目として通常の単位に認定されるわけではありませんので注意してください。

京都大学スーパーグローバル大学創生支援事業　スーパーグローバルコース数学系ユニット
https://ktgu.math.kyoto-u.ac.jp

TOP GLOBAL UNIVERSITY JAPAN

D. Voiculescu, A Note on Cyclic Gradients, IUMJ, 2000.

D. Voiculescu, Cyclomorphy, IMRN, 2002.

D. Voiculescu, A hydrodaynamic exercise in free probability: Setting up free Euler equations, Expositiones Mathematicae, 2020.

# NC divergence-free vector fields

$M$ : vN-alg $\tau$ : faithful normal tracial state

$X = (X_1, \cdots, X_n) \in M^n$ $X_i = X_i^*$ algebraically free

$\boxed{f} \in \mathbb{C}\langle X_1, \cdots, X_n \rangle^n$ $g \in \mathbb{C}\langle X_1, \cdots, X_n \rangle$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \tau\left[ g(X + \varepsilon f) - g(X) \right] = 0 \quad \forall g$$

→ divergence-free (Incompressible)

$= \tau(D_f g)$ where $D_f g = \sum_{g = AX_iB} A f_i B$ (when $g$ is monomial)

$\tau$ : trace

$= \sum_{i=1}^{n} \tau(f_i \partial_i g)$ where $\partial_i g = \sum_{g = AX_iB} BA$

NC divergence-free vector fields

$$\mathrm{Vect}\,\mathbb{C}^{\times}_{\langle n|\tau\rangle} := \left\{ f \in \mathbb{C}\langle x_1,\dots,x_n\rangle^n \;\middle|\; \sum_{i=1}^{n} \tau(f_i\,\delta_i g) = 0 \quad \forall g \in \mathbb{C}\langle x_1,\dots,x_n\rangle \right\}$$

$$\subset L^2(\mathbb{C}\langle x\rangle, \tau)^{\oplus n}$$

Rem • $(\delta_i g)^* = \delta_i g^*$ implies

$$\mathrm{Vect}\,\mathbb{C}^{\times}_{\langle n|\tau\rangle} = \mathbb{C}\langle x_1,\dots,x_n\rangle^n \cap \delta\,\mathbb{C}\langle x_1,\dots,x_n\rangle^{\perp}$$

$$\delta\,\mathbb{C}\langle x_1,\dots,x_n\rangle = \left\{ \begin{pmatrix} \delta_1 p \\ \vdots \\ \delta_n p \end{pmatrix} \;\middle|\; p \in \mathbb{C}\langle x\rangle \right\}$$

• $\mathrm{Vect}\,\mathbb{C}^{\times}_{\langle n|\tau\rangle}$ is a Lie algebra

with $\{P, Q\} = (D_P Q_i - D_Q P_i)_{i=1}^{n}$

- $\tau(D_f(ab)) = 0 = \tau((D_f a)b) + \tau(a D_f b)$

$f \in \mathrm{Vect}\,\mathbb{C}^{\times}\langle n|\tau\rangle \Rightarrow \tau((D_f a)b) = -\tau(a D_f b)$

$a, b \in \mathbb{C}\langle x_1, \dots, x_n\rangle$

Thm (Voiculescu 2002)

$f = f^* \in \mathrm{Vect}\,\mathbb{C}^{\times}\langle n|\tau\rangle \to \exists (\alpha_t)$ 1-parameter Auto. grp. on $W^*(x)$

$\exp(tD_f) = (\alpha_t)$

s.t. $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$

$\tau \circ \alpha_t = \tau, \quad \frac{d}{dt}\alpha_t P\Big|_{t=0} = D_f P$

Example $\quad {}^\forall P \in \mathbb{C}\langle x_1, \cdots, x_n \rangle$

$$\begin{pmatrix} [P, x_1] \\ \vdots \\ [P, x_n] \end{pmatrix} \in \mathrm{Vect}\,\mathbb{C}^{\times}\langle n|\tau\rangle$$

$\left\{ \begin{pmatrix} [P, x_1] \\ \vdots \\ [P, x_n] \end{pmatrix} \middle| P \right\}$ inner part of

because ${}^\forall Q \in \mathbb{C}\langle x_1, \cdots, x_n \rangle$

$$D_{\begin{pmatrix} [P, x_1] \\ \vdots \\ [P, x_n] \end{pmatrix}} Q = [P, Q] \xrightarrow{\tau} 0$$

$\mathrm{Vect}\,\mathbb{C}^{\times}\langle n|\tau\rangle$