Asymptotic freeness in tracial ultraproducts

(joint work with Cyril Houdayer)

Notation Let (M_1, τ_1) and (M_2, τ_2) be tracial von Neumann algebras.

Let $M = M_1 * M_2$ be their free product.

If $\mathcal{U}$ is an ultrafilter on a set I, then $M \subset M^{\mathcal{U}}$ is the ultrapower vN algebra.

Remark If $u_1 \in \mathcal{U}(M_1)$, $u_2 \in \mathcal{U}(M_2)$ are Haar unitaries

$\overset{\text{Popa '83}}{\Longrightarrow} \{u_1\}' \cap M \subset M_1$ and $\{u_2\}' \cap M \subset M_2$.

Hence, $\{u_1\}' \cap M$ and $\{u_2\}' \cap M$ are freely independent.

Theorem (A) Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. and $u_1 \in \mathcal{U}(M_1^{\mathcal{U}})$, $u_2 \in \mathcal{U}(M_2^{\mathcal{U}})$ be Haar unitaries. Then $\{u_1\}' \cap M^{\mathcal{U}}$ and $\{u_2\}' \cap M^{\mathcal{U}}$ are freely independent.

Example Let $M = L(\mathbb{F}_2)$, where $\mathbb{F}_2 = \langle a_1, a_2 \rangle$

Let $u_1 = u_{a_1}$ and $u_2 = u_{a_2}$.

$\overset{\text{Thm (A)}}{\Longrightarrow} \{u_1\}' \cap M^{\mathcal{U}}$ and $\{u_2\}' \cap M^{\mathcal{U}}$ are freely independent.

Remark Represent $u_1 = (u_{1,n})$, $u_2 = (u_{2,n})$ where $u_{1,n} \in U(M_1)$, $u_{2,n} \in U(M_2)$ are of finite rank.

Then
$$\{u_1\}' \cap M^{\mathcal{U}} \supset \prod_{\mathcal{U}} (\{u_{1,n}\}' \cap M)$$
$$\{u_2\}' \cap M^{\mathcal{U}} \supset \prod_{\mathcal{U}} (\{u_{2,n}\}' \cap M)$$

where $\{u_{1,n}\}' \cap M$, $\{u_{2,n}\}' \subset M$ $\subset M$ are of finite index, $\forall n$.

Jekel – Kunnawalkam Elayavalli (2024)

random matrix approach of to Thm (A) where M_1 and M_2 are Connes embeddable.

Sketch of the proof of Theorem (A).

$M = M_1 * M_2$

$$L^2(M) = \mathbb{C}1 \oplus \bigoplus_{\substack{n \geq 1 \\ i_1 \neq i_2 \neq \dots \neq i_n}} (L^2(M_{i_1}) \ominus \mathbb{C}\cdot 1) \otimes (L^2(M_{i_2}) \ominus \mathbb{C}\cdot 1) \otimes \dots \otimes (L^2(M_{i_n}) \ominus \mathbb{C}\cdot 1)$$

For $i \in \{1,2\}$, let $W_i \subset L^2(M) \ominus \mathbb{C}1$ be the $\|\cdot\|_2$-closure of the span of words starting and ending in $L^2(M_i) \ominus \mathbb{C}1$.

Let $e_i : L^2(M) \to L^2(M_i)$ be the orthogonal projection.

Claim ① Let $i \in \{1,2\}$ and $x = (x_n) \in \{u_i\}' \cap M^{\mathcal{U}}$
(Popa '83)
Assume $\tau(x) = \lim_{n \to \mathcal{U}} \tau(x_n) = 0$.

Then $\lim_{n \to \mathcal{U}} \| x_n - e_i(x_n) \|_2 = 0$

Proving Thm Ⓐ amounts to showing that if

$$\left.\begin{array}{l} x_1 = (x_{1,n}) \in \{u_{i_1}\}' \cap M^{\mathcal{U}} \\ x_2 = (x_{2,n}) \in \{u_{i_2}\}' \cap M^{\mathcal{U}} \\ \vdots \\ x_k = (x_{k,n}) \in \{u_{i_k}\}' \cap M^{\mathcal{U}} \\ i_1 \neq i_2 \neq \dots \neq i_k \\ \tau(x_1) = \tau(x_2) = \dots = \tau(x_k) = 0 \end{array}\right\}$$

$$\Longrightarrow \quad \tau(x_1 x_2 \dots x_k) = 0$$

$$\tau(x_1 x_2 \dots x_k) = \lim_{n \to \mathcal{U}} \tau(x_{1,n} x_{2,n} \dots x_{k,n})$$

$\forall\ 1 \le j \le k$

Claim ① $\Longrightarrow \| x_{j,n} - e_{i_j}(x_{j,n}) \|_2 \xrightarrow[n \to \mathcal{U}]{} 0$

Claim ② $\| x_{1,n} x_{2,n} \cdots x_{k,n} - e_{i_1}(x_{1,n}) \cdots e_{i_k}(x_{k,n}) \|_1 \to 0$

Claim ③ $\tau(e_{i_1}(x_{1,n}) e_{i_2}(x_{2,n}) \cdots e_{i_k}(x_{k,n})) = 0$
$\forall n.$

Claims ② & ③ $\Longrightarrow \tau(x_{1,n} x_{2,n} \cdots x_{k,n}) \longrightarrow 0$
which finishes the proof.

To prove Claim ② & ③ we combine Claim ①
and the following result:

Theorem (Mei-Ricard, 2017)
$\forall\ i \in \{1,2\},\ p \in (1,\infty),\ \exists\, c_p > 0$ s.t.
$$\| e_i(x) \|_p \le c_p \|x\|_p \quad \forall\, x \in M,$$ i.e.,
e_i extends to a bounded operator
$$L^p(M) \longrightarrow L^p(M) \quad \forall\ 1 < p < \infty.$$

Fact Let $y_1, \dots, y_k \in M$, $k \ge 2$, $\|y_1\|, \dots, \|y_k\| \le 1$
Let $p \ge 2$ such that
$$\frac{k-1}{p} + \frac{1}{2} = 1 \qquad \left(p = \frac{2}{k-1} \ge 2 \right)$$
$$\underbrace{\frac{1}{p} + \cdots + \frac{1}{p}} + \frac{1}{2} = 1$$

k-1 times

Then

$$\|y_1 y_2 \cdots y_k - e_{i_1}(y_1)\, e_{i_2}(y_2) \cdots e_{i_k}(y_k)\|_1$$

$$\le \sum_{l=1}^{k} \|y_1 \cdots y_{l-1}\big(y_l - e_{i_l}(y_l)\big)\, e_{i_{l+1}}(y_{l+1}) \cdots e_{i_k}(y_k)\|_1$$

$$\underset{\text{(NC Hölder)}}{\le} \sum_{l=1}^{k} \underbrace{\|y_1\|_p}_{\le 1} \cdots \underbrace{\|y_{l-1}\|_p}_{\le 1} \|y_l - e_{i_l}(y_l)\|_2 \cdots \underbrace{\|e_{i_k}(y_k)\|_p}_{\le C_p \|y_k\|_p \le C_p}$$

$$\le C_p^{k-1} \sum_{l=1}^{k} \|y_l - e_{i_l}(y_l)\|_2$$

Fact $\Longrightarrow$ Claim 2.

Farah - Hart - Sherman 2011

model theory of II_1 factors

Two II_1 factors M and N are elementarily equivalent (e.e.) if $M^{\mathcal{U}} \cong N^{\mathcal{V}}$, for some ultrafilters $\mathcal{U}$ and $\mathcal{V}$

Remark Property Gamma and McDuff's property are invariant under e.e.

Boutonnet - Chifan - Ioana (2015)

$\exists$ uncountably many separable II_1 factor $\{M_\alpha\}_{\alpha \in \mathbb{R}}$ which are pairwise non-e.e.

M_α has McDuff's property hence property Gamma $\forall \alpha$.

Theorem (Chifan - I - Kunnawalkam Elayavalli) 2022

$\exists$ a separable non-Gamma II_1 factor M such that M is not e.e. to $N = N_1 * N_2$, $\forall N_1, N_2$ diffuse & Connes embeddable (any N with $h(N) > 0$ in the sense of Jung and Hayes)

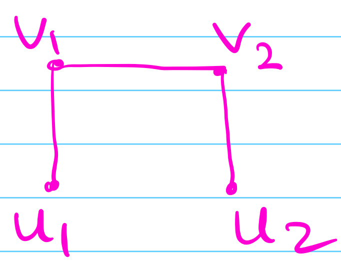
Theorem B (Houdayer–I, 2023)

$\exists$ a separable non-Gamma II_1 factor M which is not e.e. to $N = N_1 * N_2$, $\forall N_1, N_2$ diffuse.

① $\exists$ a separable non-Gamma II_1 factor M such that

$\forall u_1, u_2 \in \mathcal{U}(M^{\mathcal{U}})$ with $\{u_1\}''$, $\{u_2\}''$ 2-independent

$\exists$ Haar unitaries $v_1, v_2 \in \mathcal{U}(M^{\mathcal{U}})$ s.t.



$$[u_1, v_1] = [v_1, v_2] = [v_2, u_2] = 0.$$

② Assume $M \underset{e.e.}{\equiv} N \Longrightarrow M^{\mathcal{U}} \cong N^{\mathcal{V}}$.

Let $u_1 \in \mathcal{U}(N_1)$, $u_2 \in \mathcal{U}(N_2)$ Haar unitaries

Since $N^{\mathcal{V}} \cong M^{\mathcal{U}} \xRightarrow{①} \exists\, v_1, v_2 \in \mathcal{U}(N^{\mathcal{V}})$ Haar such that $[u_1, v_1] = [v_1, v_2] = [v_2, u_2] = 0$

Then $\left.\begin{array}{l} v_1 \in \{u_1\}' \cap N^{\mathcal{V}} \\ v_2 \in \{u_2\}' \cap N^{\mathcal{V}} \\ \tau(v_1) = \tau(v_2) = 0 \end{array}\right\} \xRightarrow{\text{Thm } Ⓐ}$ v_1 and v_2 are freely independent

$$\Longrightarrow [v_1, v_2] \neq 0.$$

<u>Question</u> Show that

$$L(\mathbb{F}_2),\ L(\mathbb{F}_2 \times \mathbb{F}_2),\ L(SL_3(\mathbb{Z}))$$

are not e.e.

The proof of ① uses a perturbation lemma: if $X, Y \subset M \ominus \mathbb{C}\cdot 1$ are finite (M is a II_1 factor) and X, Y are "almost" 2-independent.

$\Longrightarrow \exists v \in U(M)$ such that

$\|v - 1\|_2 \simeq 0$ and

$v X v^* \perp Y$.

This has the following consequence:

Theorem C Let M be a II_1 factor and $X, Y \subset M \ominus \mathbb{C}\cdot 1$ be finite sets. Then $\exists u \in U(M)$ such that

$$u X u^* \perp Y.$$

(i.e. $\tau(u x u^* y^*) = 0 \ \forall x \in X, y \in Y$)

This strengthens the fact that $\forall \varepsilon > 0, \exists u_\varepsilon \in U(M)$ s.t. $u_\varepsilon X u_\varepsilon^* \perp_\varepsilon Y$

$\iff \mathcal{U}(M) \overset{\mathrm{Ad}(\cdot)}{\curvearrowright} M$ is weakly mixing.