Self-normalized sums in free probability theory

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Overview

Self-normalized sums in classical probability theory

Self-normalized sums in free probability theory

Bounded self-normalized sums

Unbounded self-normalized sums

Self-normalized sums in the non-i.d. case

SNS in classical probability theory – Definition

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of classical random variables. Define

$$S_n := \sum_{i=1}^{n} X_i, \qquad V_n^2 := \sum_{i=1}^{n} X_i^2.$$

Then:

$$S_n/V_n = \textit{self-normalized sum}, \qquad S_n/V_n := 0 \text{ on } \{V_n = 0\}.$$

Question: How do self-normalized sums behave in the limit?

From now on: Only consider the i.i.d. case!

SNS in classical probability theory – Intuitive approach

- for $(X_i)_{i\in\mathbb{N}}$ i.i.d. with $\mathbb{E}X_1 = 0$ and $\mathrm{Var}X_1 = 1$:

$$\frac{S_n}{\sqrt{n}} \to \mathcal{N}(0,1) \text{ in distribution as } n \to \infty \text{ due to CLT}$$

$$\frac{V_n^2}{n} \to 1 \text{ in probability as } n \to \infty \text{ due to weak LLN}$$

Slutsky's theorem

Let $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$, and X be classical random variables and $c \in \mathbb{C}, c \neq 0$. Assume that

$$X_n \to X \quad \text{in distribution as } n \to \infty,$$
$$Y_n \to c \quad \text{in probability as } n \to \infty.$$

Then:

$$\frac{X_n}{Y_n} \to \frac{X}{c} \quad \text{in distribution as } n \to \infty.$$

SNS in classical probability theory – Intuitive approach

- for $(X_i)_{i\in\mathbb{N}}$ i.i.d. with $\mathbb{E}X_1 = 0$ and $\mathrm{Var}X_1 = 1$:

$$\frac{S_n}{\sqrt{n}} \to \mathcal{N}(0,1) \text{ in distribution as } n \to \infty \text{ due to CLT}$$
$$\frac{V_n^2}{n} \to 1 \text{ in probability as } n \to \infty \text{ due to weak LLN}$$

- by Slutsky's theorem:

$$\frac{S_n}{V_n} = \frac{\frac{S_n}{\sqrt{n}}}{\frac{V_n}{\sqrt{n}}} \to \mathcal{N}(0,1) \text{ in distribution as } n \to \infty$$

- so far: Self-normalization did not come into play!

SNS in classical probability theory – Two results

- Giné, Götze, Mason (1997): for i.i.d. random variables $(X_i)_{i\in\mathbb{N}}$:

  $$\frac{S_n}{V_n} \to \mathcal{N}(0,1) \text{ in distribution as } n \to \infty \quad \Leftrightarrow \quad X_1 \in \text{DAN}, \mathbb{E}X_1 = 0,$$

  where DAN $=$ domain of attraction of the normal law
- Bentkus, Götze (1996): if $\mathbb{E}X_1 = 0$, $\mathrm{Var}X_1 = 1$, $\mathbb{E}|X_1|^3 < \infty$:

  $$\Delta(\mu_{S_n/V_n}, \mathcal{N}(0,1)) \lesssim \frac{\mathbb{E}|X_1|^3}{\sqrt{n}},$$

  where

  $$\Delta(\nu_1,\nu_2) := \sup_{x\in\mathbb{R}} |\nu_1((-\infty,x]) - \nu_2((-\infty,x])|$$

  and μ_{S_n/V_n} $=$ distribution of S_n/V_n
- Berry-Esseen theorem: rate for $\frac{S_n}{\sqrt{n}}$ to $\mathcal{N}(0,1)$ given by $\frac{1}{\sqrt{n}}$ too!

SNS in free probability theory – Definition

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of self-adjoint non-commutative random variables. Define

$$S_n := \sum_{i=1}^{n} X_i, \qquad V_n^2 := \sum_{i=1}^{n} X_i^2.$$

The *free* analog of a *self-normalized sum* is given by

$$U_n := V_n^{-1/2} S_n V_n^{-1/2}.$$

We will see later that U_n is well-defined!

Question: How does U_n behave in the limit?

From now on: Only consider the case of free i.d. self-adjoint random variables!

SNS in free probability theory – Intuitive approach

- in classical case: limiting behavior of S_n/V_n determined by behavior of S_n (via CLT) and V_n (via LLN)
- in free case: for $(X_i)_{i\in\mathbb{N}}$ free i.d. self-adjoint with mean 0 and variance 1:
  - analytic distr. of $\frac{S_n}{\sqrt{n}} \Rightarrow$ Wigner semicircle distr. ω as $n \to \infty$ due to free CLT
  - analytic distr. of $\frac{V_n^2}{n} \Rightarrow \delta_1$ as $n \to \infty$ due to free LLN
- problem: How can we combine these two results?
- solution: Replace Slutsky's theorem by machinery of Cauchy transforms!
- realization of solution depends on whether the random variables are bounded or unbounded

Recap: Bounded and unbounded random variables

- bounded random variable X = element in C^*-probability space $(\mathcal{A}, \varphi)$
  - expressions of the form $\varphi(X^k)$ always exist
  - for self-adjoint X: analytic distribution μ_X is compactly supported
- unbounded random variable X = unbounded operator affiliated with finite von Neumann algebra $\mathcal{A}$; $(\mathcal{A}, \varphi)$ is tracial W^*-probability space
  - $\varphi(X)$ only defined as an extension if X positive or $\varphi(|X|) < \infty$
  - expressions of the form $\varphi(X^k)$ may not exist
  - for self-adjoint X: analytic distribution μ_X is not necessarily compactly supported

Bounded self-normalized sums

Theorem (N., 2024)

Given a tracial faithful C^*-probability space $(\mathcal{A}, \varphi)$ with norm $\|\cdot\|$, let $(X_i)_{i\in\mathbb{N}} \subset \mathcal{A}$ be a sequence of free i.d. self-adjoint random variables with

$$\varphi(X_1) = 0 \text{ and } \varphi(X_1^2) = 1.$$

For sufficiently large n, we have:

- $U_n = V_n^{-1/2} S_n V_n^{-1/2}$ is well-defined in $\mathcal{A}$.
- If μ_n denotes the analytic distribution of U_n, then $\mu_n \Rightarrow \omega$ as $n \to \infty$ with

$$\Delta(\mu_n, \omega) \lesssim \|X_1\|^3 \frac{\log n}{\sqrt{n}}.$$

For comparison: rate of convergence for $\frac{S_n}{\sqrt{n}}$ to ω and for classical SNS to $\mathcal{N}(0,1)$ given by $\frac{1}{\sqrt{n}}$

Bounded SNS: Idea of the proof

We argue in three steps:

- Show that U_n is well-defined in $\mathcal{A}$.
- Show that Cauchy transform of U_n is close to Cauchy transform of ω.
- Derive weak convergence $\mu_n \Rightarrow \omega$ and rate of convergence.

From now on: Consider normalized versions of S_n and V_n^2, i.e.:

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i, \qquad V_n^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2.$$

Bounded SNS: Proof – Step 1

Goal: $U_n = V_n^{-1/2} S_n V_n^{-1/2}$ is well-defined in $\mathcal{A}$ for large n.

Sufficient: V_n^2 is invertible in $\mathcal{A}$ for large n.

- Voiculescu (1986): For free self-adjoint $a_1, a_2, \ldots, a_k \in \mathcal{A}$ with $\varphi(a_i) = 0$ for all $i = 1, \ldots, k$, we have:

$$\|a_1 + \cdots + a_k\| \leq \max_{i=1,\ldots,k} \|a_i\| + 2 \left(\sum_{i=1}^{k} \varphi\big(a_i^2\big) \right)^{1/2}$$

- for large n:

$$\left\|V_n^2 - 1\right\| = \left\| \frac{1}{n} \sum_{i=1}^{n} X_i^2 - 1 \right\| \lesssim \frac{\|X_1\|^2}{\sqrt{n}}$$

- with Neumann series-type argument: V_n^2 is invertible in $\mathcal{A}$ for large n

Bounded SNS: Proof – Step 2

Goal: Cauchy transform G_n of $U_n = V_n^{-1/2} S_n V_n^{-1/2}$ is close to G_ω.

- for $z \in \mathbb{C}^+$ with G_{S_n} = Cauchy transform of S_n:

$$\begin{aligned} G_n(z) = \varphi((z - U_n)^{-1}) &= \varphi\big(V_n^{1/2}(zV_n - S_n)^{-1} V_n^{1/2}\big) \\ &= \varphi(V_n(zV_n - S_n)^{-1}) \ \approx G_{S_n}(z) \end{aligned}$$

- precise estimates for $z \in \mathbb{C}^+$ with $1 \geq \Im z \geq \frac{1}{\sqrt{n}}, |\Re z| \leq 8$:

$$\begin{aligned} |G_n(z) - G_\omega(z)| &\leq |G_n(z) - G_{S_n}(z)| + |G_{S_n}(z) - G_\omega(z)| \\ &\lesssim \frac{\|V_n - 1\|}{\Im z} + \frac{\Delta(\mu_{S_n}, \omega)}{\Im z} \lesssim \frac{1}{\sqrt{n}} \frac{1}{\Im z} \end{aligned}$$

reminder: $\Delta(\mu_{S_n}, \omega) := \sup_{x \in \mathbb{R}} |\mu_{S_n}((-\infty, x]) - \omega((-\infty, x])| \lesssim \frac{1}{\sqrt{n}}$

Bounded SNS: Proof – Step 3

Goal: Derive weak convergence $\mu_n \Rightarrow \omega$ and rate of convergence.

We know: $|G_n(z) - G_\omega(z)| \lesssim \frac{1}{\sqrt{n}} \frac{1}{\Im z}$ for z with $1 \geq \Im z \geq \frac{1}{\sqrt{n}}, |\Re z| \leq 8$.

- weak convergence $\mu_n \Rightarrow \omega$ can be derived by some modifications of the inequality above
- for rate of convergence:
    - with superconvergence: $\|U_n\| \leq \|S_n\| \|V_n^{-1}\| < 3$ for large n
    - with Bai's ineq. for compactly supported pm's:

$$\Delta(\mu_n, \omega) \lesssim v + \sup_{x \in [-2,2]} \int_v^1 |G_n(x+iy) - G_\omega(x+iy)| dy$$

$$+ \int_{-3}^{3} |G_n(u+i) - G_\omega(u+i)| du \lesssim \frac{\log n}{\sqrt{n}}$$

Bounded SNS: Superconvergence

A refinement of the inequality $\|U_n\| \leq \|S_n\|\|V_n^{-1}\| < 3$ leads to:

Corollary (N., 2024)

In the setting of the last theorem and for sufficiently large n, we have:

$$\operatorname{supp} \mu_n \subset \left(-2 - \frac{58\|X_1\|^2}{\sqrt{n}},\, 2 + \frac{58\|X_1\|^2}{\sqrt{n}}\right)$$

In the setting of the free CLT, the corresponding rate is of order $\frac{1}{\sqrt{n}}$ too!

Unbounded self-normalized sums

From now on: Aff($\mathcal{A}$) = algebra of operators affiliated with finite von Neumann algebra $\mathcal{A}$

Theorem (N., 2024)

Given a tracial W^*-probability space $(\mathcal{A}, \varphi)$, let $(X_i)_{i\in\mathbb{N}} \subset$ Aff($\mathcal{A}$) be a sequence of free i.d. self-adjoint (possibly unbounded) random variables with

$$\varphi(X_1) = 0,\ \varphi(X_1^2) = 1,\ \text{and } \varphi(|X_1|^4) < \infty.$$

For sufficiently large n, we have:

- $U_n = V_n^{-1/2} S_n V_n^{-1/2}$ is well-defined in Aff($\mathcal{A}$).
- If μ_n denotes the analytic distribution of U_n, then $\mu_n \Rightarrow \omega$ as $n \to \infty$ with

$$\Delta(\mu_n, \omega) \lesssim \frac{\varphi(|X_1|^4)^{5/4}}{n^{1/4}}.$$

Unbounded SNS: Idea of the proof

We argue in three (already familiar) steps:

- Show that U_n is well-defined in $\mathrm{Aff}(\mathcal{A})$.
- Show that Cauchy transform of U_n is close to Cauchy transform of ω.
- Derive weak convergence $\mu_n \Rightarrow \omega$ and rate of convergence.

Realization of steps is different compared to bounded setting!

From now on: Consider normalized versions of S_n and V_n^2.

Unbounded SNS: Proof – Step 1

Goal: $U_n = V_n^{-1/2} S_n V_n^{-1/2}$ is well-defined in Aff($\mathcal{A}$) for large n.

Sufficient: V_n^2 is invertible in Aff($\mathcal{A}$) for large n.

- with functional calculus: self-adjoint operator $T \in$ Aff($\mathcal{A}$) is invertible in Aff($\mathcal{A}$) if $\mu_T(\{0\}) = 0$, where $\mu_T =$ analytic distribution of T
- we have to prove:

$$\text{for large } n: \quad 0 = \mu_{V_n^2}(\{0\}) = \left(\mu_{\frac{X_1^2}{n}} \boxplus \cdots \boxplus \mu_{\frac{X_n^2}{n}}\right)(\{0\})$$

- use result due to Bercovici, Voiculescu (1998):

$$\gamma \text{ is atom of } \nu_1 \boxplus \nu_2$$

$$\Leftrightarrow$$

$$\exists \text{ atoms } \alpha, \beta \text{ of } \nu_1, \nu_2 : \gamma = \alpha + \beta, \nu_1(\{\alpha\}) + \nu_2(\{\beta\}) > 1$$

$$\text{In this case: } (\nu_1 \boxplus \nu_2)(\{\gamma\}) = \nu_1(\{\alpha\}) + \nu_2(\{\beta\}) - 1$$

Unbounded SNS: Proof – Step 2a

Goal: Cauchy transform G_n of $U_n = V_n^{-1/2} S_n V_n^{-1/2}$ is close to G_ω.

- in bounded setting:

$$G_n(z) = \varphi((z - U_n)^{-1}) = \varphi\big(V_n^{1/2}(zV_n - S_n)^{-1} V_n^{1/2}\big)$$
$$= \varphi(V_n(zV_n - S_n)^{-1}) = \dots$$

- in unbounded setting: $\varphi(T)$ only defined via extensions, $\varphi(ST) = \varphi(TS)$ not necessarily true

- new idea: for all $z \in \mathbb{C}^+$:

$$|G_n(z) - G_\omega(z)| \leq |G_n(z) - G_{S_n}(z)| + |G_{S_n}(z) - G_\omega(z)|,$$

$$|G_n(z) - G_{S_n}(z)| = |\varphi((z - U_n)^{-1}(U_n - S_n)(z - S_n)^{-1})| \leq \frac{\varphi(|U_n - S_n|)}{(\Im z)^2}$$

Unbounded SNS: Proof – Step 2b

- we know: $|G_n(z) - G_{S_n}(z)| \le \frac{\varphi(|U_n - S_n|)}{(\Im z)^2}$ for all $z \in \mathbb{C}^+$
- $U_n - S_n = V_n^{-1/2} S_n V_n^{-1/2} - S_n = (1 - V_n^{1/2}) U_n + S_n V_n^{-1/2} (1 - V_n^{1/2})$
- with Hölder's inequality and $\|T\|_2 := \varphi(|T|^2)^{1/2}$:

$$\varphi(|U_n - S_n|) \le \|1 - V_n^{1/2}\|_2 \big(\|U_n\|_2 + \|S_n V_n^{-1/2}\|_2 \big)$$

- with free LLN: $\|1 - V_n^{1/2}\|_2 \le \|1 - V_n^2\|_2 \lesssim \frac{\sqrt{\varphi(|X_1|^4)}}{\sqrt{n}}$
- with self-normalizing effect: $|U_n| \le \sqrt{n}$

in classical setting with Cauchy's inequality:

$$|S_n| = \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \right| \le \left(\sum_{i=1}^{n} X_i^2 \right)^{1/2} = \sqrt{n} V_n \;\Rightarrow\; |S_n / V_n| \le \sqrt{n},$$

in free setting with inequality due to Bikchentaev, Sabirova (2012)

Unbounded SNS: Proof – Step 2b

- we know: $|G_n(z) - G_{S_n}(z)| \leq \frac{\varphi(|U_n - S_n|)}{(\Im z)^2}$ for all $z \in \mathbb{C}^+$
- $U_n - S_n = V_n^{-1/2} S_n V_n^{-1/2} - S_n = (1 - V_n^{1/2}) U_n + S_n V_n^{-1/2}(1 - V_n^{1/2})$
- with Hölder's inequality and $\|T\|_2 := \varphi(|T|^2)^{1/2}$:

$$\varphi(|U_n - S_n|) \leq \|1 - V_n^{1/2}\|_2 \big(\|U_n\|_2 + \|S_n V_n^{-1/2}\|_2\big)$$

- with free LLN: $\|1 - V_n^{1/2}\|_2 \leq \|1 - V_n^2\|_2 \lesssim \frac{\sqrt{\varphi(|X_1|^4)}}{\sqrt{n}}$
- with self-normalizing effect: $|U_n| \leq \sqrt{n}$
- again with self-normalizing effect: $\|U_n\|_2 \lesssim 2 + \sqrt{n}\|V_n^2 - 1\|_2$

Idea: $V_n^2 \approx 1$ "whp" $\Rightarrow$ $U_n \approx S_n$ $\Rightarrow$ $\|U_n\|_2 \approx \|S_n\|_2 = 1$

Unbounded SNS: Proof – Step 2b

- we know: $|G_n(z) - G_{S_n}(z)| \leq \frac{\varphi(|U_n - S_n|)}{(\Im z)^2}$ for all $z \in \mathbb{C}^+$
- $U_n - S_n = V_n^{-1/2} S_n V_n^{-1/2} - S_n = (1 - V_n^{1/2}) U_n + S_n V_n^{-1/2}(1 - V_n^{1/2})$
- with Hölder's inequality and $\|T\|_2 := \varphi(|T|^2)^{1/2}$:

$$\varphi(|U_n - S_n|) \leq \|1 - V_n^{1/2}\|_2 \big(\|U_n\|_2 + \|S_n V_n^{-1/2}\|_2\big)$$

- with free LLN: $\|1 - V_n^{1/2}\|_2 \leq \|1 - V_n^2\|_2 \lesssim \frac{\sqrt{\varphi(|X_1|^4)}}{\sqrt{n}}$
- with self-normalizing effect: $|U_n| \leq \sqrt{n}$
- again with self-normalizing effect: $\|U_n\|_2 \lesssim 2 + \sqrt{n}\|V_n^2 - 1\|_2$
- with similar arguments: $\|S_n V_n^{-1/2}\|_2 \lesssim \sqrt{2} + \sqrt{n}\,\big\|V_n^2 - 1\big\|_2$
- finally: $|G_n(z) - G_{S_n}(z)| \lesssim \frac{1}{\sqrt{n}} \frac{1}{(\Im z)^2}$ for all $z \in \mathbb{C}^+$
- in particular with free CLT: $\lim_{n\to\infty} G_n(z) = G_\omega(z)$ for all $z \in \mathbb{C}^+$

Unbounded SNS: Proof – Step 3

Goal: Derive weak convergence $\mu_n \Rightarrow \omega$ and rate of convergence.

For $z \in \mathbb{C}^+$: $|G_n(z) - G_{S_n}(z)| \lesssim \frac{1}{\sqrt{n}} \frac{1}{(\Im z)^2}$ and $\lim_{n\to\infty} G_n(z) = G_\omega(z)$

- weak convergence $\mu_n \Rightarrow \omega$ is clear
- rate of convergence follows with modified version of Bai's inequality
- rate is only of order $n^{-1/4}$ for two reasons: bound on the difference of the Cauchy transforms does not depend on $\Re z$ and contains $(\Im z)^{-2}$ as a factor

The non-i.d. cases: Bounded SNS

Theorem (N., 2024)

Given a tracial faithful C^*-probability space $(\mathcal{A}, \varphi)$ with norm $\|\cdot\|$, let $(X_i)_{i\in\mathbb{N}} \subset \mathcal{A}$ be a sequence of free self-adjoint random variables with

$$\varphi(X_i) = 0, \qquad \varphi(X_i^2) = \sigma_i^2, \qquad i \in \mathbb{N}.$$

Moreover, let

$$B_n^2 := \sum_{i=1}^n \sigma_i^2, \qquad L_{S,3n} := \frac{\sum_{i=1}^n \|X_i\|^3}{B_n^3}, \qquad L_{S,4n} := \frac{\sum_{i=1}^n \|X_i\|^4}{B_n^4}.$$

We have:

- If $L_{S,4n} < 1/16$, then $U_n = V_n^{-1/2} S_n V_n^{-1/2}$ is well-defined in $\mathcal{A}$.
- If $L_{S,4n} < 1/16$, $L_{S,3n} < 1/2e$, and μ_n denotes the distr. of U_n, then

$$\Delta(\mu_n, \omega) \lesssim \max\left\{ |\log L_{S,3n}| L_{S,3n}, |\log L_{S,4n}| L_{S,4n}^{1/2} \right\}.$$

- If $\lim_{n\to\infty} L_{S,4n} = 0$, then $\mu_n \Rightarrow \omega$ as $n \to \infty$.

The non-i.d. cases: Unbounded SNS

Theorem (N., 2024)

Given a tracial W^*-probability space $(\mathcal{A}, \varphi)$, let $(X_i)_{i\in\mathbb{N}} \subset \mathsf{Aff}(\mathcal{A})$ be a sequence of free self-adjoint (possibly unbounded) random variables with

$$\varphi(X_i) = 0, \qquad \varphi(X_i^2) = \sigma_i^2, \qquad \varphi(|X_i|^4) < \infty, \qquad i \in \mathbb{N}.$$

Moreover, let

$$B_n^2 := \sum_{i=1}^{n} \sigma_i^2, \qquad L_{4n} := \frac{\sum_{i=1}^{n} \varphi(|X_i|^4)}{B_n^4}$$

and assume that Lindeberg's condition holds. For large n, we have:

- $U_n = V_n^{-1/2} S_n V_n^{-1/2}$ is well-defined in $\mathsf{Aff}(\mathcal{A})$.
- If μ_n denotes the analytic distribution of U_n, then

$$\Delta(\mu_n, \omega) \lesssim L_{4n}^{1/4} + \sqrt{n} L_{4n}^{3/4} + n L_{4n}^{5/4}.$$

- If $\lim_{n\to\infty} \sqrt{n} L_{4n} = 0$, then $\mu_n \Rightarrow \omega$ as $n \to \infty$.

Summary and Outlook

- intuitive approach to self-normalized sums works in the free setting with some modifications
- How can we prove free self-normalized limit theorems under weaker moment assumptions? How can we exploit the self-normalization?

Thank you for your attention!

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