S-transform in finite free probability

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Probabilistic Operator Algebra Seminar

August 23, 2024
Joint work with Octavio Arizmendi, Katsunori Fujie and Yuki Ueda
(arXiv:2408.09337).

Polynomials

$\mathcal{P}_d(S) :=$ monic polynomials of degree d with all its roots contained in the set $S \subset \mathbb{C}$.

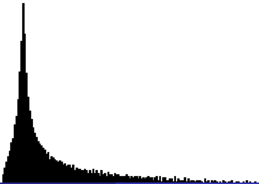
For $p \in \mathcal{P}_d(\mathbb{C})$ we use the notation:

$$p(x) \quad = \prod_{k=1}^{d} (x - \lambda_k(p)) \quad = \sum_{k=0}^{d} x^{d-k} (-1)^k \binom{d}{k} \tilde{e}_k(p).$$

Roots: $\lambda_1(p), \dots, \lambda_d(p)$.

Coefficients: $\tilde{e}_k(p) := \frac{1}{\binom{d}{k}} \sum_{1 \le i_1 < \cdots < i_k \le d} \lambda_{i_1}(p) \cdots \lambda_{i_k}(p)$.

Empirical root distribution: $\mu[\![p]\!] := \frac{1}{d} \sum_{i=1}^{d} \delta_{\lambda_i(p)}$



Main Theorem

For $\mu \in \mathcal{M}(\mathbb{R}_{\geq 0})$, its S-transform [Voiculescu '87] is

$$S_\mu(z) := \frac{1+z}{z}\Psi_\mu^{-1}(z) \qquad \text{for } z \in (\mu(\{0\})-1, 0),$$

where

$$\Psi_\mu(z) := \int_0^\infty \frac{tz}{1-tz}\mu(dt), \quad z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}.$$

Main tool for **free multiplicative convolution:** $S_{\mu \boxtimes \nu}(z) = S_\mu(z) S_\nu(z)$.

Theorem (Arizmendi, Fujie, P, Ueda '24)

$(p_d)_{d\in\mathbb{N}}$ sequence with $p_d \in \mathbb{P}_d(\mathbb{R}_{\geq 0})$ and $\nu \in \mathcal{M}(\mathbb{R}_{\geq 0})$. The following are equivalent:

1. *$\mu[\![p_d]\!] \to \nu$ weakly as $d \to \infty$.*
2. *For every $t \in (0, 1-\mu(\{0\}))$,*

$$\lim_{\substack{d\to\infty \\ \frac{k}{d}\to t}} \frac{\widetilde{e}_{k-1}^{(d)}(p_d)}{\widetilde{e}_k^{(d)}(p_d)} = S_\nu(-t).$$

Corollary. Under some assumptions,

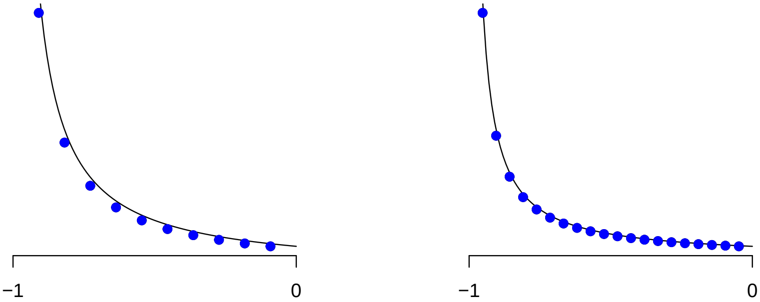
$$\left(\widetilde{e}_k^{(d)}(p_d)\right)^{\frac{1}{d}} = \exp\left(-\int_0^t \log S_\nu(-x)dx\right).$$

Theorem (Arizmendi, Fujie, P, Ueda '24)

$(p_d)_{d\in\mathbb{N}}$ sequence with $p_d \in \mathbb{P}_d(\mathbb{R}_{\geq 0})$ and $\nu \in \mathcal{M}(\mathbb{R}_{\geq 0})$. Then

$$\mu\left[\!\left[p_d \right]\!\right] \to \nu \qquad \Leftrightarrow \qquad \lim_{\substack{d\to\infty \\ \frac{k}{d}\to t}} \frac{\widetilde{e}^{(d)}_{k-1}(p)}{\widetilde{e}^{(d)}_{k}(p)} = S_\nu(-t). \qquad \textit{for } t \in (0, 1-\mu(\{0\}))$$

Laguerre of parameter 1 for $d = 10$ and $d = 20$.



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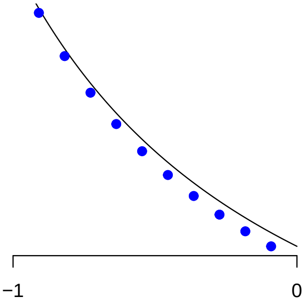
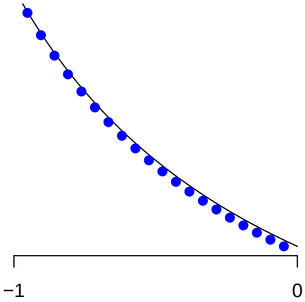
$$\mu\left[\!\left[p_d\right]\!\right] \to \nu \qquad \Leftrightarrow \qquad \lim_{\substack{d\to\infty \\ \frac{k}{d}\to t}} \frac{\widetilde{e}^{(d)}_{k-1}(p)}{\widetilde{e}^{(d)}_{k}(p)} = S_\nu(-t). \qquad \text{for } t \in (0, 1-\mu(\{0\}))$$

Laguerre of parameter 2 for $d = 10$ and $d = 20$.





Finite Free Probability

Finite free convolutions

Definition

Given $p, q \in \mathcal{P}_d$, their **multiplicative and additive convolutions** are the polynomials $p \boxtimes_d q \in \mathcal{P}_d$ and $p \boxplus_d q \in \mathcal{P}_d$ with coefficients

$$\tilde{e}_k(p \boxtimes_d q) = \tilde{e}_k(p)\tilde{e}_k(q), \qquad \text{for } k = 1, 2, \dots, d,$$

$$\tilde{e}_k(p \boxplus_d q) = \sum_{i+j=k} \binom{k}{i} \tilde{e}_i(p) \cdot \tilde{e}_j(q), \qquad \text{for } k = 1, 2, \dots, d.$$

$\boxplus_d$ **preserves real roots (Walsh 1922)** If $p, q \in \mathcal{P}_d(\mathbb{R})$ then $p \boxplus_d q \in \mathcal{P}_d(\mathbb{R})$.

$\boxtimes_d$ **preserves positive real roots (Szegö 1922)** If $p, q \in \mathcal{P}_d(\mathbb{R}_{>0}) \quad \Rightarrow \quad p \boxtimes_d q \in \mathcal{P}_d(\mathbb{R}_{>0})$.

Basic properties: interpretation in terms of differential operators, bilinear, commutative, associative, identity (x^d for $\boxplus_d$, $(x-1)^d$ for $\boxtimes_d$), preserve interlacing, preserve root separation.

Preserves interlacing: If $p, \widetilde{p}, q \in \mathcal{P}_d(\mathbb{R})$, then $p \preccurlyeq \widetilde{p} \quad \Rightarrow \quad p \boxplus_d q \preccurlyeq \widetilde{p} \boxplus_d q$.

$$p \preccurlyeq \widetilde{p} \quad \text{means} \quad \lambda_1(p) \leq \lambda_1(\widetilde{p}) \leq \lambda_2(p) \leq \lambda_2(\widetilde{p}) \leq \cdots \leq \lambda_d(p) \leq \lambda_d(\widetilde{p})$$

Connection to Random Matrices and Free Probability

[Marcus, Spielman, Srivastava '15] Let A and B be $d \times d$ selfadjoint matrices with characteristic polynomials $p = \chi(A)$ and $q = \chi(B)$. Then

$$p \boxplus_d q = \mathbb{E}_Q[\chi(A + QBQ^*)] \qquad \text{and} \qquad p \boxtimes_d q = \mathbb{E}_Q[\chi(AQBQ^*)]$$

where $Q \sim$ Haar measure over unitary (orthogonal or permutation) matrices.

[Marcus '16] Establishes a connection with free probability:
When $d \to \infty$, finite free convolutions should tend to free convolutions ($\boxplus_d \to \boxplus$).
Studies *Finite Free Probability* and gives basic examples like LLN, CLT, Poisson limit.

Formally, consider sequences $\mathfrak{p} = (p_d)_{d=1}^{\infty}$ and $\mathfrak{q} = (q_d)_{d=1}^{\infty}$ with $p_d, q_d \in \mathbb{P}_d(\mathbb{R})$ and limiting measures $\nu(\mathfrak{p}), \nu(\mathfrak{q}) \in \mathcal{M}_c(\mathbb{R})$: $\quad \mu[\![p_d]\!] \longrightarrow \nu(\mathfrak{p}) \quad$ and $\quad \mu[\![q_d]\!] \longrightarrow \nu(\mathfrak{q})$.

[Marcus '16, Arizmendi, P '16]. Then $\mu[\![p_d \boxplus_d q_d]\!] \longrightarrow \nu(\mathfrak{p}) \boxplus \nu(\mathfrak{q})$

[Arizmendi, Garza-Vargas, P '21]. Then $\mu[\![p_d \boxtimes_d q_d]\!] \longrightarrow \nu(\mathfrak{p}) \boxtimes \nu(\mathfrak{q})$.

Finite S-transform

Theorem (Arizmendi, Fujie, P, Ueda '24)

$(p_d)_{d\in\mathbb{N}}$ sequence with $p_d \in \mathbb{P}_d(\mathbb{R}_{\geq 0})$ and $\nu \in \mathcal{M}(\mathbb{R}_{\geq 0})$. Then

$$\mu\left[\!\left[p_d \right]\!\right] \to \nu \qquad \Leftrightarrow \qquad \lim_{\substack{d\to\infty \\ \frac{k}{d}\to t}} \frac{\widetilde{e}^{(d)}_{k-1}(p)}{\widetilde{e}^{(d)}_{k}(p)} = S_\nu(-t). \qquad \text{for } t \in (0, 1-\mu(\{0\}))$$

Definition

The **finite S-transform** of $p \in \mathbb{P}_d(\mathbb{R}_{>0})$ is the map

$$S^{(d)}_p : \left\{-\tfrac{d}{d}, -\tfrac{d-1}{d}, \dots, -\tfrac{2}{d}, -\tfrac{1}{d}\right\} \to \mathbb{R}_{>0}$$

$$-\frac{k}{d} \quad \mapsto \quad \frac{\widetilde{e}^{(d)}_{k-1}(p)}{\widetilde{e}^{(d)}_{k}(p)}$$

Properties:

- $S^{(d)}_{p\boxtimes_d q}\left(-\frac{k}{d}\right) = S^{(d)}_p\left(-\frac{k}{d}\right) S^{(d)}_q\left(-\frac{k}{d}\right).$ (direct from def of $\boxtimes_d$)
- $S^{(d)}_p\left(-\frac{k+1}{d}\right) > S^{(d)}_p\left(-\frac{k}{d}\right).$ (by Newton inequalities)
- $S^{(d)}_p\left(-\frac{k}{d}\right) S^{(d)}_{p^{\langle -1\rangle}}\left(-\frac{d+1-k}{d}\right) = 1.$

Proof of main theorem

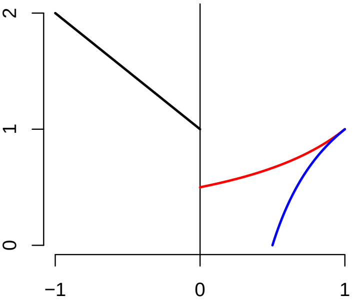
Given $\mu \in \mathcal{M}(\mathbb{R}_{>0})$, the *(shifted) T-transform* is the function $T_\mu : (0,1) \to \mathbb{R}_{\geq 0}$ with

$$T_\mu(t) = \frac{1}{S_\mu(t-1)} \qquad \text{for } t \in (0,1).$$

[Tucci '10, Haagerup and Möller '13]
there exists a limiting measure

$$\Phi(\mu) := \lim_{m\to\infty} (\mu^{\boxtimes m})^{\langle 1/m \rangle}, \qquad \text{char. by}$$

$$F_{\Phi(\mu)}\left(T_\mu(t)\right) = t \qquad \text{for all } t \in (0,1).$$



[Fujie and Ueda '23]
Given $p \in \mathbb{P}_d(\mathbb{R}_{\geq 0})$, there exists

$$\Phi_d(p) := \lim_{m\to\infty} (p^{\boxtimes_d m})^{\langle 1/m \rangle},$$

$$\text{with roots} \quad \lambda_k\left(\Phi_d(p)\right) = \frac{\widetilde{e}_k^{(d)}(p)}{\widetilde{e}_{k-1}^{(d)}(p)}.$$

Basic tool: differentiation

For $k \leq d$, denote by $\partial_{k|d} : \mathbb{P}_d \to \mathbb{P}_k$ the operation $\partial_{k|d}\, p := \frac{p^{(d-k)}}{(d)_{\underline{d-k}}}$. Then:

- $\widetilde{e}_j^{(k)}(\partial_{k|d}\, p) = \widetilde{e}_j^{(d)}(p) \qquad$ for $1 \leq j \leq k$

$$\partial_{k|d}\left(p \boxplus_d q\right) = \left(\partial_{k|d}\, p\right) \boxplus_k \left(\partial_{k|d}\, q\right)$$

$$\partial_{k|d}\left(p \boxtimes_d q\right) = \left(\partial_{k|d}\, p\right) \boxtimes_k \left(\partial_{k|d}\, q\right)$$

- $\kappa_m^{(j)}\left(\partial_{j|d}\, p\right) = \kappa_m^{(d)}\left(\mathrm{Dil}_{\frac{j}{d}}\, p^{\boxplus_d \frac{d}{j}}\right)$, where

$$\kappa_j^{(d)}(p) := d^{j-1} \sum_{\pi \in P(j)} c(\pi)\, \widetilde{e}_\pi^{(d)}(p)$$

This yields a **new proof** of the following:

[Hoskins–Kabluchko '21, Arizmendi–Garza-Vargas–P '23]
Let $\mathfrak{p} = (p_d)_{d=1}^{\infty}$ be a sequence of polynomials such that $\mu\,[\![p_d]\!] \longrightarrow \nu(\mathfrak{p})$ real measure determined by moments. Then

$$\partial_{k|d}\, p_d \longrightarrow \mathrm{Dil}_t(\mu^{\boxplus 1/t}) \qquad \text{as} \qquad d \to \infty, \text{ and } \frac{k}{d} \to t \in (0,1)$$

Case 1: compact support away from 0

Theorem

Assume the roots of the polynomials are contained in $C = [\alpha, \beta] \subset (0, \infty)$. Then

$$\lim_{\substack{d \to \infty \\ \frac{k}{d} \to t}} S_{p_d}^{(d)}\left(-\frac{k}{d}\right) = S_\mu(-t).$$

$$\begin{aligned} S_p^{(d)}\left(-\frac{k}{d}\right) &= \frac{\widetilde{e}_{k-1}^{(d)}(p)}{\widetilde{e}_k^{(d)}(p)} \\ &= \frac{\widetilde{e}_{k-1}^{(k)}(\partial_{k|d}\, p)}{\widetilde{e}_k^{(l)}(\partial_{k|d}\, p)} \\ &= \frac{1}{k}\sum_{j=1}^{k} \frac{1}{\lambda(\partial_{k|d}\, p)} = -G_{\mu[\![\partial_{k|d}\, p]\!]}(0) \end{aligned}$$

Then take limits and use relation from free probability that roughly says:

$$S_\mu(-t) = S_{\mathrm{Dil}_t(\mu^{\boxplus 1/t})}(-1) = \int_0^\infty x^{-1}\mu_t(dx) = -G_{\mathrm{Dil}_t(\mu^{\boxplus 1/t})}(0)$$

Proof of General case

Requires several steps, that gradually generalizing the previous ones.

- **Compact interval containing 0.** To avoid problems with $\frac{1}{k}\sum_{j=1}^{k}\frac{1}{\lambda(\partial_{k|d}\, p)}$ we show that for $t \approx \frac{k}{d}$ small enough, there exists $\varepsilon > 0$ such that $\partial_{k|d}\, p \in \mathbb{P}(\mathbb{R}_{\geq \varepsilon})$ (roots are uniformly bounded away from 0).

  Intuition: For $\mu \in \mathcal{M}(\mathbb{R}_{\geq 0})$ and t small enough, then $\mu^{\boxplus 1/t} \in \mathcal{M}(\mathbb{R}_{\geq \varepsilon})$.

  The proof of the uniform bound requires two main ingredients:

  1. Show the bound for polynomials of the form $x^j(1-x)^{d-j}$.
     (Classic result on the asymptotics of Jacobi polynomials)
  2. Use a partial order on polynomials to reduce to the previous case.

- **Case with unbounded support.** We use cut-up and cut-down measures, and the partial order on polynomials to reduce to the bounded case.
  Note: We can also generalize the results on differentiation and fractional convolution to allow unbounded support.

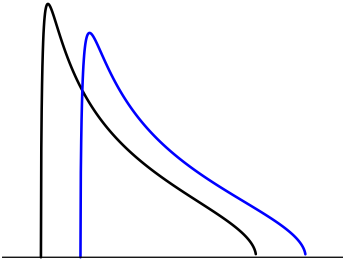
- **Converse implication**. Follows from the first implication and Helly's Selection Theorem.

A partial order on polynomials

Partial order on measures. Given $\mu, \nu \in \mathcal{M}(\mathbb{R})$ we say that $\mu \ll \nu$ if their cumulative distribution functions satisfy $F_\mu(t) \geq F_\nu(t) \qquad$ for all $t \in \mathbb{R}$.

[Bercovici, Voiculescu '93] for measures $\mu, \nu \in \mathcal{M}(\mathbb{R})$ such that $\mu \ll \nu$,

- if $\rho \in \mathcal{M}(\mathbb{R})$, then $(\mu \boxplus \rho) \ll (\nu \boxplus \rho)$, and
- if $\rho \in \mathcal{M}(\mathbb{R}_{\geq 0})$, then $(\mu \boxtimes \rho) \ll (\nu \boxtimes \rho)$.



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A partial order on polynomials: Given $p, q \in \mathcal{P}_d(\mathbb{R})$ we say that q, denoted $p \ll q$ if $\lambda_i(p) \leq \lambda_i(q)$ for all $i = 1, 2, \ldots, d$. Equivalently if $\mu[\![p]\!] \ll \mu[\![q]\!]$.

Theorem (Arizmendi, Fujie, P, Ueda '24)

$\boxplus_d$ and $\boxtimes_d$ preserve $\ll$. Namely, for $p, q \in \mathcal{P}_d(\mathbb{R})$ such that $p \ll q$.

1. *If $r \in \mathbb{P}_d(\mathbb{R})$, then $(p \boxplus_d r) \ll (q \boxplus_d r)$.*
2. *If $r \in \mathbb{P}_d(\mathbb{R}_{\geq 0})$, then $(p \boxtimes_d r) \ll (q \boxtimes_d r)$.*
3. *$(\partial_{k|d}\, p) \ll (\partial_{k|d}\, q)$.*
4. *$\Phi_d(p) \ll \Phi_d(q)$.*

Applications

Main applications

Theorem (Arizmendi, Fujie, P, Ueda '24)

$$\mu [\![p_d]\!] \to \mu \qquad \Leftrightarrow \qquad \mu [\![\Phi_d(p_d)]\!] \to \Phi(\mu).$$

Conjectured in [Fujie and Ueda, '23].

Proposition (Arizmendi, Fujie, P, Ueda '24)

Consider sequences $\mathfrak{p} = (p_d)_{d=1}^{\infty}$ and $\mathfrak{q} = (q_d)_{d=1}^{\infty}$ with $p_d, q_d \in \mathbb{P}_d(\mathbb{R}_{\geq 0})$ and limiting measures $\nu(\mathfrak{p}), \nu(\mathfrak{q}) \in \mathcal{M}(\mathbb{R}_{\geq 0})$. Then $\mu [\![p_d \boxtimes_d q_d]\!] \longrightarrow \nu(\mathfrak{p}) \boxtimes \nu(\mathfrak{q})$.

An extension of the result in [Arizmendi, Garza-Vargas, P'23] to **unbounded** measures, but only when supported in $\mathbb{R}_{\geq 0}$.

Hypergeometric polynomials

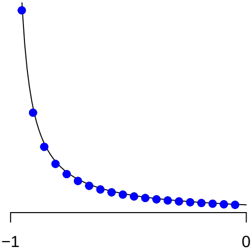
For $a_1, \dots, a_i, b_1, \dots, b_j \in \mathbb{R}$, consider $p_d = \mathcal{H}_d \left[\begin{smallmatrix} b_1, \dots, b_j \\ a_1, \dots, a_i \end{smallmatrix} \right] \in \mathbb{P}_d$ with coefficients

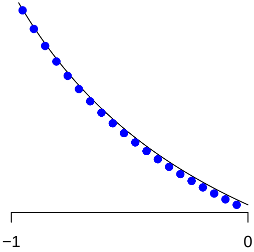
$$\widetilde{\mathrm{e}}_k^{(d)} \left(\mathcal{H}_d \left[\begin{smallmatrix} b_1, \dots, b_j \\ a_1, \dots, a_i \end{smallmatrix} \right] \right) := d^{k(i-j)} \frac{\prod_{s=1}^{j} (b_s d)^{\underline{k}}}{\prod_{r=1}^{i} (a_r d)^{\underline{k}}} \qquad \text{where} \;\; (d)^{\underline{k}} := \frac{d!}{(d-k)!}.$$

Then its finite S-transform is

$$S_{p_d}^{(d)} \left(-\frac{k}{d} \right) = \frac{\prod_{r=1}^{i} (a_r - \frac{k-1}{d})}{\prod_{s=1}^{j} (b_s - \frac{k-1}{d})} \quad \longrightarrow \quad \frac{\prod_{r=1}^{i} (a_r + t)}{\prod_{s=1}^{j} (b_s + t)} = S_\mu(t)$$

This limit was also obtained recently in **[Morales, Martinez-Finkelshtein, P ' 24]**





Finite free multiplicative Poisson's law of small numbers

[Bercovici, Voiculescu '92]: for $\lambda \geq 0$ and $\beta \in \mathbb{R} \setminus [0,1]$ there exists a measure $\Pi_{\lambda,\beta} \in \mathcal{M}(\mathbb{R}_{\geq 0})$ with S-transform given by

$$S_{\Pi_{\lambda,\beta}}(t) = \exp\left(\frac{\lambda}{t+\beta}\right).$$

It can be understood as a free multiplicative Poisson's law.

Proposition (Arizmendi, Fujie, P, Ueda '24)

Let $\lambda \geq 0$ and $\beta \in \mathbb{R} \setminus [0,1]$, and

$$p_d(x) := \left(x - \tfrac{\beta-1}{\beta}\right)(x-1)^{d-1} = \mathcal{H}_d\begin{bmatrix} \beta - \frac{1}{d} \\ \beta \end{bmatrix}.$$

Then

$$\mu\left[\!\left[p_d^{\boxtimes_d k} \right]\!\right] \to \Pi_{\lambda,\beta} \qquad \text{as } d \to \infty \quad \text{with } \frac{k}{d} \to \lambda.$$

Finite Free max-convolution powers

[Ben Arous, Voiculescu '06] Given $\nu_1, \nu_2 \in \mathcal{M}(\mathbb{R})$, define the free max-convolution measure $\nu_1 \boxvee \nu_2$, with

$$F_{\nu_1 \boxvee \nu_2}(x) := \max\{F_{\nu_1}(x) + F_{\nu_2}(x) - 1, 0\} \qquad \text{for all } x \in \mathbb{R}.$$

[Ueda '21] Given $\nu \in \mathcal{M}(\mathbb{R})$ and $t \geq 1$, define the convolution powers $\nu^{\boxvee t}$

$$F_{\nu^{\boxvee t}}(x) := \max\{tF_\nu(x) - (t-1), 0\},$$

Then, for $\mu \in \mathcal{M}(\mathbb{R}_{\geq 0})$, one has that $\Phi(\mathrm{Dil}_{1/t}(\mu^{\boxplus t})) = \Phi(\mu)^{\boxvee t}$.

Definition (Arizmendi, Fujie, P, Ueda '24)

Given $p \in \mathbb{P}_d(\mathbb{R})$ and $1 \leq k \leq d$, define $p^{\boxvee \frac{d}{k}} \in \mathbb{P}_k(\mathbb{R})$ with roots

$$\lambda_j\left(p^{\boxvee \frac{d}{k}}\right) = \lambda_j(p) \qquad \text{for } j = 1, \dots, k.$$

Proposition (Arizmendi, Fujie, P, Ueda '24)

Let $p \in \mathbb{P}_d(\mathbb{R}_{\geq 0})$. Then $\Phi_k(\partial_{k|d}\, p) = (\Phi_d(p))^{\boxvee \frac{d}{k}}$.

Other results

- Using hypergeometric polynomials and multiplicative convolution, we can provide some **finite analogue of free stable laws**.

- We can extend the definition of the finite S-transform to include **symmetric polynomials** in the real line. Similar to how it is done in [Arizmendi, Pérez-Abreu '09] for free probability.

Can we generalize this type of results to measures on the complex plane?

Thanks!