On the singular abelian rank of ultraproduct II_1 factors

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Background

Definition

Let M be a II_1 factor. A MASA $A \subset M$ is **singular** if any unitary $u \in \mathcal{U}(M)$ with $uAu^* = A$ has $u \in A$.

- This definition is originally due to Dixmier (54) who showed R contains a singular MASA.
- Popa (81) showed every separable II_1 factor M contains a singular MASA.
- If $A \subset M$ is singular, $uAu^* = vAv^*$ iff $u^*v \in A$. So in fact, any separable II_1 factor M contains uncountably many singular MASA.

Background

Definition (Popa)

For $P, Q \subset M$ subalgebras, we say $P \prec_M Q$ if there are projections $p \in P$ and $q \in Q$, a $*$-homomorphism $\theta : pPp \to qQq$, and a partial isometry $v \neq 0$ in M with $\theta(x)v = vx$.

If $A_1, A_2 \subset M$ are singular MASAs:

- $A_1 \prec_M A_2$ iff $A_2 \prec_M A_1$.
- $A_1 \prec_M A_2$ iff there exists a unitary $u \in M$ and projections $p \in A_1$, $q \in A_2$ with $uA_1pu^* = A_2q$.
- $A_1 \not\prec_M A_2$ (or A_1 and A_2 are **disjoint**) iff there is no partial isometry $v \in M$ with $vA_1v^* \subset A_2$.

Background

Theorem (Popa 16)

Let M be a separable II_1 factor. Then there is an uncountable family $\{A_i\}_{i\in I}$ of singular MASAs $A_i \subset M$ with $A_i \nprec A_j$ for all $i \neq j$.

Background

Definition (Boutonnet, Drimbe, Ioana, Popa 23)

- The **singular abelian core** of M is the (unique up to unitary conjugacy) largest abelian $*$-subalgebra $\mathcal{A} \subset \mathcal{M} := \mathcal{B}(\ell^2 K) \otimes M$ (for some $|K| \geq 2^{|\mathcal{U}(M)|}$) generated by finite projections with $\mathcal{A} \subset 1_{\mathcal{A}} \mathcal{M} 1_{\mathcal{A}}$ singular.
- The **singular abelian rank** of M is

$$r(M) := \mathrm{Tr}(1_{\mathcal{A}})$$

- $r(M)$ is an invariant
- $r(M)$ "counts" the largest number of of disjoint singular MASA that fit in M.
- The previous result of Popa implies $r(M) = \mathfrak{c}$ for M separable.

Background

Definition (Boutonnet, Drimbe, Ioana, Popa 23)

- The **non-separable singular abelian core**, or sans core, of M is the (unique up to unitary conjugacy) largest purely non-separable abelian $*$-subalgebra $\mathcal{A}_{\rm ns} \subset \mathcal{M} := \mathcal{B}(\ell^2 K) \otimes M$ (for some $|K| \geq 2^{|\mathcal{U}(M)|}$) generated by finite projections with $\mathcal{A} \subset 1_{\mathcal{A}} \mathcal{M} 1_{\mathcal{A}}$ singular.
- The **sans rank** of M is

$$r_{\rm ns}(M) := \mathrm{Tr}(1_{\mathcal{A}_{\rm ns}})$$

- $r_{\rm ns}(M)$ is an invariant that can distinguish between II_1 factors
- E.g. [BDIP 23] showed $r_{\rm ns}(A^{*n}) = n r_{\rm ns}(A)$.

Background

- It was observed in [BDIP 23] that for a separable II_1 factor M that

$$r_{\mathrm{ns}}(M^\omega) = r(M^\omega) \geq r(M) = \mathfrak{c}$$

Question

– Is $r_{\mathrm{ns}}(M) > \mathfrak{c}$ for all ultraproduct II_1 factors? (E.g. for M^ω or $\prod_\omega M_n(\mathbb{C})$)

– As an invariant, can $r_{\mathrm{ns}}(M)$ distinguish between ultraproduct II_1 factors?

Background

- **Answer:** Yes. $r_{\mathrm{ns}}(M) > \mathfrak{c}$ for all ultraproduct II_1 factors, but it is not a good invariant.

Theorem (HP24)

Let $\{M_n\}_{n\geq 1}$ be a sequence of separable tracial factors with $\dim(M_n) \to \infty$ and ω a free ultrafilter on $\mathbb{N}$. Denote $M = \Pi_\omega M_n$ the associated ultraproduct II_1 factor. If we assume the continuum hypothesis then $\mathrm{r}(M) > \mathfrak{c}$. If we further assume the strong continuum hypothesis, then $\mathrm{r}(M) = 2^{\mathfrak{c}}$.

- In fact, the above result depends only a bicommutant property of ultraproducts, namely $(A' \cap M)' \cap M = A$ for any separable abelian $A \subset M$.
- We will show the above result holds for a broader class of II_1 factors.

U factors

Definition (HP24)

Let M be a II_1 factor. We say M has property

- U_0 if $(A' \cap M)' \cap M = A$ for any separable abelian subalgebra $A \subset M$.
- U_1 if any trace preserving automorphism $\Phi : A_1 \to A_2$ between separable diffuse abelian subalgebras in M is implemented by some $u \in \mathcal{U}(M)$.

- Any ultraproduct II_1 factor M satisfies U_0 and U_1. (Popa, Connes)

U factors

Theorem (HP24)

A II_1 factor is U_0 iff it is U_1. Moreover, U_0 and U_1 are stable properties. (i.e. M has U_0 iff M^t has U_0 for all $t > 0$)

- We say M is a **U-factor** if M has U_0 (respectively U_1).
- Any ultraproduct II_1 factor $M = \prod_\omega M_n$ is a U factor.
- A result of Gao, Kunnawalkam Elayavalli, Patchell, Tan (24) shows there are U factors that are not ultraproduct II_1 factors

U factors

Definition (HP24)

Let M be a II_1 factor. We say M has property

- U_1 if any trace preserving automorphism $\Phi : A_1 \to A_2$ between separable diffuse abelian subalgebras in M is implemented by some $u \in \mathcal{U}(M)$.
- U_2 if for any A_1, A_2 separable diffuse abelian subalgebras of M there is a unitary $u \in \mathcal{U}(M)$ with $uA_1u^* = A_2$.

- I.e. U_2 says for $A_1 \cong A_2 \subset M$ separable abelian at least one automorphisms $\Phi : A_1 \to A_2$ is implemented by a unitary.
- $U_0 \Leftrightarrow U_1 \Rightarrow U_2$

U factors

Theorem (Structural Results for U factors (Popa))

If M has U_2 (E.g. M is a U factor).

- Given any MASA A in M, there exists a diffuse abelian von Neumann subalgebra $B_0 \subset M$ orthogonal to A.
- Any separable abelian von Neumann subalgebra $A_0 \subset M$ has type II_1 relative commutant $A_0' \cap M$.
- Any MASA in M is purely non-separable.
- M has no Cartan MASA.
- M is prime.

Main Result

Theorem (HP24)

Let M be a II_1 U-factor with the property that the cardinality of its unitary group $\mathcal{U}(M)$ is equal to $\mathfrak{c}$. If the continuum hypothesis, $\mathfrak{c} = \aleph_1$, is assumed, then M contains more than $\mathfrak{c}$ many mutually disjoint singular MASAs, i.e., $\mathrm{r}(M) > \mathfrak{c}$. Moreover, if the strong continuum hypothesis $2^{\mathfrak{c}} = \aleph_2$ is assumed, then $\mathrm{r}(M) = 2^{\mathfrak{c}}$.

Main Result

Sketch of the proof:

- Inspired by a technique of Popa
- Imagine we want to construct a singular MASA $B \subset M$ with some properties (i.e. $B \not\prec A_i$ for some family of algebrs $\{A_i\}_{i \in I}$).
- We use transfinite induction to build up $B = \overline{\bigcup_{i \in I} B_i}^{\text{wo}}$ as a tower of separable abelian algebras B_i.
- At each stage, we choose $B_{i+1} \subset B_i' \cap M$ to look "more singular" and "more disjoint" from the A_i.
- The key ingredients will be
  - $B_i' \cap M$ is type II_1
  - $(B_i' \cap M)' \cap M = B_i$

Main Result

What does it mean to be "more singular"?

- At step i, we prevent a new partial isometry $v_i \in M$ from giving rise to a non-trivial element of the normalizer $\mathcal{N}_M(B)$.
- I.e. we construct B_i so that if $v_i^* v_i, v_i v_i^* \in B_i$ then $[v_i B_i v_i^*, B_i v_i v_i^*] \neq 0$.

What does it mean to be "more disjoint"?

- At step i, we prevent a new partial isometry $v_i \in M$ from intertwining B into A_i.
- I.e. we construct B_i so that if $v_i^* v_i \in B$ and $v_i v_i^* \in A_i$, then $[v_i B_i v_i^*, A_i v_i v_i^*] \neq 0$.

Proof of Main Result

Setup:

- Let $(I, \leq)$ be the set of all ordinals $\alpha < \aleph_1$.
- By CH, we can index the partial isometries of M with $\mathfrak{c}$ repetition by $\{v_i\}_{i \in I}$.
- Take $\{A_i\}_{i \in I}$ a family of singular MASAs in M.

Claim

There is an increasing family of separable abelian subalgebras $\{B_i\}_{i \in I}$ such that

- $B = \overline{\bigcup_{i \in I} B_i}^{\mathrm{wo}}$ is a singular MASA in M.
- $B \not\prec A_j$ for all $j \in I$

- We proceed by transfinite induction.
- Assume we have $\{B_j\}_{j\in I}$.

Induction Step 1: Let $B_i^0 = \overline{\cup_{j<i} B_j}^{\mathrm{wo}}$

- Note B_i^0 is still separable abelian
- B_i^0 is larger than the previous algebras B_j for $j < i$.

Proof of Main Result

Induction Step 2: Prevenet v_i from intertwining B into A_j for $j \leq i$.

- I.e. prevent $v_i^* v_i \in B$, $v_i v_i^* \in A_j$, and $v_i B v_i^* \subset A_j$ (so v_i moves a corner of B into a corner of into A_j).
- This could happen if $v_i^* v_i \in Q := (B_i^0)' \cap M$ and $v_i v_i^* \in A_j$.
- But look at the countable set

$$K = \{j \leq i \mid v_i^* v_i \in Q, v_i v_i^* \in A_j\}$$

- Then since Q is type II_1, $Q \setminus \bigcup_{j \in K} v_i^* A_j v_i$ is dense G_δ in Q.
- So $\exists x \in Q \setminus \bigcup_{j \in K} v_i^* A_j v_i$ self adjoint.
- Call $B_i^1 = B_i^0 \vee \{x\}''$

Proof of Main Result

Induction Step 3: Make B "more singular"

- I.e. We want to avoid $v_i^* v_i, v_i v_i^* \in B$, and $v_i B v_i^* = B v_i v_i^*$, but $v_i \notin B$. (so v_i moves a corner of B to another corner of B).
- Calling $Q := (B_i^1)' \cap M$, this could happen if either
- **Case A:** $v_i \in Q$ but $v_i \notin B_i^1$. (i.e. v_i could be in B but we haven't added it in yet).
- **Case B:** $v_i \notin Q$ but $v_i v_i^*, v_i^* v_i \in Q$. (i.e. v_i can't be in B, but the support projects of v_i might be in B)
- For Case A, just take $B_i^2 = B_i^1 \vee \{v_i\}''$.

Proof of Main Result

For Case B:

- If $v_i B_i^1 v_i^* \neq B_i^1 v_i v_i^*$ then by U_0, $\exists a_1 \in B_i^1$ and $a_2 \in Q$ self adjoint with $[v_i a_1 v_i^*, a_2] \neq 0$. Then we take $B_i^2 = B_i^1 \vee \{a_2\}''$
- If $v_i B_i^1 v_i^* = B_i^1 v_i v_i^*$, take orthogonal central projections $z_1, z_2 \in Q$ with $z_1 \leq v_i^* v_i$, $z_2 \leq v_i v_i^*$ and $v_i z_1 v_i^* = z_2$.
- Qz_1 is type II_1, so contains non-commuting self adjoints u, w. Letting $c = u + v_i w v_i^*$ gives $[v_i(cz_1)v_i^*, cz_2] \neq 0$. We then let $B_2^i = B_i^1 \vee \{c\}''$.

Induction Step 4:

- We let $B_i = B_i^2$.
- We claim this family $\{B_i\}_{i\in I}$ satisfies our earlier claim.
- I.e. let $B = \overline{\bigcup_{i\in I} B_i}^{\mathrm{wo}}$. Then
- B is a MASA: any partial isometry $v = v_i \in B' \cap M \subset B_i' \cap M$ would have been added to B at step i.
- B is singular: follows from step 3
- B is disjoint from all A_i: follows from step 2.

Therefore, $\{A_i\}_{i\in I}$ can't be a maximal family of disjoint singular MASAs. Hence $r(M) > \mathfrak{c}$.

Main Result

Theorem

Let M be a II_1 U-factor M with the property that the cardinality of its unitary group $\mathcal{U}(M)$ is equal to $\mathfrak{c}$. If the continuum hypothesis, $\mathfrak{c} = \aleph_1$, is assumed, then M contains more than $\mathfrak{c}$ many mutually disjoint singular MASAs, i.e., $\mathrm{r}(M) > \mathfrak{c}$. Moreover, if the strong continuum hypothesis $2^{\mathfrak{c}} = \aleph_2$ is assumed, then $\mathrm{r}(M) = 2^{\mathfrak{c}}$.