Voiculescu's Theorem in Properly Infinite Factors

Jointly with

Minghui Ma and Junhao Shen

Let H be a separable Hilbert space, and $\mathcal{M} \subset B(H)$ be a factor.

An operator $T \in \mathcal{M}$ is called **reducible**, if there exists a projection $P \neq 0, I$ in $\mathcal{M}$, such that $TP = PT$. Let $Red(\mathcal{M})$ be the collection of reducible operators in $\mathcal{M}$.

Problem 1 *Is $\mathcal{M}$ the norm closure of $Red(\mathcal{M})$?*

In 1976 D. Voiculescu gave an affirmative answer when $\mathcal{M} = B(H)$, which answered a question of Halmos.

Another consequence of Voiculescu's noncommutative Weyl-von Neumann theorem is his Bicommutant theorem for the Calkin algebra:

For every separable unital C*-subalgebra $\mathcal{D} \subset B\left(\ell^2\right)/\mathcal{K}\left(\ell^2\right)$ we have $\mathcal{D} = \mathcal{D}''$.

Arveson extended this result to show that if $\mathcal{A}$ is a separable unital norm-closed subalgebra of $B\left(\ell^2\right)/\mathcal{K}\left(\ell^2\right)$, and $t \in B\left(\ell^2\right)/\mathcal{K}\left(\ell^2\right)$, then there is a projection $p \in B\left(\ell^2\right)/\mathcal{K}\left(\ell^2\right)$ such that

$$(1-p)\,\mathcal{A}p = \{0\} \text{ and } \|(1-p)\,tp\| = dist\,(t, \mathcal{A})\,.$$

G. Pedersen in 1988 asked if $\mathcal{B}$ is a σ-unital C*-algebra with multiplier algebra $\mathcal{M}(\mathcal{B})$ and corona algebra $C(\mathcal{B}) = \mathcal{M}(\mathcal{B})/\mathcal{B}$, **under what conditions on** $\mathcal{B}$, does every separable unital C*-algebra $\mathcal{D} \subset C(\mathcal{B})$ satisfy

$$\mathcal{D} = \mathcal{D}'' \text{ ?}$$

T. Giordano and P. W. Ng in 2019 gave an affirmative answer when $\mathcal{B}$ is stable, simple and purely infinite

Problem 2 *Suppose $\mathcal{M}$ is a separable type II_∞ factor and $\mathcal{K}_\mathcal{M}$ is the closed ideal generated by the finite projections in $\mathcal{M}$. Does Pedersen's question have an affirmative answer?*

Since $\mathcal{K}_\mathcal{M}$ is the unique nontrivial closed ideal in $\mathcal{M}$, this question is more closely related to Voiculescu's result, and $\mathcal{K}_\mathcal{M}$ is not purely infinite, so T. Giordano and P. W. Ng's result does not apply.

Voiculescu's non-commutative Weyl-von Neumann theorem gives a characterization of approximate (unitary) equivalence of unital representations of a separable C*-algebra $\mathcal{A}$ into $B\left(\ell^2\right)$.

We say that representations $\pi, \rho : \mathcal{A} \to B\left(\ell^2\right)$ are **approximately equivalent**, denoted by $\pi \sim_a \rho$, if and only if there is a sequence $\{U_n\}$ of unitary operators such that, for every $a \in \mathcal{A}$,

$$\lim_{n\to\infty} \left\|U_n^* \pi(a) U_n - \rho(a)\right\| = 0.$$

If in addition, we have, for every $n \geq 1$ and every $a \in \mathcal{A}$, we have

$$U_n^* \pi(a) U_n - \rho(a) \in \mathcal{K}\left(\ell^2\right),$$

we say that π and ρ are approximately equivalent **relative to** $\mathcal{K}\left(\ell^2\right)$ and denote it by $\pi \sim_a \rho \left(\mathcal{K}\left(\ell^2\right)\right)$.

Voiculescu's theorem (1976) says that the following are equivalent:

1. $\pi \sim_a \rho$

2. $\pi \sim_a \rho\left(\mathcal{K}\left(\ell^2\right)\right)$

3. We have

   (a) $\ker \pi = \ker \rho$

   (b) $\pi^{-1}\left(\mathcal{K}\left(\ell^2\right)\right) = \rho^{-1}\left(\mathcal{K}\left(\ell^2\right)\right)$,

   (c) The nonzero parts of $\pi|_{\pi^{-1}(\mathcal{K}(\ell^2))}$ and $\rho|_{\pi^{-1}(\mathcal{K}(\ell^2))}$ are unitarily equivalent.

In 1981, I proved that condition (3) in Voiculescu's theorem was equivalent to: For every $a \in \mathcal{A}$,

$$rank\left(\pi\left(a\right)\right) = rank\left(\rho\left(a\right)\right).$$

This formulation emphasizes the power of Voiculescu's theorem by showing that a simple purely algebraic condition is equivalent to a very strong geometric one.

I also proved that if $\mathcal{A}$ is any unital C*-algebra, and H is any Hilbert space and $\pi, \rho : \mathcal{A} \to B\left(H\right)$ are unital representations, the $\pi \sim_a \rho$ (using nets of unitaries) if and only if, for every $a \in \mathcal{A}$, $rank\left(\pi\left(a\right)\right) = rank\left(\rho\left(a\right)\right)$. Moreover, if $\mathcal{A}$ is separable and $m = \dim H > \aleph_0$, and $\mathcal{K}_m = \left\{K \in B\left(H\right) : rank\left(K\right) < m\right\}^{-\|\|}$ is the unique maximal ideal in $B\left(H\right)$, then $\pi \sim_a \rho$ is equivalent to $\pi \sim_a \rho \left(\mathcal{K}_m\right)$ if and only if m is countably cofinal (i.e., m is the sup of a sequence of smaller cardinals). In this case $B\left(H\right)/\mathcal{K}_m$ yields an affirmative answer to Pedersen's question.

A key ingredient in the proof of Voiculescu's theorem is the following:

Lemma 3 *Suppose $\mathcal{A}$ is a separable unital C*-subalgebra of $B\left(\ell^2\right)$ and H is a separable Hilbert space and $\varphi : \mathcal{A} \to B(H)$ is a unital completely positive map such that $\mathcal{A} \cap \mathcal{K}\left(\ell^2\right) \subset \ker \varphi$. Then there is a sequence $\{V_n\}$ of isometries such that*

1. *For every $A \in \mathcal{A}$, $\lim_{n\to\infty} \|V_n^* A V_n - \varphi(A)\| = 0$, and*

2. *For every $A \in \mathcal{A}$ and every $n \geq 1$, $V_n^* A V_n - \varphi(A) \in \mathcal{K}(H)$.*

Moreover, if φ is a unital $*$-homomorphism, then $id_{\mathcal{A}} \sim_a id_{\mathcal{A}} \oplus \varphi\left(\mathcal{K}\left(\ell^2\right)\right)$

This is the result we want to generalize.

Throughout we assume that $\mathcal{M}$ is a separable properly infinite factor and $\mathcal{A}$ is a separable unital C*-subalgebra of $\mathcal{M}$, and $\mathcal{K}_{\mathcal{M}}$ is the norm closed ideal generated by the finite projections in $\mathcal{M}$.

Definition 4 *(Factorable Map) Let ψ: $\mathcal{A} \to \mathcal{M}$ be a completely positive map with $\psi|_{\mathcal{A}\cap\mathcal{K}_{\mathcal{M}}} = 0$. If $\psi = \eta \circ \sigma$ for some $n \geq 1$ and some completely positive maps $\sigma : \mathcal{A} \to \mathbb{M}_n(\mathbb{C})$ and $\eta : \mathbb{M}_n(\mathbb{C}) \to \mathcal{M}$ with $\sigma|_{\mathcal{A}\cap\mathcal{K}_{\mathcal{M}}} = 0$, then we say that ψ is* **factorable** *with respect to $\mathcal{K}_{\mathcal{M}}$. Let $\mathcal{F} = \mathcal{F}(\mathcal{A}, \mathcal{M}, \mathcal{K}_{\mathcal{M}})$ denote the set of all factorable maps with respect to $\mathcal{K}_{\mathcal{M}}$ from $\mathcal{A}$ into $\mathcal{M}$.*

The following results by A. Ciuperca, T. Giordano, P. W. Ng, and

Z. Niu in 2013 played a critical role in the proof of our result.

Lemma 5 *Let $\mathcal{M}$ be a separable properly infinite factor, $\mathcal{A}$ a unital separable C*-subalgebra of $\mathcal{M}$, and $P \in \mathcal{K}_{\mathcal{M}}$ a finite projection. Then every map $\psi \in \mathcal{F}(\mathcal{A},\mathcal{M},\mathcal{K}_{\mathcal{M}})$ can be approximated in the pointwise-norm topology by maps of the form $A \mapsto V^*AV$, where $V \in \mathcal{M}$ and $PV = 0$. In particular, V can be selected as a partial isometry such that $V^*V = \psi(I)$ when $\psi(I)$ is a projection.*

Definition 6 *Let* $\mathcal{SF} = \mathcal{SF}(\mathcal{A}, \mathcal{M}, \mathcal{K}_{\mathcal{M}})$ *denote the set of all maps of the form* $\sum_{n \in \mathbb{N}} \psi_n$*, where* $\psi_n \in \mathcal{F}(\mathcal{A}, \mathcal{M}, \mathcal{K}_{\mathcal{M}})$ *for every* $n \in \mathbb{N}$*, and the series* $\sum_{n \in \mathbb{N}} \psi_n(I)$ *converges in the strong-operator topology. Let* $c\mathcal{SF} = c\mathcal{SF}(\mathcal{A}, \mathcal{M}, \mathcal{K}_{\mathcal{M}})$ *denote the closure of* $\mathcal{SF}$ *in the pointwise-norm topology.*

Theorem 7 *Let* $\mathcal{M}$ *be a separable properly infinite factor,* $\mathcal{A}$ *a unital C*-subalgebra of* $\mathcal{M}$*, and* $P \in \mathcal{K}_{\mathcal{M}}$ *be a finite projection. Then any* $\psi \in c\mathcal{SF}$ *can be approximated in the pointwise-norm topology by maps of the form* $A \mapsto V^*AV$*, where* $V \in \mathcal{M}$ *and* $PV = 0$*. In particular,* V *can be selected as a partial isometry such that* $V^*V = \psi(I)$ *when* ■(I) *is a projection.*

When$\mathcal{M}$ is a semifinite properly infinite factor, we have the following result.

Theorem 8 *Let $\mathcal{M}$ be a separable properly infinite semifinite factor, $\mathcal{A}$ a separable unital C*-subalgebra of $\mathcal{M}$, and $P \in \mathcal{K}_{\mathcal{M}}$ a finite projection. Then for any $\psi \in c\mathcal{SF}$, there is a sequence $\{V_k\}$ in $\mathcal{M}$, such that*

(1) $PV_k = 0$ for every $k \in \mathbb{N}$.

(2) $\lim_{k\to\infty} \left\| \psi(A) - V_k^* A V_k \right\| = 0$ for every $A \in \mathcal{A}$.

(3) $\psi(A) - V_k^* A V_k \in \mathcal{K}_{\mathcal{M}}$ for every $k \in \mathbb{N}$ and every $A \in \mathcal{A}$.

In particular, V_k can be selected as a partial isometry such that $V_k^* V_k = \psi(I)$ when $\psi(I)$ is a projection.

Here are our main results.

Theorem 9 *Let $\mathcal{M}$ be a separable properly infinite factor, $\mathcal{A}$ a separable unital C*-subalgebra of $\mathcal{M}$. If $\pi \in c\mathcal{SF}$ is a unital $*$-homomorphism, then*

$$id_{\mathcal{A}} \sim_a id_{\mathcal{A}} \oplus \pi \ .$$

Furthermore, if $\mathcal{M}$ is semifinite, then

$$id_{\mathcal{A}} \sim_a id_{\mathcal{A}} \oplus \pi \ \left(\mathcal{K}_{\mathcal{M}}\right).$$

This doesn't give us the full analogue of Voiculescu's key lemma. However, $\mathcal{M}$ contains a copy $\mathcal{B}$ of $B\left(\ell^2\right)$, and we have the following analogue.

Theorem 10 *Let $\mathcal{M}$ be a separable properly infinite factor, $\mathcal{A}$ a separable unital C*-subalgebra of $\mathcal{M}$. If $\pi : \mathcal{A} \to \mathcal{B}$ is a unital $*$-homomorphism and $\mathcal{A} \cap \mathcal{K}_{\mathcal{M}} \subset$* $\ker \pi$, *then*

$$id_{\mathcal{A}} \sim_a id_{\mathcal{A}} \oplus \pi \ .$$

Furthermore, if $\mathcal{M}$ is semifinite, then

$$id_{\mathcal{A}} \sim_a id_{\mathcal{A}} \oplus \pi \ \left(\mathcal{K}_{\mathcal{M}}\right).$$

Corollary 11 *If* $\mathcal{M}$ *be is separable properly infinite factor, then* $Red\,(\mathcal{M})$ *is dense in* $\mathcal{M}$*.*

Corollary 12 *If* $\mathcal{M}$ *be is separable properly infinite semifinite factor, and* $\mathcal{A}$ *is a unital separable norm closed subalgebra of* $\mathcal{M}/\mathcal{K}_{\mathcal{M}}$ *and* $t \in \mathcal{M}/\mathcal{K}_{\mathcal{M}}$*, then there is a projection* $p \in \mathcal{M}/\mathcal{K}_{\mathcal{M}}$ *such that*

$$(1-p)\,\mathcal{A}p = \{0\} \text{ and } \|(1-p)\,tp\| = dist\,(t, \mathcal{A})\,.$$

If $\mathcal{A}$ *is a C*-algebra, then* $\mathcal{A} = \mathcal{A}''$*.*

When $\mathcal{M}$ is a separable type III factor, then $\mathcal{K}_{\mathcal{M}} = 0$ and we obtain asymptotic versions of the above corollaries.

Theorem 13 *Suppose $\mathcal{M}$ is a separable type III factor and $\mathcal{A}$ is a unital closed subalgebra of $\mathcal{M}$ and $T \in \mathcal{M}$. Then there is a net $\{P_\lambda\}$ of projections such that*

1. $\lim_\lambda \|(1 - P_\lambda) A P_\lambda\| = 0$ *for every $A \in \mathcal{A}$, and*

2. $\lim_\lambda \|(1 - P_\lambda) T P_\lambda\| = dist\,(T, \mathcal{A})$.

Moreover, if $\mathcal{A}$ is a C-algebra, then $\mathcal{A}$ is equal to its approximate bicommutant in $\mathcal{M}$.*

The theorem says that $\mathcal{A}$ is approximately hyperreflexive in $\mathcal{M}$. For $B(H)$ the first author proved that if $\mathcal{A}$ is a unital C*-subalgebra of $B(H)$ and $T \in B(H)$, there is a net $\{P_\lambda\}$ of projections in $B(H)$ such that (1) holds and $\lim_\lambda \|(1 - P_\lambda) T P_\lambda\| \geq \frac{1}{29} dist\,(T, \mathcal{A})$.

Suppose $\mathcal{A}$ is a subalgebra $B\left(\ell^2\right)$. We define

$$\textit{ApprAlgLat}\,(\mathcal{A})$$

to be the set of all operators T for which $\lim_\lambda \|(1 - P_\lambda)\, T P_\lambda\| = 0$ whenever $\{P_\lambda\}$ is a net of projections in $B\left(\ell^2\right)$ such that, for every $A \in \mathcal{A}$, $\lim_\lambda \|(1 - P_\lambda)\, A P_\lambda\| = 0$. The algebra $\mathcal{A}$ is called **strongly reductive** whenever $\mathcal{A}^* \subset \textit{ApprAlgLat}(\mathcal{A})$. In 1976 C. Apostol, C. Foiaș, D. Voiculescu proved that every commutative strongly reductive algebra is a C*-algebra. Later, Bebe Prunaru proved the same for any subalgebra of $B\left(\ell^2\right)$. Our last theorem proves that any unital norm closed subalgebra of a separable type III von Neumann algebra $\mathcal{M}$ that is strongly reductive in $\mathcal{M}$ is a C*-algebra.

After our paper was typed up, we learned from our private communication with P. W. Ng that Giordano, Kaftal and Ng also obtained similar results with different proofs.

Some of our results are true in more general algebras.

Suppose φ is a state on a unital C*-algebra $\mathcal{A}$. We say that φ can be **excised by projections** if there is a net $\{P_\lambda\}$ of nonzero projections such that, for every $A \in \mathcal{A}$,

$$\lim_{\lambda} \|P_\lambda A P_\lambda - \varphi(A) P_\lambda\| = 0.$$

Nate Brown 2005 proved that if $\mathcal{A}$ is simple and $RR0$, then every state on $\mathcal{A}$ can be excised by projections.

We say that a unital C*-algebra $\mathcal{A}$ is **projectionally prime** if whenever $P, Q \in \mathcal{A}$ and $P\mathcal{A}Q = 0$, we must have $P = 0$ or $Q = 0$. A von Neumann algebra is projectionally prime if and only if it is a factor. An $RR0$ unital C*-algebra $\mathcal{A}$ is projectionally prime if and only if each pair P, Q of nonzero projections in $\mathcal{A}$ have nonzero subprojections that are Murray von Neumann equivalent.

Lemma 14 *Suppose $\mathcal{A}$ is a unital $RR0$ C*-algebra. The following are equivalent:*

(1) *Every state on $\mathcal{A}$ can be excised by projections.*

(2) *$\mathcal{A} = \mathbb{C}$ or $\mathcal{A}$ is projectionally prime and has no minimal projections.*

Lemma 15 *Suppose $\mathcal{A}$ is a unital $RR0$ C*-algebra and J is an ideal in $\mathcal{A}$ such that*

(1) *$\mathcal{A}/J$ is projectionally prime and has no minimal projections.*

(2) *Every nonzero projection in $\mathcal{A}/J$ is equivalent to I.*

Then, for every state φ on $\mathcal{A}$ such that $\mathcal{A} \cap J \subset \ker \varphi$, there is a net $\{V_\lambda\}$ of isometries in $\mathcal{A} \backslash J$ such that, for every $A \in \mathcal{A}$,

$$\lim_{\lambda} \|V_\lambda^* A V_\lambda - \varphi(A) I\| = 0.$$

Lemma 16 *Suppose $\mathcal{A}$ is a unital $RR0$ C*-algebra and J is an ideal in $\mathcal{A}$ such that*

(1) *$\mathcal{A}/J$ is projectionally prime and has no minimal projections.*

(2) *Every nonzero projection in $\mathcal{A}/J$ is equivalent to I.*

(3) *For some $n \in \mathbb{N}$, $\mathcal{A}$ is isomorphic to $\mathbb{M}_n(\mathcal{A})$ such that the matrix unit e_{11} is equivalent to I.*

Then, for every unital completely positive map $\varphi : \mathcal{A} \to \mathbb{M}_n(\mathbb{C})$ with $\mathcal{A} \cap J \subset \ker \varphi$, there is a net $\{V_\lambda\}$ of isometries in $\mathcal{A}$ such that, for every $A \in \mathcal{A}$,

$$\lim_{\lambda} \|V_\lambda^* A V_\lambda - \varphi(A)\| = 0.$$

Suppose $\mathcal{A}$ is a purely infinite factor von Neumann algebra on any Hilbert space, and J is a maximal ideal of $\mathcal{A}$. Then the preceding lemma applies.

Theorem 17 *Suppose $\mathcal{M}$ is a purely infinite factor von Neumann algebra on any Hilbert space and J is a maximal ideal in $\mathcal{M}$ and J is sequentially weak* dense in $\mathcal{M}$. Then there is a von Neumann subalgebra $\mathcal{B}$ of $\mathcal{M}$ such that $\mathcal{B}$ is isomorphic to a countable direct sum of copies of $B\left(\ell^2\right)$,and such that, if $\mathcal{A}$ is a separable unital C*-subalgebra of $\mathcal{M}$ and $\pi : \mathcal{A} \to \mathcal{B}$ is a unital $*$-homomorphism such that $\mathcal{A} \cap J \subset \ker \pi$, then*

$$id_{\mathcal{A}} \sim_a id_{\mathcal{A}} \oplus \pi \ (J).$$

Corollary 18 *Suppose $\mathcal{M}$ is a purely infinite factor von Neumann algebra on any Hilbert space and J is a maximal ideal in $\mathcal{M}$ and J is sequentially weak* dense in $\mathcal{M}$. If $\mathcal{A}$ is a unital separable norm closed subalgebra of $\mathcal{M}/J$ and $T \in \mathcal{M}/J$, then there is a projection $P \in \mathcal{M}/J$ such that*

$$P\mathcal{A}P = 0 \text{ and } \|(1-P)\,TP\| = dist\,(T,\mathcal{A}).$$

If $\mathcal{A}$ is a C-algebra, then $\mathcal{A} = \mathcal{A}''$.*