The Fuglede-Kadison determinant of matrix-valued semicircular elements and the capacity of completely positive maps

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Basics from noncommutative probability theory

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Throughout the following, let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space, i.e., a finite von Neumann algebra $\mathcal{M}$ which is endowed with a faithful normal tracial state $\tau : \mathcal{M} \to \mathbb{C}$.

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Definition

For $x = x^* \in \mathcal{M}$, the (analytic) distribution of x is the Borel probability measure μ_x on the real line $\mathbb{R}$ which is uniquely determined by

$$\phi\big((z\mathbf{1} - x)^{-1}\big) = \mathcal{G}_{\mu_x}(z) \qquad \text{for all } z \in \mathbb{C}^+.$$

Notation: For any Borel probability measure μ on $\mathbb{R}$, we denote by

$$\mathcal{G}_\mu :\ \mathbb{C}^+ \to \mathbb{C}^-, \quad z \mapsto \int_{\mathbb{R}} \frac{1}{z-t}\, d\mu(t),$$

where $\mathbb{C}^\pm := \{z \in \mathbb{C} \mid \pm \operatorname{Im}(z) > 0\}$, the Cauchy transform of μ.

Matrix-valued semicircular elements I

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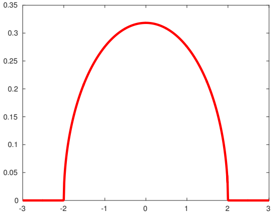
Definition

An operator $s = s^* \in \mathcal{M}$ which satisfies

$$d\mu_s(t) = \frac{1}{2\pi}\sqrt{4-t^2}\,\mathbf{1}_{[-2,2]}(t)\,dt$$

is called (standard) semicircular element.



Let us consider now the "augmented" tracial W^*-probability space $(M_m(\mathbb{C}) \otimes \mathcal{M}, \mathrm{tr}_m \otimes \tau)$, where $\mathrm{tr}_m = \frac{1}{m}\mathrm{Tr}_m$, $\mathrm{Tr}_m((a_{ij})_{i,j=1}^m) = \sum_{i=1}^m a_{ii}$.

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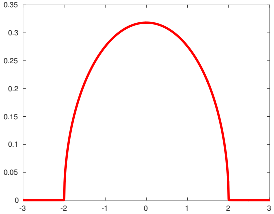

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Definition

A matrix-valued semicircular element is a noncommutative random variable in $(M_m(\mathbb{C}) \otimes \mathcal{M}, \mathrm{tr}_m \otimes \tau)$ which is of the form

$$S := a_1 \otimes s_1 + \cdots + a_n \otimes s_n$$

with
1. $s_1, \dots, s_n$ freely independent semicircular elements in $\mathcal{M}$;
2. $a_1, \dots, a_n$ selfadjoint matrices in $M_m(\mathbb{C})$.

Matrix-valued semicircular elements II

$S := a_1 \otimes s_1 + \cdots + a_n \otimes s_n$ is a centered operator-valued semicircular element with the covariance $\eta : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ which is given by

$$\eta(b) := \mathbb{E}[SbS] = \sum_{j=1}^{n} a_j b a_j \qquad \text{with} \qquad \mathbb{E} := \mathrm{id}_{M_m(\mathbb{C})} \otimes \tau.$$

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- The operator-valued Cauchy transform of S, which is given by

$$G_S : \mathbb{H}^+(M_m(\mathbb{C})) \to \mathbb{H}^-(M_m(\mathbb{C})), \quad b \mapsto \mathbb{E}\big[(b \otimes \mathbf{1} - S)^{-1}\big],$$

where $\mathbb{H}^\pm(M_m(\mathbb{C})) := \{b \in M_m(\mathbb{C}) \mid \pm \mathrm{Im}(b) > 0\}$, is determined uniquely by the Dyson equation

$$bG_S(b) = \mathbf{1}_m + \eta(G_S(b))G_S(b) \qquad \text{for all } b \in \mathbb{H}^+(M_m(B)).$$

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- The scalar-valued Cauchy transform of μ_S is related to G_S by
$$\mathcal{G}_{\mu_S}(z) = \mathrm{tr}_m(G_S(z\mathbf{1}_m)) \quad \text{for all } z \in \mathbb{C}^+.$$

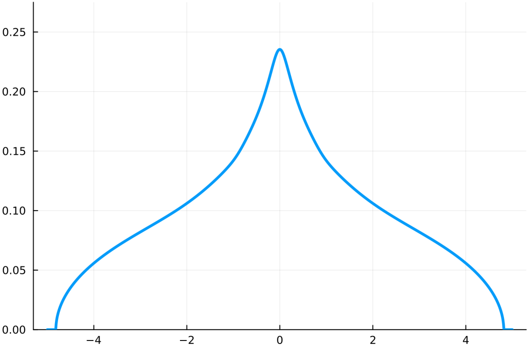
We obtain μ_S from $\mathcal{G}_S$ with the help of Stieltjes inversion.

Example

Consider the matrix-valued semicircular element

$$S = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes s_1 + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \otimes s_2 + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \otimes s_3.$$

We obtain for μ_S the following (approximate) density:



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Definition

Let $A \in M_m(\mathbb{C}\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle)$ be given.

- For $A \neq 0$, the (inner) rank $\operatorname{rank}(A)$ of A is the least integer $k \geq 1$ for which A can be written as $A = R_1 R_2$ with some rectangular matrices
$$R_1 \in M_{m \times k}(\mathbb{C}\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle) \quad \text{and} \quad R_2 \in M_{k \times m}(\mathbb{C}\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle).$$
In the particular case $A = 0$, we put $\operatorname{rank}(A) = 0$.
- We say that A is full if it has full rank, i.e., if $\operatorname{rank}(A) = m$.

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Noncommutative Edmonds' problem

Decide fullness (or, more generally, compute the inner rank) of
$$A = a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n \in M_m(\mathbb{C}\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle).$$

☞ [Garg, Gurvits, Oliveira, Wigderson (2016,2020)], ...

An analytic approach using matrix-valued semicirculars I

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$$A^{\mathrm{h}} = \begin{bmatrix} 0 & a_1 \\ a_1^* & 0 \end{bmatrix} \mathbf{x}_1 + \cdots + \begin{bmatrix} 0 & a_n \\ a_n^* & 0 \end{bmatrix} \mathbf{x}_n$$

  in $M_{2m}(\mathbb{C}\langle\mathbf{x}_1,\ldots,\mathbf{x}_n\rangle)$, we may assume without loss of generality that A is hermitian; note that $\operatorname{rank}(A^{\mathrm{h}}) = 2\operatorname{rank}(A)$.

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- To $A = A^*$, we associate the matrix-valued semicircular element $S = a_1 \otimes s_1 + \cdots + a_n \otimes s_n$ in $(M_m(\mathbb{C}) \otimes \mathcal{M}, \operatorname{tr}_m \otimes \tau)$, where $s_1,\dots,s_n$ are freely independent semicircular elements in $(\mathcal{M},\tau)$.

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Theorem (Shlyakhtenko, Skoufranis (2015); M., Speicher, Yin (2023))

1. The analytic distribution μ_S of S is of "regular type".
2. The only possible values of $\mu_S(\{0\})$ are $\{\frac{k}{m} \mid k = 0,1,\dots,m\}$.
3. We have $\operatorname{rank}(A) = m\big(1 - \mu_S(\{0\})\big)$.

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Proposition (M., Speicher, Hoffmann (2023))

Let μ be a Borel probability measure on $\mathbb{R}$. We define the function

$$\theta_\mu : \mathbb{R}^+ \to \mathbb{R}^+, \quad y \mapsto -y\,\mathrm{Im}(\mathcal{G}_\mu(iy)) = \mathrm{Re}(iy\mathcal{G}_\mu(iy))$$

on $\mathbb{R}^+ := (0,\infty)$, where $\mathcal{G}_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ is the Cauchy transform of μ. Then, the following statements hold true:

1. We have $\lim_{y\to\infty} \theta_\mu(y) = 1$ and $\lim_{y\to 0} \theta_\mu(y) = \mu(\{0\})$.

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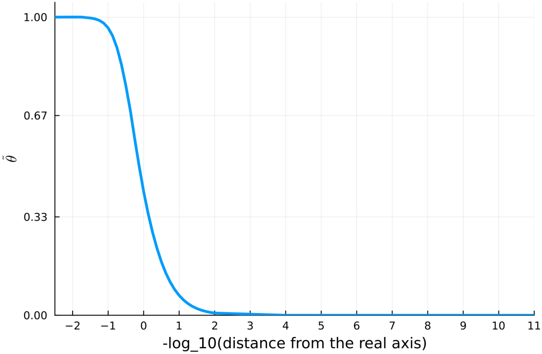
3. We have $\mu(\{0\}) \leq \theta_\mu(y)$ for all $y \in \mathbb{R}^+$.

Example

Consider again the matrix-valued semicircular element

$$S = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes s_1 + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \otimes s_2 + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \otimes s_3.$$

We can obtain arbitrarily good approximations $\tilde{\theta}$ for $\theta_{\mu_S}(y)$:



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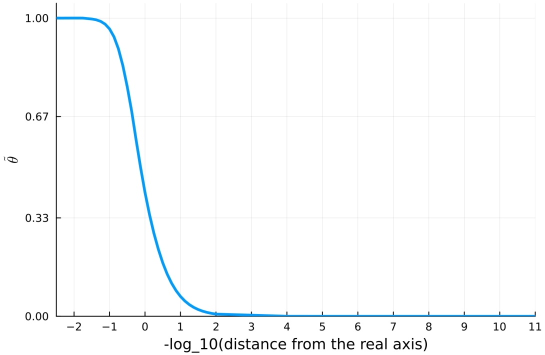

How small must $y > 0$ be chosen so that we can decide whether $\mu_S(\{0\}) < \frac{1}{m}$ or not?

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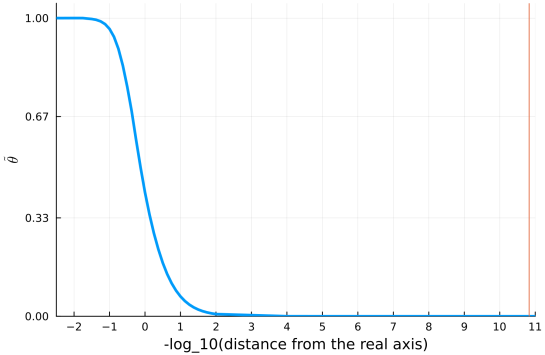

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How close do we have to get to the real axis?

How close do we have to get to the real axis?

Definition

A Borel probability measure μ is said to be of regular type if it is of the form

$$\mu = \mu(\{0\})\delta_0 + \nu$$

for some finite Borel measure ν for which there are $c \geq 0$, $\beta \in (0,1]$ and $r_0 > 0$ such that

$$\nu([-r,r]) \leq cr^\beta \qquad \text{for all } 0 < r < r_0.$$

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Proposition (M., Speicher, Hoffmann (2023))

Let μ be of regular type. Put $\gamma := \frac{2}{2+\beta}$ and $y_0 := r_0^{1/\gamma}$; then

$$\theta_\mu(y) - \mu(\{0\}) \leq (c + \nu(\mathbb{R}))y^{\frac{2\beta}{2+\beta}} \qquad \text{for all } 0 < y < y_0.$$

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Idea: Use Fuglede-Kadison determinant instead!

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Definition (Fuglede, Kadison (1952))

For a (not necessarily selfadjoint) operator $x \in \mathcal{M}$ its Fuglede-Kadison determinant is defined by

$$\Delta(x) := \exp\left(\int_0^\infty \log(t)\, d\mu_{|x|}(t)\right) \in [0, \infty),$$

where $|x| := (x^*x)^{1/2}$.

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Fact: For $(\mathcal{M}, \tau) = (M_m(\mathbb{C}), \operatorname{tr}_m)$, we get $\Delta(b) = |\det(b)|^{1/m}$.

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Lemma (M., Speicher, Hoffmann (2024))

Let $x = x^* \in \mathcal{M}$ be invertible. Then, for all $0 < \varepsilon < \|x\|$, we have

$$\mu_x([-\varepsilon, \varepsilon]) \leq \frac{\log \|x\| - \log \Delta(x)}{\log \|x\| - \log \varepsilon}.$$

… of matrix-valued semicircular elements

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Definition (Gurvits (2004))

The capacity of a positive map $\eta : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ is defined by

$$\operatorname{cap}(\eta) := \inf\{\det(\eta(b)) \mid b \in M_m(\mathbb{C}), b > 0, \det(b) = 1\}.$$

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Theorem (M., Speicher (2024))

Consider a (not necessarily selfadjoint) matrix-valued semicircular element

$$S = a_1 \otimes s_1 + \cdots + a_n \otimes s_n \quad \text{for} \quad a_1, \dots, a_n \in M_m(\mathbb{C})$$

with the associated covariance map

$$\eta : \; M_m(\mathbb{C}) \to M_m(\mathbb{C}), \quad b \mapsto \sum_{i=1}^{n} a_i b a_i^*.$$

Then we have for the Fuglede-Kadison determinant of S that

$$\Delta(S) = \operatorname{cap}(\eta)^{\frac{1}{2m}} e^{-\frac{1}{2}}.$$

The dual covariance map

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To a matrix-valued semicircular element $S = a_1 \otimes s_1 + \cdots + a_n \otimes s_n$, we can associate two completely positive maps $\eta, \eta^* : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ by

$$\eta(b) = \mathbb{E}[SbS^*] = \sum_{i=1}^{n} a_i b a_i^* \qquad \text{and} \qquad \eta^*(b) = \mathbb{E}[S^*bS] = \sum_{i=1}^{n} a_i^* b a_i.$$

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- These maps $\eta, \eta^* : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ are related by

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- Since $\Delta(S) = \Delta(S^*)$, our theorem readily implies $\mathrm{cap}(\eta) = \mathrm{cap}(\eta^*)$. This is consistent with the fact that

  $$\mathrm{cap}(\eta)^{\frac{1}{m}} = \inf\{\mathrm{tr}_m(\eta(b_1)b_2) \mid b_1, b_2 > 0, \det(b_1) = \det(b_2) = 1\}.$$

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The selfadjoint, doubly stochastic case

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Proposition (M., Speicher (2024))

Consider a selfadjoint matrix-valued semicircular element

$$S = a_1 \otimes s_1 + \cdots + a_n \otimes s_n \quad \text{for} \quad a_1, \ldots, a_n \in M_m(\mathbb{C})_{\text{sa}}$$

with the associated self-dual covariance map

$$\eta: \ M_m(\mathbb{C}) \to M_m(\mathbb{C}), \quad b \mapsto \sum_{i=1}^{n} a_i b a_i.$$

If η is doubly stochastic, that is, if $\eta(\mathbf{1}_m) = \mathbf{1}_m$, then μ_S is the standard semicircle distribution [Nica, Shlyakhtenko, Speicher (2002)] and thus

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Proof.

$\mathrm{d}\mu_S(t) = \frac{1}{2\pi}\sqrt{4-t^2}\,\mathbf{1}_{[-2,2]}(t)\,\mathrm{d}t$ yields $\mathrm{d}\mu_{|S|}(t) = \frac{1}{\pi}\sqrt{4-t^2}\,\mathbf{1}_{[0,2]}(t)\,\mathrm{d}t$; hence, we get $\Delta(S) = \exp\big(\int_{[0,2]} \log(t)\,\mathrm{d}\mu_{|S|}(t)\big) = e^{-\frac{1}{2}}$. □

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A positive linear map $\eta : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ is said to be

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3. If η is indecomposable (and hence rank non-decreasing), then
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is doubly stochastic.

Operator scaling

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Lemma (M., Speicher (2024))

Let $S = \sum_{i=1}^n a_i \otimes s_i$ be a matrix-valued semicircular element with the associated pair (η, η^*) of dual covariance maps and let $c_1, c_2 \in M_m(\mathbb{C})$. Then the associated pair of dual covariance maps of

$$\widetilde{S} := \sum_{i=1}^n (c_1 a_i c_2) \otimes s_i$$

is given by $(\eta_{c_1,c_2}, (\eta^*)_{c_2,c_1})$ and we have that

$$\Delta(\widetilde{S}) = |\det(c_1)|^{\frac{1}{m}} |\det(c_2)|^{\frac{1}{m}} \Delta(S).$$

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$$S^{\mathrm{h}} := \sum_{i=1}^n \begin{bmatrix} 0 & a_i \\ a_i^* & 0 \end{bmatrix} \otimes s_i$$

has the self-dual covariance map $\eta^{\mathrm{h}} : M_{2m}(\mathbb{C}) \to M_{2m}(\mathbb{C})$ given by

$$\eta^{\mathrm{h}}\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) = \begin{bmatrix} \eta(b_{22}) & \rho(b_{21}) \\ \rho^*(b_{12}) & \eta^*(b_{11}) \end{bmatrix}$$

with $\rho(b) := \sum_{i=1}^n a_i b a_i$. We call S^{h} the hermitization of S. Then

$$\Delta(S^{\mathrm{h}}) = \Delta(S).$$

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Let $S = \sum_{i=1}^{n} a_i \otimes s_i$ be a matrix-valued semicircular element for which the associated covariance map $\eta : M_m(\mathbb{C}) \to M_m(\mathbb{C}), b \mapsto \sum_{i=1}^{n} a_i b a_i^*$ is indecomposable (and hence rank non-decreasing). Then we have that

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- Then $\widetilde{\eta}^{\mathrm{h}}$ satisfies $\widetilde{\eta}^{\mathrm{h}}(\mathbf{1}_{2m}) = \mathbf{1}_{2m}$; hence, $\Delta(\widetilde{S}^{\mathrm{h}}) = e^{-\frac{1}{2}}$. □

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- Let $S^\flat$ be a selfadjoint matrix-valued semicircular element whose covariance map is the completely depolarizing channel

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- For $t \geq 0$, $S_t := S + \sqrt{t} S^\flat$ is a matrix-valued semicircular element with the associated pair (η_t, η_t^*) of dual covariance maps given by

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- For all $t > 0$, η_t is indecomposable. Thus, $\Delta(S_t) = \mathrm{cap}(\eta_t)^{\frac{1}{2m}} e^{-\frac{1}{2}}$.

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- We conclude that $\Delta(S) \geq \operatorname{cap}(\eta)^{\frac{1}{2m}} e^{-\frac{1}{2}}$ using the following result.

Proposition

Let T_n $(n \in \mathbb{N})$ and T be positive operators in some tracial W^*-probability space $(\mathcal{M}, \tau)$, such that $(T_n)_{n \in \mathbb{N}}$ converges in distribution to T, that is

$$\lim_{n \to \infty} \tau(T_n^k) = \tau(T^k) \qquad \text{for all } k \in \mathbb{N}.$$

Assume that all T_n and T have trivial kernel and that $\sup_n \|T_n\| < \infty$. Then we have

$$\Delta(T) \geq \limsup_{n \to \infty} \Delta(T_n).$$

Positive maps close to being doubly stochastic

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Definition (Gurvits (2004))

For a positive linear map $\eta : M_m(\mathbb{C}) \to M_m(\mathbb{C})$, we define

$$\mathrm{ds}(\eta) := \mathrm{Tr}_m\left((\eta(\mathbf{1}_m) - \mathbf{1}_m)^2\right) + \mathrm{Tr}_m\left((\eta^*(\mathbf{1}_m) - \mathbf{1}_m)^2\right).$$

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Let $\eta : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ be completely positive with $\mathrm{cap}(\eta) > 0$ and let $c > 0$ be an "approximate minimizer" of $\mathrm{cap}(\eta)$ in the sense that

$$\mathrm{cap}(\eta) \geq e^{-\delta} \cdot \frac{\det(\eta(c))}{\det(c)}$$

for some $\delta \in (0, \frac{1}{6}]$. Then

$$\mathrm{ds}(\eta_{\eta(c)^{-1/2}, c^{1/2}}) = \mathrm{Tr}_m\left((c\eta^*(\eta(c)^{-1}) - \mathbf{1}_m)^2\right) \leq 6\delta.$$

Recall: $\eta_{\eta(c)^{-1/2}, c^{1/2}}(b) = \eta(c)^{-1/2}\eta(c^{1/2}bc^{1/2})\eta(c)^{-1/2}$.

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- For the corresponding matrix-valued semicircular element $\widetilde{S}_\delta$, we get
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- From $\lim_{\delta \searrow 0} \operatorname{ds}(\eta_\delta) = 0$, it follows that $\lim_{\delta \searrow 0} \operatorname{ds}(\eta_\delta^{\mathrm{h}}) = 0$.

The general case: proof of the upper bound

Let $S = \sum_{i=1}^{n} a_i \otimes s_i$ be a matrix-valued semicircular element with the associated pair (η, η^*) of dual covariance maps. Suppose $\operatorname{cap}(\eta) > 0$.

- For $\delta \in (0, \frac{1}{6}]$, choose an "approximate minimizer" $c_\delta > 0$ of $\operatorname{cap}(\eta)$ in the sense that
$$\operatorname{cap}(\eta) \geq e^{-\delta} \cdot \frac{\det(\eta(c_\delta))}{\det(c_\delta)}.$$
- Define $\eta_\delta := \eta_{\eta(c_\delta)^{-1/2}, c_\delta^{1/2}}$; then $\operatorname{ds}(\eta_\delta) \leq 6\delta$.
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- From $\lim_{\delta \searrow 0} \operatorname{ds}(\eta_\delta^{\mathrm{h}}) = 0$, we infer that $\lim_{\delta \searrow 0} \|\eta_\delta^{\mathrm{h}}(\mathbf{1}_{2m}) - \mathbf{1}_{2m}\| = 0$.

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- Since $\lim_{\delta \searrow 0} \|\eta_\delta^{\mathrm{h}}(\mathbf{1}_{2m}) - \mathbf{1}_{2m}\| = 0$, the next proposition shows that the hermitizations $\widetilde{S}_\delta^{\mathrm{h}}$ converge in distribution to a standard semicircular element s as $\delta \searrow 0$.

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Proposition (M., Speicher (2024))

Let $S = \sum_{i=1}^n a_i \otimes s_i$ be selfadjoint with the associated self-dual covariance map $\eta : M_m(\mathbb{C}) \to M_m(\mathbb{C}), b \mapsto \sum_{i=1}^n a_i b a_i$. For all $k \in \mathbb{N}$, we have

$$\left|(\mathrm{tr}_m \otimes \tau)(S^{2k}) - \int_{-2}^{2} t^{2k}\, d\mu_s(t)\right| \leq C_k \left(\sum_{j=0}^{k-1} \|\eta\|^j\right) \|\eta(\mathbf{1}_m) - \mathbf{1}_m\|,$$

where C_k denotes the k-th Catalan number.

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- We infer that $\limsup_{\delta \searrow 0} \Delta(\widetilde{S}_{\delta}) = \limsup_{\delta \searrow 0} \Delta(\widetilde{S}^{\mathrm{h}}_{\delta}) \leq \Delta(s) = e^{-\frac{1}{2}}$.

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- We infer that $\limsup_{\delta\searrow 0} \Delta(\widetilde{S}_\delta) = \limsup_{\delta\searrow 0} \Delta(\widetilde{S}_\delta^{\mathrm{h}}) \leq \Delta(s) = e^{-\frac{1}{2}}$.
- Since $\Delta(S) \leq \operatorname{cap}(\eta)^{\frac{1}{2m}} e^{\frac{\delta}{2m}} \Delta(\widetilde{S}_\delta)$, we get $\Delta(S) \leq \operatorname{cap}(\eta)^{\frac{1}{2m}} e^{-\frac{1}{2}}$.

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- On the other hand, $\Delta(\widetilde{T})^2 = \Delta(\widetilde{T}\widetilde{T}^*) \leq (\mathrm{tr}_m \otimes \tau)(\widetilde{T}\widetilde{T}^*) = 1$. □

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We claim that in the situation described above, we even have that

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Thank you!