Operator-valued twisted Araki-Woods algebras

(joint w/ Rahul Kumar R)

1 Free Araki-Woods factors

H Hilbert space, $\mathcal{F}(H) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}$

$a^*(\xi)(\eta_1 \otimes \dots \otimes \eta_n) = \xi \otimes \eta_1 \otimes \dots \otimes \eta_n$

$a(\xi) = a^*(\xi)^*$

$s(\xi) = a^*(\xi) + a(\xi)$

[standard subspaces: $J: H \to H$ anti-unitary invol.,
(U_t) strongly cont. unitary group, $[U_t, J] = 0$]

$\Gamma_0(H, J, (U_t)) = \{ s(\xi) \mid JU_{-i/2}\xi = \xi \}''$

2 Tomita correspondences

(M, φ) von Neumann alg., φ n.s.f. weight

- **Def.** $(\mathcal{H}, \mathcal{J}, (U_t))$ Tomita correspondence if
$\mathcal{H}$ Hilbert M-bimodule $(\langle \cdot , \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C})$,
$\mathcal{J} : \mathcal{H} \to \mathcal{H}$ anti-lin. invol., (U_t) C_0-unitary group on $\mathcal{H}$
s.t.
  - $[\mathcal{J}, U_t] = 0$
  - $\mathcal{J}(x \xi y) = y^*(\mathcal{J}\xi)x^*$
  - $U_t(x \xi y) = \sigma_t^\varphi(x)(U_t \xi)\sigma_t^\varphi(y)$

Ex.: •) $M \subset N$, $E: N \to M$ n.f. cond. exp.

$\mathcal{H} = L^2(N)$, $\mathcal{J} = J_{\varphi \circ E}$, $\mathcal{U}_t = \Delta_{\varphi \circ E}^{it}$

•) $\mathcal{H} = {}_M L^2(M)_M \otimes H$, $\mathcal{J} = J_\varphi \otimes J$, $\mathcal{U}_t = \Delta_\varphi^{it} \otimes U_t$

$(H, J, (U_t))$ "standard subspace"

•) $(H, J, (U_t))$ "standard subspace"

$\pi: G \to \mathcal{U}(H)$ unitary rep., $[\pi(g), J], [\pi(g), U_t] = 0$

$\mathcal{H}_\pi = \ell^2(G) \otimes H$, $\lambda_g \cdot (\delta_s \otimes \xi) \cdot \lambda_h = \delta_{gsh} \otimes \pi(g) \xi$

$\rightsquigarrow$ $L(G)$-bimodule

$\mathcal{J}(\delta_s \otimes \xi) = \delta_{s^{-1}} \otimes \pi(s^{-1}) J \xi$, $\mathcal{U}_t = 1_{\ell^2(G)} \otimes U_t$

3 Operator-valued free Araki-Woods algebras

(M, φ), $(\mathcal{H}, \mathcal{J}, (\mathcal{U}_t))$ Tomita correspondence over (M, φ)

$$\mathcal{F}_0({}_M\mathcal{H}_M) = {}_M L^2(M)_M \oplus \bigoplus_{n=1}^{\infty} {}_M\mathcal{H}^{\otimes_\varphi n}{}_M \quad \text{(Pimsner)}$$

$\xi \in \mathcal{H}$ left φ-bounded:

$$a_0^*(\xi)(\eta_1 \otimes_\varphi \dots \otimes_\varphi \eta_n) = \xi \otimes_\varphi \eta_1 \otimes \dots \otimes \eta_n$$

$$a_0(\xi) = a_0^*(\xi)^*, \quad s_0(\xi) = a_0^*(\xi) + a_0(\xi)$$

$$\Gamma_0(\mathcal{H}, \mathcal{J}, (\mathcal{U}_t)) = M \vee \{ s(\xi) \mid \xi \text{ left-bdd.}, \mathcal{J}\mathcal{U}_{-i/2}\xi = \xi \}''$$

$$\subset \mathbb{B}(\mathcal{F}_0(\mathcal{H}))$$

4 Braided twists

$J : \mathcal{H} \otimes_\varphi \mathcal{H} \longrightarrow \mathcal{H} \otimes_\varphi \mathcal{H}$ self-adj., $\|J\| \leq 1$, bimodule map

$$J_k = 1_{\mathcal{H}}^{\otimes_\varphi (k-1)} \otimes_\varphi J \otimes_\varphi 1_{\mathcal{H}}^{\otimes_\varphi (n-k-1)} \in \mathbb{B}(\mathcal{H}^{\otimes_\varphi n})$$

$$J_1 = J \otimes_\varphi 1 \otimes_\varphi 1 \otimes_\varphi \cdots$$

$$J_2 = 1 \otimes_\varphi J \otimes_\varphi 1 \otimes_\varphi \cdots$$

J braided twist if it satisfies the Yang-Baxter eq.:

$$\boxed{J_1 J_2 J_1 = J_2 J_1 J_2}$$

$$T_1 T_2 T_1 = T_2 T_1 T_2$$

$\pi_i \in S_n$ transpos. of i and $i+1$

Lemma (Boz., Speicher '94):

$$\Phi_n : S_n \to B(\mathcal{H}^{\otimes n})$$

$\Phi_n(e) = 1$, $\Phi_n(\pi_{i(1)} \cdots \pi_{i(k)}) = T_{i(1)} \cdots T_{i(k)}$ for reduced words

$\leadsto$ well-defined, pos. def.:

$$\sum_{j,k} \langle \xi_j, \Phi_n(\sigma_j^{-1} \sigma_k) \xi_k \rangle \geq 0$$

twisted inner prod. on $\mathcal{H}^{\otimes_\varphi n}$:

$$\langle \xi, \eta \rangle_{J,n} = \frac{1}{n!} \Big\langle \xi, \sum_{\sigma,\pi \in S_n} \Phi_n(\sigma^{-1}\pi)\, \eta \Big\rangle_{\mathcal{H}^{\otimes_\varphi n}}$$

$$= \Big\langle \xi, \sum_{\sigma \in S_n} \Phi_n(\sigma)\, \eta \Big\rangle_{\mathcal{H}^{\otimes_\varphi n}}$$

$\rightsquigarrow$ $\mathcal{H}_{J,n}$ completion of $\mathcal{H}^{\otimes_\varphi n}$ w.r.t. $\langle \cdot, \cdot \rangle_{J,n}$

<u>Warning:</u> $\langle \cdot, \cdot \rangle_{J,n}$ may be <u>degenerate</u>

Ex. / Non-Ex.:

- $\mathcal{H} \otimes_\varphi \mathcal{H} \to \mathcal{H} \otimes_\varphi \mathcal{H}$, $\xi \otimes_\varphi \eta \mapsto \eta \otimes_\varphi \xi$

  not bimodule map, not bounded in general

- $\mathcal{H} = {}_M L^2(M)_M \otimes H$, $\mathcal{H} \otimes_\varphi \mathcal{H} \cong {}_M L^2(M)_M \otimes H \otimes H$

  $\mathcal{T} = 1 \otimes T$, $T \in \mathbb{B}(H \otimes H)$ braided twist

  ($\rightsquigarrow$ Skeide '98)

- $\mathcal{H} = {}_M L^2(M) \otimes L^2(M)_M$, $\mathcal{H} \otimes_\varphi \mathcal{H} \cong {}_M L^2(M) \otimes L^2(M) \otimes L^2(M)_M$

  $\mathcal{T} = p \otimes 1 \otimes q$, $p \in M'$, $q \in M$ proj.

5 Operator-valued twisted Araki-Woods alg.

$$\mathcal{F}_J(\mathcal{H}) = L^2(M) \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_{J,n}$$

again creation/annihilation, field op. (bdd. if $\|J\| < 1$)

$$\Gamma_J(\mathcal{H}, \mathcal{J}, (\mathcal{U}_t)) = M \vee \{ s_J(\xi) \mid \xi \text{ left } \varphi\text{-bdd.}, \mathcal{J}\mathcal{U}_{-i/2}\xi = \xi \}''$$

makes sense even for $s_J(\xi)$ unbdd.

J <u>compatible twist</u> if $[J, U_t \otimes_\varphi U_t] = 0$, $[J, \mathcal{Y}^{(2)}] = 0$

$$\mathcal{Y}^{(n)}(\xi_1 \otimes_\varphi \cdots \otimes_\varphi \xi_n) = \mathcal{Y}\xi_n \otimes_\varphi \cdots \otimes_\varphi \mathcal{Y}\xi_1$$

η right φ-bdd. $\rightsquigarrow$ right field op. $d_J(\eta)$

J <u>local twist</u> if $[s_J(\xi), d_J(\eta)] = 0$ if ξ right bdd.,

η left bdd., $\mathcal{Y} U_{-i/2} \xi = \xi$

$$\mathcal{Y} U_{i/2} \eta = \eta$$

6 Modular theory

$L^2(M) \subset \mathcal{F}_J(\mathcal{H}) \rightsquigarrow P$ proj. onto $L^2(M)$

$$E: \mathbb{B}(\mathcal{F}_J(\mathcal{H})) \longrightarrow \mathbb{B}(L^2(M)), \quad x \longmapsto Px|_{L^2(M)}$$

Thm. (Kumar, W. '24): If J local, compatible, braided twist, then E restricts to a normal faithful cond. exp. from $\Gamma_J(\mathcal{H}, \mathcal{J}, (\mathcal{U}_t))$ onto M.

Let $\hat{\varphi} = \varphi \circ E$. Then $L^2(\Gamma_J(\mathcal{H}, \mathcal{J}, (\mathcal{U}_t)), \hat{\varphi}) = \mathcal{F}_J(\mathcal{H})$ and

$$\Delta_{\hat{\varphi}}^{it}|_{\mathcal{H}_{J,n}} = \mathcal{U}_t^{\otimes_\varphi n}$$

$$J_{\hat{\varphi}}|_{\mathcal{H}_{J,n}} = \mathcal{J}^{(n)}$$

Cor.: $J = 0$ local, compatible, braided

$\Gamma_J(\mathcal{H}, \mathcal{J}, (\mathcal{U}_t))$ is of the form $\Phi(M, \eta)$ (Shlyakhtenko)

and the converse is true if $E_\eta : \Phi(M, \eta) \to M$ is faithful

7 Disintegration of Tomita correspondences

$(\mathcal{H}, \mathcal{J}, (\mathcal{U}_t))$ Tomita correspondence over (M, φ)

M semi-finite, $\varphi = \tau(\cdot h)$

Thm. (Kumar, W. '24):

- $\mathcal{H} = \mathcal{H}_0 \oplus \int_{\mathbb{R}_+}^{\oplus} (\mathcal{H}_\omega \oplus \mathcal{H}_{-\omega})\, d\mu(\omega)$ as bimodules
- $\mathcal{U}_t = 1_{\mathcal{H}_0} \oplus \int_{\mathbb{R}_+}^{\oplus} \begin{pmatrix} e^{i\omega t} & \\ 0 & e^{-i\omega t} \end{pmatrix} h^{it} \cdot h^{-it}\, d\mu(\omega)$
- $\mathcal{J} = \mathcal{J}_0 \oplus \int_{\mathbb{R}_+}^{\oplus} \begin{pmatrix} 0 & \mathcal{J}_{-\omega} \\ \mathcal{J}_\omega & 0 \end{pmatrix} d\mu(\omega)$

Thm. (Kumar, W. '24): M type I factor

$\exists\ (H, J, (U_t))$ over $\mathbb{C}$, $T: H \otimes H \to H \otimes H$ twist s.t.

$$\Gamma_J(\mathcal{H}, \mathcal{J}, (\mathcal{U}_t)) = M \overline{\otimes}\ \Gamma_T(H, J, (U_t))$$

8 Factoriality

Thm. (Kumar, W. '24): If $\mathcal{H}$ is mixing bimodule, techn. ass., centralizer M^{φ} contains diffuse element

- $M' \cap \Gamma_J(\mathcal{H}, J, (U_t)) = \mathcal{Z}(M)$
- $\mathcal{Z}(\Gamma_J(\mathcal{H}, J, (U_t))) = \{ z \in \mathcal{Z}(M) \mid z\xi = \xi z \text{ for all } \xi \in \mathcal{H} \}$

In particular, if M is a factor of $\mathcal{H} = {}_M L^2(M) \otimes L^2(M)_M$, then $\Gamma_J(\mathcal{H}, J, (U_t))$ is a factor.

$$\mathcal{F}_{-1}(H) \otimes \mathcal{F}_1(K)$$

$$\Gamma_{-1}(H) \,\overline{\otimes}\, \Gamma_1(K)$$

$$\mathcal{F}_{-1}\left(H \otimes_{\Gamma_1(K)} \mathcal{F}_1(K)_{\Gamma_1(K)} \right) \text{ over } \Gamma_1(K)$$

$$\mathcal{F}_{-1}\left({}_{\Gamma_1(K)}\mathcal{H}_{\Gamma_1(K)} \right)$$