Schreier's Formula of some Free Probability Invariants:

Thm [Nielsen-Schreier] Any $H \leq F$ subgp of free gp is itself a free gp. If $F = \mathbb{F}_n$ and $[F:H] < \infty$, $H = \mathbb{F}_k$ with $n-1 = [F:H]^{-1}(k-1)$.

Tracial vNa (M, τ) $\rightsquigarrow$ faithful, normal, tracial state

- Notion of index:

Ex. Given a countable discrete gp Γ, on $\ell^2(\Gamma)$ with basis $\delta_h, h \in \Gamma$

$\lambda(g)\delta_h = \delta_{gh} \rightsquigarrow L(\Gamma) = \overline{\operatorname{span}\{\lambda(g) : g \in \Gamma\}}^{WOT} \subset B(\ell^2(\Gamma))$

$$\tau(x) = \langle x\delta_e, \delta_e \rangle$$

In $[L(\mathbb{F}_n) : L(\mathbb{F}_k)] = [\mathbb{F}_n : \mathbb{F}_k]$.

Ex: $G \overset{\alpha}{\curvearrowright} (M, \tau)$ trace-preserving action, $\alpha : G \to \operatorname{Aut}(M)$ is a homomorp. (finite gp)

$\tau(\alpha_g(x)) = \tau(x)$. On $L^2(M, \tau)$ $\quad x \mapsto \alpha_g(x)$ with $x \in M$, $g \in G$

we can extend it to u_g unitary. $M \rtimes G = (M \cup \{u_g : g \in G\})''$

$x \in M \rtimes_\alpha G \qquad x = \sum_{g \in G} x_g u_g, \; x_g \in M$, the $\tau(x) = \tau(x_e)$

In index: $[M \rtimes_\alpha G : M] = |G|$.

Free Probability Theory: $\delta^* \rightsquigarrow$ set of generators $X = X^* \subset M$.

Fact: there exist a set of generators $S \subset L(\mathbb{F}_n)$ s.t. $\delta^*(S) = n$.

For $M \subset M \rtimes_\alpha G$, for any generating set S_0 of M, there exists a generating set S_1 of $M \rtimes_\alpha G$ s.t. $\delta^*(S_1)-1 = |G|^{-1}(\delta^*(S_0)-1)$

In Shly 23, Given $\varepsilon > 0$, there exists a generating S_0 for M and S_1 for $M \rtimes G$ (G abelian and action is outer) s.t.

$$\delta^*(S_1)-1 \leq |G|^{-1}(\delta^*(S_0)-1)+\varepsilon.$$

We want to look at $M \subset M \rtimes_\alpha G$ (G finite)

$\Delta(A,\tau), \sigma(A,\tau)$, $A = \mathbb{C}\langle X\rangle$, $X = X^* \subset M$

CShly (05) free stein CN (21)

$\mathrm{Der}_c(A,\tau)$ Shly (09)

Detour: Derivation sp:

Let (M,τ) tracial vNa and $X = X^*$ generating with $A = \mathbb{C}\langle X\rangle$.
A'' vNa generated by X and $L^2(A,\tau)$ GNS construction (A,τ)

- (A°,τ°) the opposite algebra of A, $(x^\circ y^\circ) = (yx)^\circ$.

$\mathrm{Der}(A,\tau) := \{ d: A \to L^2(A\otimes A^\circ, \tau\otimes\tau^\circ) :$ d is linear and satisfies the Leibniz rule. $\}$

for $a\otimes b^\circ \in L^2(A\otimes A^\circ)$, $x,y \in A$, $x\cdot(a\otimes b)\cdot y = (xa)\otimes(by)^\circ = (x\otimes y^\circ)(a\otimes b^\circ)$.

- Defining an action: $m \in (A\otimes A^\circ)''$ and $d \in \mathrm{Der}(A,\tau)$,

$[d \cdot m](x) = d(x)m$, $x \in A$

$\mathrm{Der}(A,\tau)$ is a right $(A\otimes A^\circ)''$-module.

Now $\phi_X : \mathrm{Der}(A,\tau) \to L^2(A\otimes A^\circ)^X$ via $d \mapsto (d(x))_{x\in X}$

Notice ϕ_x is right $(A\otimes A^o)''$-linear injective, closed rang.

$$\dim \operatorname{Der}(A,\tau)_{(A\otimes A^o)''} := \dim \phi_x\left(\operatorname{Der}(A,\tau)_{(A\otimes A^o)''}\right)$$

Some subspaces:

1) $\operatorname{InnDer}(A,\tau) := \{[\cdot,\xi] : \xi\in L^2(A\otimes A^o)\}$

2) $B\subset A$ unital $*$-subalgebra $\operatorname{Der}(B\subset A) := \{d : d|_B = 0\}$

Ex: On $\ell^2(\mathbb{Z}_n)$, consider $L(\mathbb{Z}_n)$

$$\dim \operatorname{Der}(\ell^\infty(\mathbb{Z}_n),\tau) = \dim \operatorname{InnDer}(\ell^\infty(\mathbb{Z}_n),\tau)$$
$$= 1 - \left(\begin{array}{l}\text{elements in } \ell^2(\mathbb{Z}_n)\otimes \ell^2(\mathbb{Z}_n)^o \text{ s.t.}\\ \text{they commute with the action } \ell^\infty(\mathbb{Z}_n)\end{array}\right)$$
$$= 1 - \frac{1}{n}. \quad \square$$

Result:

$G \overset{\alpha}{\curvearrowright} (M,\tau)$ trace-preserving action by a finite gp.

- Assume A is globally invarian $\alpha_g(A)\subset A$ $\forall g\in G$.

$A\rtimes_\alpha^{alg} G = \mathbb{C}\langle A \vee \{u_g : g\in G\}\rangle$

$$a\in A\rtimes_\alpha G \rightsquigarrow a = \sum_{g\in G} a_g u_g \rightsquigarrow L^2(A\rtimes_\alpha G) = \bigoplus_{g\in G} L^2(A)u_g$$

Thm [6.6.23] The following is a right $(A\otimes A^o)''$-module isom.

$$\operatorname{Der}(\mathbb{C}[G]\subset A\rtimes_\alpha G) \longleftrightarrow \bigoplus_{g\in G} \operatorname{Der}(A,\tau)_{1\otimes\alpha_g^{-1}}$$

$$L^2\big((A \rtimes_\alpha G) \otimes (A \rtimes_\alpha G)^\circ\big) = \bigoplus_{g,h \in G} \underline{L^2(A \otimes A^\circ)(u_g \otimes u_h^\circ)}$$

$L^2(A \otimes A^\circ)$ is identified with $L^2(A \otimes A^\circ) u_e \otimes u_e^\circ$

$$P_{g,h} = [L^2(A \otimes A^\circ)(u_g \otimes u_h^\circ)] \in B\big(L^2((A \rtimes_\alpha G) \otimes (A \rtimes_\alpha G)^\circ)\big)$$

$D \in \mathrm{Der}(\mathbb{C}[G] \subset A \rtimes_\alpha G)$, $\quad P_{e,e} D|_A \in \mathrm{Der}(A, \tau)$

$\Leftarrow$: $d \in \mathrm{Der}(A, \tau) \rightsquigarrow \underline{D_d(u_g) = 0}$

$\Big[\, d(\alpha_g(x)) = u_g \cdot d(x) \cdot u_g^{\circ *}$ for $d \in \mathrm{Der}(A, \tau)$

$$D_d\Big(\sum a_x u_x\Big) = \sum_{x \in G} \sum_{g \in G} u_g^* \, d(\alpha_g(x)) \cdot (u_g u_x)^\circ \in \mathrm{Der}(A \rtimes G, \tau),$$

Thm [6.6] Furthermore

$$\dim \mathrm{Der}(\mathbb{C}[G] \subset A \rtimes_\alpha G)_{(A \otimes A^\circ)''} = |G| \dim \mathrm{Der}(A, \tau)_{(A \otimes A^\circ)''}$$

Thm [Charlesworth, Nelson 22] Let $B \subset A$ be a unital finite dimensional subspace of A,

$$\dim \mathrm{Der}(A, \tau)_{(A \otimes A^\circ)''} = \dim \mathrm{Der}(B, \tau)_{(B \otimes B^\circ)''} + \dim \mathrm{Der}(B \subset A, \tau)_{(A \otimes A^\circ)''}$$

Fact:

1) $\dim \mathrm{Der}(\mathbb{C}[G], \tau) = 1 - \frac{1}{|G|}$

2) $\dim \mathrm{Der}(\mathbb{C}[G] \subset A \rtimes_\alpha G)_{\underline{[(A \rtimes_\alpha G) \otimes (A \rtimes_\alpha G)^\circ]''}} =$

$$= \frac{1}{|G|^2} \dim \mathrm{Der}(\mathbb{C}[G] \subset A \rtimes_\alpha G)_{\underline{(A \otimes A^\circ)''}}$$

Cor:

$$\dim \mathrm{Der}(A\rtimes_\alpha G)-1=\frac{1}{|G|}\left(\dim \mathrm{Der}(A,\tau)-1\right)$$

Application to Free Probability:

In 2005, Shly, Connes $\Delta(A,\tau)$ (homology for vNas) +

In 2009, Shly, $\dim \mathrm{Der}_c(A,\tau)$ ✓

In 2021, Nel, Charl looked $\sigma(A,\tau)$ ✓

Cor [G.G 23]

- G is finite gp
  - $\sigma(A\rtimes_\alpha G,\tau)-1=\frac{1}{|G|}\left(\sigma(A,\tau)-1\right)$?
  - $\dim \mathrm{Der}_c(A\rtimes_\alpha G,\tau)-1=\frac{1}{|G|}\left(\dim \mathrm{Der}(A,\tau)-1\right)$
- G is abelian

  $\Delta(A\rtimes_\alpha G,\tau)-1=\frac{1}{|G|}\left(\Delta(A,\tau)-1\right).$

- It's known that $\delta_0 \le \delta^* \le \delta^\star$
- Assume A'' is embedded in the ultrapower of the hyperfinite II_1 factor. $-\infty < \delta_0(X)$.

$$\underline{\dim \mathrm{Der}_c(A,\tau)} \underset{2009}{\le} \delta_0(X)\le\delta^*(X)\le\delta^\star(X)\underset{2005}{\le}\Delta(A,\tau)$$

Corollary [G.G.23] Assume G is abelian: A'' embedded

- If $\underline{\dim \mathrm{Der}_c(A,\tau)=\Delta(A,\tau)}$, then for any generating set Y of $A\rtimes_\alpha G$,

$$\underline{\underline{\dim \mathrm{Der}_c(A\rtimes_\alpha G,\tau)}}=\underline{\delta_0(Y)=\delta^*(Y)=\delta^\star(Y)}=\Delta(A\rtimes_\alpha G,\tau)$$

- its known $\sigma(A,\tau)\le\delta^*(X)\le\delta^\star(X)\le\Delta(A,\tau)$

Cor: G is abelian

- If $\sigma(A,\tau) = \Delta(A,\tau)$, then for any generating set Y of $A \rtimes_\alpha G$, we have

$$\sigma(A \rtimes_\alpha G, \tau) = \delta^*(Y) = \delta^*(Y) = \Delta(A \rtimes_\alpha G, \tau).$$