Modular Structure and Inclusions of Twisted Araki-Woods Algebras

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joint work with Ricardo Correa da Silva
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UC Berkeley Probabilistic Operator Algebra Seminar
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Plan of the talk

1. Define twisted Araki-Woods Algebras $\mathcal{L}_T(H)$ on $T$-twisted Fock spaces (mostly review)
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3. Standardness and modular data
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2. Motivation and questions (background: mathematical physics, QFT)
3. Standardness and modular data
4. Inclusions of twisted Araki-Woods algebras (“twisted subfactors”)
Construction of $\mathcal{L}_T(H)$ on twisted Fock spaces

[Bożejko/Speicher '94; Jørgensen/Schmitt/Werner '95]

Setup: Fix Hilbert space $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. 

Idea: New scalar products $\langle \cdot, \cdot \rangle_T, n$ on $\mathcal{H} \otimes_n$. 

Notation: $T_k : = 1 \otimes (k - 1) \mathcal{H} \otimes T \otimes 1 \otimes (n - k - 1) \mathcal{H} \in \mathcal{B}(\mathcal{H} \otimes_n)$, $1 \leq k \leq n - 1$.

Kernels: $P_T, 1 = 1$, $P_T, 2 = 1 + T$, $P_T, 3 = 1 + T_1 + T_2 + T_1 T_2$, $P_T, n + 1 = (1 \otimes P_T, n)(1 + T_1 + T_2 + \ldots + T_n)$.

Definition twisted: $T = T^*$, $\|T\| \leq 1$, $P_T, n \geq 0$ for all $n$.

Strict twist: In addition $\ker P_T, n = \{0\}$.

Definition $T$-twisted Fock space $\mathcal{F}_T(\mathcal{H})$ : $= \bigoplus_{n \geq 0} \mathcal{H} \otimes_n / \ker P_T, n \langle \cdot, \cdot \rangle_T$.
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- **Setup:** Fix Hilbert space $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$.
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$\triangleright$ Notation: $T_k := 1 \otimes (k - 1) \mathcal{H} \otimes T \otimes 1 \otimes (n - k - 1) \mathcal{H} \in \mathcal{B}(\mathcal{H} \otimes n)$, $1 \leq k \leq n - 1$.

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$T$-twisted Fock space

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Examples

- $T = F : v \otimes w \mapsto w \otimes v$ (flip): $\mathcal{F}_F(\mathcal{H}) = \text{Bose Fock space}$
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Theorem ([Jørgensen/Schmitt/Werner; Bożejko/Speicher])

Let $T = T^* \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}), \|T\| \leq 1$.

1. If $\|T\| \leq \frac{1}{2}$, then $T$ is a strict twist.
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3. If

$$T_1T_2T_1 = T_2T_1T_2 \quad \text{(Yang-Baxter equation)}$$

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An example from QFT (“S-Matrix Model”)

$\mathcal{H} = L^2(\mathbb{R}, d\theta), s : \mathbb{R} \rightarrow S^1, s(-\theta) = \overline{s(\theta)}$. Then

$$(Tf)(\theta_1, \theta_2) = s(\theta_1 - \theta_2) \cdot f(\theta_2, \theta_1)$$

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**An example from QFT ("S-Matrix Model")**

$\mathcal{H} = L^2(\mathbb{R} \to \mathcal{K}, d\theta), \quad s : \mathbb{R} \to \mathcal{U}(\mathcal{K} \otimes \mathcal{K})$ solves YBE w.spec.par., $s(-\theta) = s(\theta)^*$. 

$$(Tf)(\theta_1, \theta_2) = s(\theta_1 - \theta_2) \cdot f(\theta_2, \theta_1) \quad \text{is a unitary twist.}$$
From now on: $\mathcal{H}$ Hilbert space, $T$ twist.
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- On $\mathcal{F}_T(\mathcal{H})$, have (left) creation/annihilation operators $a_{L,T}(\xi), \xi \in \mathcal{H}$:

  
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  a_{T,L}^*(\xi)\Omega = \xi, \quad a_{L,T}(\xi)\Omega = 0, \quad \Omega: \text{Fock vacuum}
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  a_{L,T}^*(\xi)[\Psi_n] = [\xi \otimes \Psi_n], \quad \Psi_n \in \mathcal{H}^\otimes n,
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- Relations ($\dim \mathcal{H} < \infty$, $(e_k)$ ONB, $a_i := a_{L,T}(e_i)$)

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\begin{align*}
a_i a_j^* &= \sum_{k,l} \langle e_i \otimes e_k, T(e_j \otimes e_l) \rangle a_k^* a_l + \delta_{i,j} \cdot 1
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**Left field operators:**

$$\phi_{L,T}(\xi) := a_{L,T}^*(\xi) + a_{L,T}(\xi).$$

**Left** twisted Araki-Woods Algebra (with $H \subset \mathcal{H}$)

$$\mathcal{L}_T(H) := \{\phi_{L,T}(h) : h \in H\}'' \subset \mathcal{B}(\mathcal{F}_T(\mathcal{H}))$$

w.l.o.g.: $H \subset \mathcal{H}$ closed $\mathbb{R}$-linear subspace.
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- Tomita operator

  $$S_H : H + iH \rightarrow H + iH, \quad h_1 + ih_2 \mapsto h_1 - ih_2.$$  

- Polar decomposition: $S_H = J_H \Delta_H^{1/2}$ with $J_H$ antiunitary and $\Delta_H > 0$. 
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- Tomita’s Theorem for standard subspaces:

  $$\Delta_H^{it}H = H, \quad J_HH = H' = \{h' \in \mathcal{H} : \text{Im}\langle h, h' \rangle = 0 \forall h \in H\}$$

  $H'$ is also a standard subspace, and $(H')' = H$.
Real Hilbert spaces vs. standard subspaces

**Proposition ([Shlyakhtenko ’97])**

There is a one to one correspondence between

- (real) standard subspaces of a complex Hilbert spaces, $H \subset \mathcal{H}$,
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- \( T = 0 \) and \( H = \overline{\mathbb{R}\text{-span(ONB)}} \), i.e. \( \Delta_H = 1 \) (or: \( U(t) = 1 \) on \( \mathcal{H}_\mathbb{R} \)).
  Then \( \mathcal{L}_0(H) = \mathcal{L}(\mathbb{F}_{\dim \mathcal{H}}) \). (free Gaussian functor, [Voiculescu '85])
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- $T = qF$ and $H = \mathbb{R}$-span(ONB), with $-1 < q < 1$
  $q$-Gaussian v. Neum. alg., [Bożejko/Speicher '91]. $\mathrm{II}_1$-factors [Ricard '05]
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Real Hilbert spaces vs. standard subspaces

**Proposition ([Shlyakhtenko ’97])**

There is a one to one correspondence between

- (real) standard subspaces of a complex Hilbert spaces, \( H \subset \mathcal{H} \),
- real Hilbert spaces \( \mathcal{H}_R \) with a strongly continuous one parameter orthogonal group \( U(t) \)

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H \leftrightarrow \mathcal{H}_R, \quad \Delta_H^{it}|_H \leftrightarrow U(t)
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**Examples**

- \( T = 0 \) and \( H = \overline{\mathbb{R}}\text{-span(ONB)} \), i.e. \( \Delta_H = 1 \) (or: \( U(t) = 1 \) on \( \mathcal{H}_R \)). Then \( \mathcal{L}_0(H) = \mathcal{L}(F_{\text{dim } \mathcal{H}}) \). (free Gaussian functor, [Voiculescu ’85])

- \( T = qF \) and \( H = \overline{\mathbb{R}}\text{-span(ONB)} \), with \(-1 < q < 1\)  
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  $\mathcal{L}_T(H)$ is non-injective, of type III unless $\Delta_H = 1$
Questions

- For general $T$ (and general $H$), only little is known about $\mathcal{L}_T(H)$. 

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### Main Questions

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In the following: \( H \subset \mathcal{H} \) an arbitrary standard subspace (i.e. arbitrary \( U(t) \) resp. modular group \( \Delta^i_H \)), and \( T \) a twist.
Separating vacuum

Basic assumption: $T$ and $H$ are compatible in the sense $[T, \Delta_H^{it} \otimes \Delta_H^{it}] = 0$.

**Lemma:** If $\Omega$ is separating for $\mathcal{L}_T(H)$ and $H, T$ are compatible, then the modular data $\Delta, J$ of $(\mathcal{L}_T(H), \Omega)$ restrict to $\Delta_H, J_H$ on $\mathcal{H}$. 
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In order to have $\Omega$ separating for $\mathcal{L}_T(H)$, need **KMS-property**. Consider $n$-point functions $(h_1, \ldots, h_n \in H)$

$$f_n(t) := \langle \Omega, \phi_{L,T}(h_1) \cdots \phi_{L,T}(h_{n-1}) \Delta^{it}_{L,T}(h_n) \Omega \rangle_T = \langle 1 2 \cdots (n - 1) n_t \rangle.$$
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- **Graphical notation** ($\sim$[Bożejko/Speicher])

\[
\begin{align*}
2_t & \quad 1 \\
4 & \quad 1 \\
3 & \quad 2
\end{align*}
\]

\[
\begin{align*}
6 & \quad 1 \\
5 & \quad 2 \\
4 & \quad 3
\end{align*}
\]

\[
\langle J_H h_1, \Delta_H^{it} h_2 \rangle, \quad \langle 1, 2 \rangle \cdot \langle 3, \Delta_H^{it} 4 \rangle, \quad \langle 3 \otimes T(2 \otimes 1), T(4 \otimes 5) \otimes 6_t \rangle
\]
Six-point function $\langle 1 \ 2 \ldots \ 6_t \rangle$
By imposing the KMS condition, one can extract two properties of $T$:

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![Crossing symmetry diagrams]

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   $\langle 2_t \otimes 1, T(3 \otimes 4_t) \rangle = T \quad (t \sim t - i)$

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\langle 2t \otimes 1, T(3 \otimes 4t) \rangle = T \left( \begin{array}{ccc} 2t & 1 \\ 3 & 4t \end{array} \right) = \langle 1 \otimes 4t, T(2t \otimes 3) \rangle
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This is a condition on \( T \).

2. The two possible triple crossing terms in the 6-point function differ by a Reidemeister move of type III.
By exploiting KMS condition, one can show that one must have \( \text{RHS} = \text{LHS} \) (Yang-Baxter equation.)
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$T$ is called **crossing-symmetric** (w.r.t. $H$) if for all $\psi_1, \ldots, \psi_4 \in \mathcal{H}$, the function

$$T_{\psi_3, \psi_4}^{\psi_2, \psi_1}(t) := \langle \psi_2 \otimes \psi_1, (\Delta_H^{it} \otimes 1)T(1 \otimes \Delta_H^{-it})(\psi_3 \otimes \psi_4) \rangle$$

has an analytic continuation to the strip $\mathbb{S}_{1/2}$ ($\ldots$) and

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Theorem

Let $H \subset \mathcal{H}$ be a standard subspace and $T$ a compatible twist. The following are equivalent:

a) $\Omega$ is separating for $\mathcal{L}_T(H)$.

b) $T$ is braided and crossing symmetric w.r.t. $H$. 

Braided twists and left-right duality

How does the argument “YBE+crossing $\Rightarrow \Omega$ separating” work?
Braided twists and left-right duality

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For **braided** twists (YBE $T_1 T_2 T_1 = T_2 T_1 T_2$ holds), also **right** creation/annihilation operators exist:

$$a_{R,T}^*(\xi)[\Psi_n] = [\Psi_n \otimes \xi],$$

$$a_{R,T}(\xi)[\Psi_n] = [a_{R,0}(\xi)(1 + T_n + \ldots + T_{n-1}\cdots T_1)\Psi_n]$$

$$\phi_{R,T}(\xi) := a_{R,T}^*(\xi) + a_{R,T}(\xi)$$

... generating “right” twisted Araki-Woods algebras $\mathcal{R}_T(H)$. 
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**Proposition**

*Let T be braided and crossing symmetric.*

a) **The Tomita operator $S$ of** $(\mathcal{L}_T(H), \Omega)$ **is given by**

$$
S[\psi_1 \otimes \ldots \otimes \psi_n] = [S_H\psi_n \otimes \ldots \otimes S_H\psi_1]
$$

b) **Left-right duality holds:**

$$
\mathcal{L}_T(H)' = \mathcal{R}_T(H').
$$
Remarks on standardness question

- From our perspective, the braided and crossing-symmetric twists are the most interesting ones (Classification unknown).
- Both the Yang-Baxter equation and crossing symmetry have their origins in physics, but can here be derived from modular theory.
- Definition of crossing is inspired by QFT crossing symmetry (scattering of particles vs. scattering of antiparticles, $J_H = \text{TCP operator}$)
- Result on modular data generalizes many known results [Eckmann/Osterwalder '73, Leyland/Roberts/Testard '78, Shlyakhtenko '97, Baumgärtel/Jurke/Lledo '02, Buchholz/L/Summers '11, L '12]
Inclusions

Have two maps

\[ H \hookrightarrow \mathcal{L}_T(H), \quad H \hookrightarrow \mathcal{R}_T(H) \]

from \( T \)-comp. standard subspaces \( H \subset \mathcal{H} \) to v. Neumann algebras on \( \mathcal{F}_T(\mathcal{H}) \).
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- By definition: \( K \subset H \implies \mathcal{L}_T(K) \subset \mathcal{L}_T(H), \mathcal{R}_T(K) \subset \mathcal{R}_T(H) \).
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- Inspired by QFT models: Investigate von Neumann algebra inclusions

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**Lemma:** Proper inclusions \( K \subset H \) only exist if \( \Delta_H, \Delta_K \) are unbounded. In particular \( \text{dim} \, \mathcal{H} = \infty \) is needed.
Twisted subfactors

For $T = qF$, $-1 < q < 1$, it is known that $L_{qF}(H)$ is a non-injective factor of type III if $\Delta_H$ is unbounded [Kumar, Skalski, Wasilewski ’23].
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- This is no longer true for $q = 1$, where $L_F(H) \cap L_F(H)' = L_F(H \cap H')$ (and $L_F(H) = R_F(H)$) holds [Leyland/Roberts/Testard ’78].
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- We expect that for general (braided, crossing-symmetric) twist with $\|T\| < 1$, it is still true that $\mathcal{L}_T(H)$ is a non-injective factor of type III if $\Delta_H$ is unbounded.
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$K \subset H$. Relative commutant

$$\mathcal{C}_T(K, H) := \mathcal{L}_T(K)' \cap \mathcal{L}_T(H) = \mathcal{L}_T(K)' \cap \mathcal{R}_T(H')'.$$

In the following: Two results on $\mathcal{L}_T(K) \subset \mathcal{L}_T(H)$ in different situations,

- one “negative” (singular inclusions, $\mathcal{C}_T(K, H) = \mathbb{C}1$)
- one “positive” (large relative commutant, $\mathcal{C}_T(K, H) \neq \mathbb{C}1$)
Half-sided inclusions

Let us consider a half-sided inclusion $K \subset H$ of standard subspaces:

- have unitary one-parameter group $V(x)$ with positive generator,
- $V(x)H \subset H$, $x \geq 0$. Set $K := V(1)H$.
- $[V(x) \otimes V(x), T] = 0$. 

Well-studied scenario in CFT (translations on a lightray). Known:

- $L_T(H)$ is a III$_1$ factor [Wiesbrock '93].
- Modular group acts by dilations, $\Delta^it_H V(x) \Delta^{-it_H} = V(e^{-2\pi t}x)$ [Borchers'92].

Suppose $\|T\| < 1$ and $k \in K$, $h' \in H'$. Then

$$\phi_{T,L}(k) \phi_{T,R}(h') \in L_T(K) \vee R_T(H') = C_{T,K,H} \quad \phi_{T,L}(\Delta^it_H k) \phi_{T,R}(\Delta^it_H h') \in L_T(K) \vee R_T(H') = C_{T,K,H}, \quad t < 0.$$ 

For $\|T\| < 1$, weak limit $t \to -\infty$ can be controlled. Gives vacuum projection $P_\Omega$. 

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- Modular group acts by dilations, $\Delta_H^{it} V(x) \Delta_H^{-it} = V(e^{-2\pi t}x)$ [Borchers'92].

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$$\phi_{T,L}(k)\phi_{T,R}(h') \in \mathcal{L}_T(K) \vee \mathcal{R}_T(H') = \mathcal{C}_T(K,H)'$$

$$\phi_{T,L}(\Delta_H^{it}k)\phi_{T,R}(\Delta_H^{it}h') \in \mathcal{L}_T(K) \vee \mathcal{R}_T(H') = \mathcal{C}_T(K,H)', \quad t < 0.$$  

For $\|T\| < 1$, weak limit $t \to -\infty$ can be controlled. Gives vacuum projection $P_\Omega$. 

Singular inclusions

**Theorem**

Let $K \subset H$ be a half-sided inclusion of standard subspaces and $T$ a compatible braided crossing-symmetric twist with $\|T\| < 1$. Then $C_T(K, H) = C_1$.

- For $T = 0$, the proof becomes quite easy.
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**Generalization:**

**Theorem**

Let $K \subset H$ be standard subspaces. Suppose there exist sequences of unit vectors $k_n \in K$, $h'_n \in H'$, such that

$$k_n \to 0, \quad h'_n \to 0 \quad \text{weakly}, \quad \langle k_n, h'_n \rangle \not\to 0.$$

- Then $C_T(K, H) = \mathbb{C}1$ (for $\|T\| < 1$).
- This is in particular the case when $\Delta_H^{1/4} \Delta_K^{-1/4}$ is not compact.
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This is in particular the case when $\Delta_H^{1/4} \Delta_K^{-1/4}$ is not compact.

Corollary: $\mathcal{L}_T(H)$ is a factor for $\|T\| < 1$ and $\dim \mathcal{H} = \infty$. 
The fact that many inclusions \( L^2(K) \subset L^2(H) \) are singular for \( \|T\| < 1 \) is in line with proximity to extreme situation at \( T = 0 \). Surprisingly, \( L^2(K) \subset L^2(H) \) can also have very large relative commutant for suitable \( K \subset H \) and \( \|T\| < 1 \).

Theorem

Let \( K \subset H \) be an inclusion such that
\[
\|\Delta_1/4_H - 1/4_K\|_1 < 1 \quad \text{(trace norm)}.
\]
Let \( T \) be a braided crossing symmetric compatible twist with \( \|T\| < 1 \). Then

\( a) \ \ L^2(K) \subset L^2(H) \) is split.

\( b) \ \ C^*_{T}(K,H) \cong L^2(H) \otimes R^*_{T}(K') \).

Proof uses split property \cite{Doplicher/Longo '84} and modular density conditions \cite{D'Antoni/Longo/Radulescu'01,Buchholz/D'Antoni/Longo'07}.
\( L^2 \)-inclusions

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**Theorem**

Let $K \subset H$ be an inclusion such that $\|\Delta_1/4_H - 1/4_K\|_1 < 1$ (trace norm). Let $T$ be a braided crossing symmetric compatible twist with $\|T\| < 1$. Then

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**Theorem**

Let \(K \subset H\) be an inclusion such that \(\|\Delta_H^{1/4} \Delta_K^{-1/4}\|_1 < 1\) (trace norm). Let \(T\) be a braided crossing symmetric compatible twist with \(\|T\| < 1\). Then

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For $\|T\| < 1$:

$$\|\Delta_H^{1/4} \Delta_K^{-1/4}\|_1 < 1$$

↓

$\mathcal{L}_T(K) \subset \mathcal{L}_T(H)$ split

↓

$\mathcal{C}_T(K, H) \neq \mathbb{C}$

↓

$\Delta_H^{1/4} \Delta_K^{-1/4}$ compact
For $\|T\| < 1$:

$$\|\Delta_H^{1/4} \Delta_K^{-1/4}\|_1 < 1$$

$\implies$

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$\implies$

$$\mathcal{C}_T(K, H) \neq \mathbb{C}$$

$\implies$

$$\Delta_H^{1/4} \Delta_K^{-1/4} \text{ compact}$$

- Relation between $\Delta_H^{1/4} \Delta_K^{-1/4}$ and $\mathcal{C}_T(K, H)$ is much closer for $\|T\| < 1$ than for $\|T\| = 1$. 

\[21/22\]
$\sigma(T F)$

$T = 0$
$\sigma(TF)$

$T = -F \quad T = 0 \quad T = F$
$\sigma(TF)$

$T = -F$ $T = 0$ $T = F$
\( \sigma(TF) \)

\[ T = F \quad T = 0 \quad T = -F \]

\( S \)-model
\( \sigma(TF) \)

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\( S\)-model

\( qS\)-model
Do there exist inclusions $\mathcal{L}_T(K) \subset \mathcal{L}_T(H)$ that have non-trivial relative commutant but are not split?

Interesting regime: $\Delta_H^{1/4} \Delta_K^{-1/4}$ compact, but not $\|\Delta_H^{1/4} \Delta_K^{-1/4}\|_1 < 1$. Can we say something about $C_T(K, H)$ (avoiding split)?