Monotonicity of entropy for random matrices

Based on joint works with $\begin{cases} \text{Benjamin Dadoun} \\ \text{Djalil Chafaï, } \text{—} \end{cases}$

Artstein - Ball - Barthe - Naor '04:

X r.v, $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$.

X has density f; $\mathrm{Ent}(X) = -\mathbb{E}\log f(X)$
$= -\int f \log f$.

$S_n = X_1 + \text{—} + X_n$; $(X_i)_i$ i.i.d copies.

$$\mathrm{Ent}\left(\frac{S_n}{\sqrt{n}}\right) \leq \mathrm{Ent}\left(\frac{S_m}{\sqrt{m}}\right) \qquad \forall n \leq m$$

Remark:

- Conservation law : system centred
  unit variance.
- $Z \sim N(0,1) \rightsquigarrow$ maximizes Entropy among

continuous centered, unit variance r.v

- Ent $\leadsto$ rate fct° in LDP.

Answers Q. of Shannon '40.

Stam '59, Lieb '78 : monotonicity for $n = 2^k$.

<u>Related quantities</u>:

Score function : $\rho_X = \frac{f_X'}{f_X}$

Fisher information: $\Phi_X = \mathbb{E}\,\rho_X^2$.

De Bruijn identity:

$$\mathrm{Ent}(X) = \frac{\log 2\pi e}{2} + \frac{1}{2}\int_0^\infty \left(\frac{1}{1+t} - \Phi\left(X+\sqrt{t}Z\right)\right) dt$$

Courtade's proof of monotonicity of $\Phi_{\frac{S_n}{\sqrt{n}}}$.

Take $n \le m$.

$$\rho_{S_m} = \rho_{S_n + S_m - S_n} = \mathbb{E}\left[\rho_{S_n} \mid S_m\right]$$

indep

$$\Phi_{S_m} = \mathbb{E}\, \rho_{S_m}^2 = \mathbb{E}\left[\rho_{S_n} \cdot \mathbb{E}\left[\rho_{S_n} \mid S_m\right]\right]$$

$$= \operatorname{cov}\left(g(S_n),\, h(S_m)\right)$$

Dembo-Kagan-Shepp '01:

X, Y r.v, $\quad R(X,Y) = \sup_{g,h} \dfrac{\operatorname{cov}(g(X), h(Y))}{\sigma_{g(X)} \cdot \sigma_{h(Y)}}$

$$R(S_n, S_m) = \sqrt{\frac{n}{m}} \quad \forall\, n \le m.$$

$$\le \sqrt{\frac{n}{m}}\, \sqrt{\Phi_{S_n}}\, \sqrt{\Phi_{S_m}}.$$

$\hookrightarrow \quad m\,\Phi_{S_m} \leq n\,\Phi_{S_n} \Rightarrow \Phi_{\frac{S_m}{\sqrt{m}}} \leq \Phi_{\frac{S_n}{\sqrt{n}}}$

$\boxtimes$

In Free Probability.

Classical Prob	Free Probability
$(\Omega, \mathbb{P})$	$(\mathcal{A}, \tau)$
distributions	$\tau(a^k) = \int x^k \, d\mu_a$
Indep	Freeness
Gaussian	semi-circle s.
Entropy	Free entropy $\chi^*(\mu) = \iint \log\lvert x-y\rvert \, d\mu(x)\, d\mu(y)$
Score fct°	conjugate variable ξ $\xi(z) \in L^2(z)$

$$\tau\left(\xi(z)\cdot z^{r}\right)=\sum_{k=0}^{r-1}\tau\left(z^{k}\right)\tau\left(z^{r-1-k}\right).$$

Fisher information

De Bruijn

Free Fisher

$$\Phi^{*}(z)=\|\xi(z)\|_{2}^{2}.$$

replace Z by s.

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* $\boxed{\underline{\text{Shlyakhtenko '07:}}}$ $(a_i)_i$ free NC r.v identically distributed

$$S_n=a_1+\ldots+a_n.$$

$$\chi^{*}\left(\frac{S_n}{\sqrt{n}}\right)\leq\chi^{*}\left(\frac{S_m}{\sqrt{m}}\right)\qquad\forall\, n\leq m.$$

(Shlyakhtenko-Tao '21: Fractional free convolution)

<u>Dadoun-Y '21:</u> Another proof.

Correlation coefficient:

$$\mathrm{Cov}(a,b) = \tau\left((a-\tau(a))^* \cdot (b-\tau(b))\right).$$

$$R(M_1, M_2) = \sup_{\substack{x \in M_1 \\ y \in M_2}} \frac{\mathrm{Cov}(x,y)}{\sigma_X \cdot \sigma_Y}.$$

Dadoun - Y '21: $(a_i)_i$ free identically dist

$$S_n = a_1 + \text{---} + a_n.$$

$$R(S_n, S_m) = \sqrt{\frac{n}{m}} \quad \forall n \le m.$$

proof: assume n divides m.

ξ polynomial in S_n

ξ' ——— in S_m.

Denote

$$\mathrm{Proj}_I = E\left[\,\cdot \mid (a_i)_{i \in I}\right]$$

$$\langle \xi, \xi' \rangle = \langle \xi, \mathrm{Proj}_{\{1,-,n\}} \xi' \rangle \overset{C.S}{\le} \|\xi\|_2 \cdot \|\mathrm{Proj}_{[n]} \xi'\|_2$$

$\tau(\xi) = \tau(\xi') = 0$

$$\xi^{\Gamma} = \sum_{j=1}^{m} \mathrm{Proj}_{\{1,\dots,j\}}\, \xi^{\Gamma} - \mathrm{Proj}_{\{1,\dots,j-1\}}\, \xi^{\Gamma}$$

$$\boxed{\|\xi^{\Gamma}\|_2^2 = \sum_{j=1}^{m} \left\| \mathrm{Proj}_{\{1,\dots,j\}}\, \xi^{\Gamma} - \mathrm{Proj}_{\{1,\dots,j-1\}}\, \xi^{\Gamma} \right\|^2}$$

$$\left\|\mathrm{Proj}_{[n]}\, \xi^{\Gamma}\right\|_2^2 = \sum_{l=1}^{n} \left\| \mathrm{Proj}_{\{1,\dots,l\}}\, \xi^{\Gamma} - \mathrm{Proj}_{\{1,\dots,l-1\}}\, \xi^{\Gamma} \right\|_2^2$$

$$= \frac{1}{\frac{m}{n}} \sum_{k=0}^{\frac{m}{n}-1} \sum_{l=1}^{n} \left\| \mathrm{Proj}_{\{kn+1,\dots,kn+l\}}\, \xi^{\Gamma} - \mathrm{Proj}_{\{kn+1,\dots,kn+l-1\}}\, \xi^{\Gamma} \right\|_2^2$$

Freeness $\Rightarrow \mathrm{Proj}_I \circ \mathrm{Proj}_J = \mathrm{Proj}_{I\cap J}$

$$= \frac{n}{m} \sum_{k=0}^{\frac{m}{n}-1} \sum_{l=1}^{n} \left\| \cancel{\mathrm{Proj}_{\{kn+1,\dots,kn+l\}}} \circ \left(\mathrm{Proj}_{\{1,\dots,kn+l\}}\, \xi^{\Gamma} - \mathrm{Proj}_{\{1,\dots,kn+l-1\}}\, \xi^{\Gamma} \right) \right\|_2^2$$

$$\leq \frac{n}{m} \|\xi^{\Gamma}\|_2^2$$

Entropy in RMT:

X_N $\quad N \times N$ random matrix
i.i.d entries, centered, unit variance.

Wigner semi-circle theorem:

$$W_N = \frac{X_N + X_N^*}{\sqrt{2}}$$

$$\mathbb{E}\,\mu_{\frac{1}{\sqrt{N}} W_N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i\left(\frac{1}{\sqrt{N}} W_N\right)} \xrightarrow[N \to \infty]{\cancel{a.s}} \mu_{s.c}$$

Girko's circular law:

$$\mathbb{E}\,\mu_{\frac{1}{\sqrt{N}} X_N} \xrightarrow[N \to \infty]{\cancel{a.s}} \mu_{circ}$$

Marchenko-Pastur (square case)

$$\mathbb{E}\,\mu_{\frac{1}{N} X_N X_N^*} \xrightarrow[N \to \infty]{\cancel{a.s}} \mu_{MP}$$

What is entropy?

① rate fct° in LDP

② optimizer of which functional?

Take GUE

↳ distributed on H_N according

$$e^{-\frac{N}{2}\mathrm{Tr}(A^2)} \, dA$$

↳ joint law of eig

$$C_N \, e^{-\frac{N}{2}\sum_{i=1}^{N}\lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^2$$

$$\mathbb{P}(\mu_N \approx \nu) = C_N \int_{\{\mu_N \approx \nu\}} e^{-\frac{N}{2}\sum_{i=1}^{N}\lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^2$$

$$= c_N \int_{\{\mu_N \simeq \nu\}} e^{-N^2\left(\frac{1}{2N}\sum_{i=1}^{N}\lambda_i^2 - \frac{1}{N^2}\sum_{i,j}\log|\lambda_i-\lambda_j|\right)}$$

$$\approx c_N \; e^{-N^2\left(\underbrace{\frac{1}{2}m_2(\nu)}_{\text{second moment}} - \underbrace{\iint \log|x-y|\, d\nu(x)\, d\nu(y)}\right)}$$

<u>Ben Arous-Guionnet</u> '97: made this rigorous.

$$\mathcal{E}(\mu) = -\chi^{\circ}(\mu) = -\iint \log|x-y|\, d\mu(x)\, d\mu(y)$$

$$\mu_{sc} = \underset{m_2(\mu)=1}{\operatorname{argmin}} \mathcal{E}(\mu)$$

$$= \operatorname{argmin} \mathcal{E}\left(\frac{1}{\sqrt{m_2(\mu)}}\mu\right)$$

$$= \operatorname{argmin}\left(\boxed{\mathcal{E}(\mu) + \frac{1}{2}m_2(\mu)}\right)$$

$:= \mathcal{E}_1^{(2)}(\mu)$

Question 1: Is $\mathcal{E}_1^{(2)}$ monotone (in terms of matrix dimension) along Wigner's th?

Question 2: Is $\mathcal{E}_{\frac{1}{2}}^{(2)}$ monotone in Girko's th?

Q 3: Is $\mathcal{E}^{(1)}$ monotone in MP th?

Results:

Let X_N^{Gin} $N \times N$ (complex) Gaussian matrix properly normalized

$$X_N^{GUE} = \frac{X_N^{Gin} + (X_N^{Gin})^*}{2}$$

$$X_N^{LUE} = X_N^{Gin} X_N^{Gin^*}$$

Chafaï - Dadoun - Y '22 :

$$\mathcal{E}_1^{(2)}\left(\mathbb{E}\mu_{X_N^{GUE}}\right) = \frac{3}{4} + \frac{1}{2}\left(\log N + \gamma + \frac{1}{2N} - H_N\right)$$

γ = Euler constant

$H_N = \sum_{i=1}^{N} \frac{1}{i} \rightsquigarrow N$-th harmonic nb.

$\hookrightarrow$ decreasing in N.

Chafaï - Dadoun - Y '22 :

$$\mathcal{E}_{\frac{1}{2}}^{(2)}\left(\mathbb{E}\mu_{X_N^{Gin}}\right) = \frac{3}{4} + \frac{1}{2}\left(\log N + \gamma + \frac{1}{2N} - H_N\right) \boxed{+ \frac{1}{2}\sum_{k=N+1}^{\infty} \frac{4^{-k}\binom{2k}{k}}{k(k-1)}}$$

$$\mathcal{E}^{(1)}\left(\mathbb{E}\mu_{X_N^{LUE}}\right) = \frac{3}{2} + \left(\log N + \gamma + \frac{1}{2N} - H_N\right).$$

Open problems:

- Prove / disprove Q 1, 2, 3.
- Prove it for $N = 2^k$.

Wigner $\begin{pmatrix} \text{Wigner} & \\ & \text{Wigner} \end{pmatrix}$

- Explain $\mathcal{E}^{(2)}_{1}(GUE) = \mathcal{E}^{(2)}_{\frac{1}{2}}(Gin)$

$$+$$

$$\boxed{+\frac{1}{2} \sum_{k=N+1}^{\infty} \frac{4^{-k}\binom{2k}{k}}{k(k-1)}}$$

$$\begin{pmatrix} 0 & Gin \\ Gin^{*} & 0 \end{pmatrix}$$

- $2\,\mathcal{E}^{(2)}_{1}(GUE) = \mathcal{E}^{(1)}(LUE)$.