Factoriality of q-Araki-Woods von Neumann algebras via conjugate variables (or on the power of technology)

joint work with Manish Kumar and Mateusz Wasilewski

Adam Skalski

IMPAN, Polish Academy of Sciences, Warsaw

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Plan of the talk

We will recall the construction of q-Araki-Woods von Neumann algebras, due to Hiai, combining earlier deformations of free group factors due to Bożejko, Kümmerer and Speicher on one hand, and to Shlyakhtenko on the other, and explain how we can use conjugate variables to establish their factoriality.

Fock spaces and free Gaussians

$\mathsf{H}_{\mathbb{R}}$ – a separable real Hilbert space (often finite-dimensional)

H – complexification of $\mathsf{H}_{\mathbb{R}}$

$\mathcal{F}(\mathsf{H}) := \bigoplus_{n=0}^{\infty} \mathsf{H}^{\otimes n}$ – free Fock space

$\mathcal{F}(\mathsf{H})_{\mathrm{alg}}$ – 'finite' vectors in $\mathcal{F}(\mathsf{H})$, $\Omega \in \mathsf{H}^{\otimes 0} \approx \mathbb{C} \subset \mathcal{F}(\mathsf{H})_{\mathrm{alg}}$ – the vacuum vector.

Definition

Given $\xi \in \mathsf{H}_{\mathbb{R}}$ define the (left) creation operator $a^*(\xi) : \mathcal{F}(\mathsf{H})_{\mathrm{alg}} \to \mathcal{F}(\mathsf{H})_{\mathrm{alg}}$

$$a^*(\xi)\Omega = \xi, \quad a^*(\xi)(\zeta) = \xi \otimes \zeta, \;\; \zeta \in \mathsf{H}^{\otimes n}.$$

It is easy to check it is bounded. Further the free Gaussian operator associated to ξ is

$$s(\xi) = a^*(\xi) + (a^*(\xi))^*$$

and the free Gaussian von Neumann algebra is

$$\Gamma(\mathsf{H}_{\mathbb{R}}) := \{s(\xi) : \xi \in \mathsf{H}_{\mathbb{R}}\}'' \subset B(\mathcal{F}(\mathsf{H})).$$

Free group factors – Fock space picture

$$\Gamma(\mathsf{H}_{\mathbb{R}}) := \{s(\xi) : \xi \in \mathsf{H}_{\mathbb{R}}\}'' \subset B(\mathcal{F}(\mathsf{H})).$$

The following facts go back to Voiculescu, Dykema and Nica (1992):

- the vacuum vector defines a faithful tracial state $\varphi := \omega_\Omega$ on $\Gamma(\mathsf{H}_{\mathbb{R}})$;
- if $(e_i)_{i=1}^d$ is an ONB inside $\mathsf{H}_{\mathbb{R}}$, then $(s(e_1), \ldots, s(e_d))$ are free inside $(\Gamma(\mathsf{H}_{\mathbb{R}}), \varphi)$;
- $\Gamma(\mathsf{H}_{\mathbb{R}}) \approx L(\mathbb{F}_d)$, where $d = \dim(\mathsf{H}_{\mathbb{R}})$. Thus if $d \geq 2$, $\Gamma(\mathsf{H}_{\mathbb{R}})$ is a non-injective, full II_1 factor with the Haagerup approximation property.

Write $a_i^* = a^*(e_i)$. Then $a_i a_j^* = \delta_{ij} I$, so

$$a_i a_j^* - 0 a_j^* a_i = \delta_{ij} I.$$

Free group factors – Fock space picture

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Write $a_i^* = a^*(e_i)$. Then $a_i a_j^* = \delta_{ij} I$. What about

$$a_i a_j^* - q a_j^* a_i = \delta_{ij} I \quad ?$$

q-Fock space

Fix $q \in (-1, 1)$.

Bożejko and Speicher defined in [BS94] a new scalar product on $\mathrm{H}^{\otimes n}$:

$$\langle \zeta_1 \otimes \cdots \otimes \zeta_n, \eta_1 \otimes \cdots \otimes \eta_n \rangle_q := \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} \langle \zeta_1, \eta_{\pi(1)} \rangle \cdots \langle \zeta_n, \eta_{\pi(n)} \rangle$$

and showed it is strictly positive.

Set $\mathcal{F}_q(\mathrm{H}) := \sum_{n=0}^{\infty} \mathrm{H}_q^{\otimes n}$ and for $\xi \in \mathrm{H}_{\mathbb{R}}$ define the (left) q-creation operator $a_q^*(\xi) : \mathcal{F}_q(\mathrm{H})_{\mathrm{alg}} \to \mathcal{F}_q(\mathrm{H})_{\mathrm{alg}}$

$$a_q^*(\xi)\Omega = \xi, \quad a_q^*(\xi)(\zeta) = \xi \otimes \zeta, \;\; \zeta \in \mathrm{H}_q^{\otimes n}.$$

[BS 94] showed each $a_q^*(\xi)$ is bounded, defined the q-Gaussian operators

$$s_q(\xi) = a_q^*(\xi) + (a_q^*(\xi))^*$$

and the q-Gaussian von Neumann algebra

$$\Gamma_q(\mathrm{H}_{\mathbb{R}}) := \{ s_q(\xi) : \xi \in \mathrm{H}_{\mathbb{R}} \}'' \subset B(\mathcal{F}_q(\mathrm{H})).$$

Note that we have

$$a_i a_j^* - q a_j^* a_i = \delta_{ij} P_{\Omega}.$$

q-Gaussians and factoriality

What do we know about $\Gamma_q(\mathsf{H}_\mathbb{R})$?

- The vacuum vector defines a faithful tracial state $\varphi := \omega_\Omega$ on $\Gamma_q(\mathsf{H}_\mathbb{R})$ (Bożejko+Speicher, 1994),
- if $\dim(\mathsf{H}_\mathbb{R}) = \infty$, $\Gamma_q(\mathsf{H}_\mathbb{R})$ is a factor (Bożejko+Kümmerer+Speicher, 1997),
- if $\dim(\mathsf{H}_\mathbb{R})$ is large enough (in terms of q), $\Gamma_q(\mathsf{H}_\mathbb{R})$ is a factor (Śniady, 2004),
- $\Gamma_q(\mathsf{H}_\mathbb{R})$ is a factor whenever $\dim(\mathsf{H}_\mathbb{R}) \geqslant 2$ (Ricard, 2005),
- when $\dim(\mathsf{H}_\mathbb{R}) \geqslant 2$, $\Gamma_q(\mathsf{H}_\mathbb{R})$ is a non-injective (Nou, 2004), has the Haagerup property (Wasilewski, 2017).

In many ways $\Gamma_q(\mathsf{H}_\mathbb{R})$ behaves as a free group factor $\Gamma_0(\mathsf{H}_\mathbb{R})$.

- when $\dim(\mathsf{H}_\mathbb{R}) = d < \infty$ and q is 'small enough' (depending on d) then $\Gamma_q(\mathsf{H}_\mathbb{R}) \approx \Gamma_0(\mathsf{H}_\mathbb{R})$ (Guionnet+Shlyakhtenko, 2014)...
- but in 2022 Caspers proved that $\Gamma_q(\ell^2) \not\approx \Gamma_0(\ell^2)$ if $q \neq 0$.
- Recently Kuzmin showed that when $\dim(\mathsf{H}_\mathbb{R}) < \infty$ the C^*-algebra generated by $a_q^*(\xi), \xi \in \mathsf{H}_\mathbb{R}$ inside $B(\mathcal{F}_q(\mathsf{H})$ does not depend on q.

Quasi-free deformation of Shlyakhtenko

$\mathsf{H}_{\mathbb{R}}$ – a separable real Hilbert space (often finite-dimensional)

$(U_t)_{t\in\mathbb{R}}$ – a one-parameter strongly continuous group of orthogonal transformations of $\mathsf{H}_{\mathbb{R}}$

Extend U_t to H to obtain a one-parameter unitary group $(\tilde{U}_t)_{t\in\mathbb{R}}$ and let A denote its positive (in general unbounded) generator ($\tilde{U}_t = A^{it}$, $t \in \mathbb{R}$). Then set a new inner product on H by

$$\langle \zeta, \eta \rangle_U := \langle \zeta, \frac{2}{1 + A^{-1}} \eta \rangle.$$

Let H_U denote the completion of H with respect to the new scalar product.

- the new product restricts to the original one on $\mathsf{H}_{\mathbb{R}}$, so we can think of a new 'generating' embedding of $\mathsf{H}_{\mathbb{R}} \subset \mathsf{H}_U$;
- $\tilde{U}_t$ extend also to unitaries on H_U.

Quasi-free von Neumann algebras

Given $\xi \in \mathsf{H}_{\mathbb{R}}$ we can still consider the creation operator $a^*(\xi) \in B(\mathcal{F}(\mathsf{H}_U))$.

Definition

The quasi-free Gaussian operator associated to $\xi \in \mathsf{H}_{\mathbb{R}}$ is

$$s(\xi) = a^*(\xi) + (a^*(\xi))^* \in B(\mathcal{F}(\mathsf{H}_U))$$

and the quasi-free Gaussian von Neumann algebra is

$$\Gamma(\mathsf{H}_{\mathbb{R}}, U_t) := \{s(\xi) : \xi \in \mathsf{H}_{\mathbb{R}}\}'' \subset B(\mathcal{F}(\mathsf{H}_U)).$$

Theorem (Shlyakhtenko, 1995)

The vacuum vector defines a faithful (in general non-tracial) state φ on $\Gamma(\mathsf{H}_{\mathbb{R}}, U_t)$. Moreover $\Gamma(\mathsf{H}_{\mathbb{R}}, U_t)$ is factor, of type III whenever $(U_t)_{t\in\mathbb{R}}$ is non-trivial. The precise type can be computed from spectral data of A.

Weakly mixing/almost periodic parts

The idea behind the results above is based on the following key observations:

- when $\mathsf{H}_{\mathbb{R}}$ decomposes into U_t-invariant subspaces, the resulting $\Gamma(\mathsf{H}_{\mathbb{R}}, U_t)$ decompose as free products;
- spectral decomposition of A first allows us to decompose H into almost periodic part (spanned by eigenvectors of A) and the weakly mixing part (the rest)
- the almost periodic part corresponds further to splitting $\mathsf{H}_{\mathbb{R}}$ into the part where U_t is trivial and two-dimensional subspaces where U_t acts as

$$\begin{bmatrix} \cos \log \lambda t & -\sin \log \lambda t \\ \sin \log \lambda t & \cos \log \lambda t \end{bmatrix}$$

q-Araki-Woods algebras of Hiai

Around 2002 Fumio Hiai combined the two constructions:

$\mathsf{H}_{\mathbb{R}}$ – a separable real Hilbert space (often finite-dimensional)

$(U_t)_{t\in\mathbb{R}}$ – a one-parameter strongly continuous group of orthogonal transformations of $\mathsf{H}_{\mathbb{R}}$

$q \in (-1, 1)$

Definition

The q-quasi-free Gaussian operator associated to $\xi \in \mathsf{H}_{\mathbb{R}}$ is

$$s_q(\xi) = a_q^*(\xi) + (a_q^*(\xi))^* \in B(\mathcal{F}_q(\mathsf{H}_U))$$

and the q-Araki-Woods von Neumann algebra is

$$\Gamma_q(\mathsf{H}_{\mathbb{R}}, U_t) := \{s_q(\xi) : \xi \in \mathsf{H}_{\mathbb{R}}\}'' \subset B(\mathcal{F}_q(\mathsf{H}_U)).$$

Modular structure

Proposition (Hiai, 2002)

The vacuum vector defines a faithful (in general non-tracial) state φ on $\Gamma_q(\mathsf{H}_\mathbb{R}, U_t)$; the GNS space of φ is canonically isomorphic to $\mathcal{F}_q(\mathsf{H}_U)$, and we have for all $n \in \mathbb{N}$, $\xi_1, \ldots, \xi_n \in \mathsf{H}_\mathbb{R}$

$$\Delta_\varphi(\xi_1 \otimes \cdots \otimes \xi_n) = A^{-1}\xi_1 \otimes \cdots \otimes A^{-1}\xi_n.$$

In other words, the modular group is governed by the behaviour of $(U_t)_{t\in\mathbb{R}}$, and in the almost periodic case one can use the matrix form of A to produce eigenvectors for Δ_φ.

When $\dim(\mathsf{H}_\mathbb{R}) = d < \infty$ and q is 'small enough' (depending on d) then $\Gamma_q(\mathsf{H}_\mathbb{R}, U_t) \approx \Gamma_0(\mathsf{H}_\mathbb{R}, U_t)$ (Nelson, 2015).

Expected subalgebras

If $\mathsf{K}_{\mathbb{R}} \subset \mathsf{H}_{\mathbb{R}}$ is a subspace left invariant by $(U_t)_{t\in\mathbb{R}}$, then

$$\Gamma_q(\mathsf{K}_{\mathbb{R}}, U_t|_{\mathsf{K}_{\mathbb{R}}}) \subset \Gamma_q(\mathsf{H}_{\mathbb{R}}, U_t)$$

as a φ-expected subalgebra.

The relevant conditional expectation is a second quantisation of $P : \mathsf{H}_{\mathbb{R}} \to \mathsf{K}_{\mathbb{R}}$; in general 'second-quantising' good maps on $\mathsf{H}_{\mathbb{R}}$ is a useful and important tool!

Known factoriality results

Assume $\dim(\mathsf{H}_\mathbb{R}) \geqslant 2$.

- Hiai (2003) showed that if $(U_t)_{t\in\mathbb{R}}$ is weakly mixing, then $\Gamma_q(\mathsf{H}_\mathbb{R}, U_t)$ is a non-injective factor
- in fact $\Gamma_q(\mathsf{H}_\mathbb{R}, U_t)^\varphi \subset \Gamma_q(\mathsf{H}_\mathbb{R}, U_t)_{ap}$ (AS+Wang, 2018)
- if $(U_t)_{t\in\mathbb{R}}$ is almost periodic, and $\dim(\mathsf{H}_\mathbb{R}) = \infty$, then $\Gamma_q(\mathsf{H}_\mathbb{R}, U_t)$ is a factor (Hiai, 2003, Bikram+Mukherjee, 2017)
- if $(U_t)_{t\in\mathbb{R}}$ is almost periodic and admits a fixed non-zero vector, then $\Gamma_q(\mathsf{H}_\mathbb{R}, U_t)$ is a factor (Bikram+Mukherjee, 2017, AS+Wang, 2018);
- if $\dim(\mathsf{H}_\mathbb{R}) \geqslant 3$, or if $\dim(\mathsf{H}_\mathbb{R}) = 2$ and the 'deformation' is large enough, then $\Gamma_q(\mathsf{H}_\mathbb{R}, U_t)$ is a factor (Bikram+Mukherjee+Ricard+Wang, 2022).

Why do the earlier proofs do not give a complete result?

Factoriality remained unknown in dimension 2. All the known proofs used either

- freeness – when $q = 0$;
- some sort of mixing property which allowed one to show for example that if $\xi \in \mathsf{H}_{\mathbb{R}}$ is a fixed vector for $(U_t)_t \in \mathbb{R}$ then $s_q(\xi)''$ is a masa in $\Gamma_q(\mathsf{H}_{\mathbb{R}}, U_t)$.

In general if $\xi \in \mathsf{H}_{\mathbb{R}}$ is not fixed, then $s_q(\xi)''$ is not a MASA!
(Bikram+Mukherjee, 2022)

Later Bikram, Mukherjee, Ricard and Wang worked hard to bypass this providing estimates guaranteeing the existence of a 'mixing subspace' inside the L^2-space of $\Gamma_q(\mathsf{H}_{\mathbb{R}}, U_t)$, but such analysis seems to be unable to cover all cases.

But...

Declaration

Whatever happens, we have got
The Maxim gun, and they had not!

Conjugate variables: abstract setup

(M, φ) – a von Neumann algebra with a faithful normal state φ
$A_1, \ldots, A_d$ – a self-adjoint set of elements of M, which we assume to be algebraically free.

Definition

Quasi-free difference quotients ∂_i are unique derivations from $\mathbb{C}[A_i, \ldots, A_d]$ into $\mathrm{M}\overline{\otimes}\mathrm{M}^{op}$ such that $\partial_i(A_j) := \varphi(A_j A_i)\mathbb{1} \otimes \mathbb{1}$ for all $i, j \in \{1, \ldots, d\}$. The conjugate variable for ∂_i will be a vector $\xi_i \in L^2(\mathrm{M}, \varphi)$ such that

$$\langle \xi_i, x\mathbb{1} \rangle = \langle \mathbb{1} \otimes \mathbb{1}, \partial_i(x)(\mathbb{1} \otimes \mathbb{1}) \rangle$$

for all $x \in \mathbb{C}[A_i, \ldots, A_d]$. If such a vector exists, we say that A_i has the finite free Fisher information.

We say that a non-zero $A \in \mathrm{M} \subset L^2(\mathrm{M}, \varphi)$ is an eigenoperator of the modular group of φ if there exists a $\lambda > 0$ such that $\Delta_\varphi(A) = \lambda A$.

Joys of modern technology, part I

Theorem (Nelson (2017), Theorems A and B)

Let M be a von Neumann algebra with a faithful normal state φ. Suppose M is generated by a finite set $G = G^*$, $|G| \geqslant 2$ of eigenoperators of the modular group σ^φ with finite free Fisher information. Then $(\mathsf{M}^\varphi)' \cap \mathsf{M} = \mathbb{C}$. In particular, M^φ is a II_1 factor and if $H < \mathbb{R}_*^\times$ is the closed subgroup generated by the eigenvalues of G then M is a factor of type

$$\begin{cases} \mathrm{III}_1 & \text{if } H = \mathbb{R}_*^\times \\ \mathrm{III}_\lambda & \text{if } H = \lambda^{\mathbb{Z}}, 0 < \lambda < 1 \\ \mathrm{II}_1 & \text{if } H = \{1\}. \end{cases}$$

Moreover M^φ does not have property Γ, and if M is a type III_λ factor, $0 < \lambda < 1$, then M is full.

Back to q-Araki-Woods

Fix $d \in \mathbb{N}$, $q \in (-1, 1)$ and a d-dimensional real Hilbert space $\mathsf{H}_\mathbb{R}$ with $(U_t)_{t\in\mathbb{R}}$ a one-parameter strongly continuous group of orthogonal transformations.

$$\Gamma_q(\mathsf{H}_\mathbb{R}, U_t) := \{s_q(\xi) : \xi \in \mathsf{H}_\mathbb{R}\}'' \subset B(\mathcal{F}_q(\mathsf{H}_U)).$$

Given $\zeta \in \mathsf{H}$ we can consider the (unique) operator $W(\zeta) \in \Gamma_q(\mathsf{H}_\mathbb{R}, U_t)$ such that $W(\zeta)\Omega = \zeta$. If $(e_i)_{i=1}^d$ is a linearly independent set in $\mathsf{H} = \mathbb{C}^d$ and $A_i = W(e_i)$, $i \in \{1, \ldots, d\}$, then

$$\Gamma_q(\mathsf{H}_\mathbb{R}, U_t) = \{A_1, \ldots, A_d\}''.$$

Form of the modular operator implies that we can choose $(e_1, \ldots, e_d)$ so that each A_i is an eigenoperator of the modular group, and the set $\{A_1, \ldots, A_d\}$ is self-adjoint.

In 2022 Miyagawa and Speicher constructed conjugate variables for $\Gamma_q(\mathsf{H}_\mathbb{R})$ (in the tracial case), using the dual variable approach.

From dual variables to conjugate variables

Definition

A tuple $(D_1, \dots, D_d)$ of unbounded operators on $\mathcal{F}_q(H_U)$ with $\mathcal{F}_q(\mathrm{H})_{\mathrm{alg}}$ contained in their domains and $\mathbb{1}$ contained in the domains of their adjoints is a (normalized) dual system for $(A_1, \dots, A_d)$ if for all $i, j \in \{1, \dots, d\}$

$$[D_i, A_j] = \langle \bar{e}_j, e_i \rangle_U P_{\mathbb{C}\Omega} = \varphi(A_j A_i) P_{\mathbb{C}\Omega} \quad \text{and } D_i \Omega = 0.$$

Here $\bar{\xi}$ denotes the usual conjugate of a vector ξ in $\mathbb{C}^d$.

Lemma

Let $\{e_i\}$ and $\{f_i\}$ be two linearly independent sets in H, set $A_i = W(e_i)$, $C_i = W(f_i)$. A dual system for $(A_i)_{i=1}^d$ exists if and only if one for $(C_i)_{i=1}^d$ does.

Proposition

Suppose that $(D_1, \dots, D_d)$ is a normalized dual system for $(A_1, \dots, A_d)$. Then $(D_1^* \mathbb{1}, \dots, D_d^* \mathbb{1})$ are conjugate variables for $(A_1, \dots, A_d)$.

In the tracial case this is due to Miyagawa and Speicher; here the proof is essentially the same.

Joys of modern technology, part II: algebra...

Fix now $(e_1, \ldots, e_d)$ – an orthonormal set of elements in H (with respect to undeformed scalar product), and set $B_{ij} = \langle \bar{e}_i, e_j \rangle$, so that $\varphi(A_i A_j) = B_{ij}$.

Lemma

The algebraic formula for the dual variables of $(A_1, \ldots, A_d)$ is given as follows ($i \in \{1, \ldots, d\}$, $n \in \mathbb{N}$, $j_1, \ldots, j_n \in \{1, \ldots, d\}$):

$$D_i(e_{j_n} \ldots e_{j_1}) = \sum_{\pi \in P(n+1)} (-1)^{\pi(0)-1} q^{\mathrm{cross}(\pi)} \delta^{B}_{p(\pi)} e_{s(\pi)},$$

where $\delta^{B}_{p(\pi)} := \prod_{\substack{(l,m)\in\pi \\ l>m}} B_{j_l, j_m}$.

Recall that we know that

$$[D_i, A_j] = \langle \bar{e}_j, e_i \rangle_U P_{\mathbb{C}\Omega} = \varphi(A_j A_i) P_{\mathbb{C}\Omega} \qquad D_i \Omega = 0.$$

$P(n+1)$ above denotes a class of partitions of the set indexed by $(j_1, \ldots, j_n, i)$ into pairs and singletons, following specific rules, introduced by Miyagawa and Speicher. The proof is inductive, and follows closely the tracial case.

... and estimates

Proposition

For each $i \in \{1, \ldots, d\}$ we have $\mathbb{1} \in \mathrm{Dom}\ D_i^*$. Thus $(D_1^*\mathbb{1}, \ldots, D_d^*\mathbb{1})$ forms a set of conjugate variables for $(A_1, \ldots, A_d)$.

Here again the proof follows Miyagawa and Speicher, but needs some tweaks.

- we analyse expressions of the form

$$\Big\langle \Omega, D_i\big(\sum_w \alpha_w e_w\big)\Big\rangle_{\mathcal{F}_q(\mathsf{H}_U)} = \sum_{m=1}^{\infty} \sum_{\pi \in P(2m), \pi(0)=m} \sum_{|w|=2m-1} \alpha_w (-1)^{m-1} q^{\mathrm{cross}(\pi)} \delta^B_{p(\pi),w};$$

- partitions in $P(2m)$ with $\pi(0) = m$ correspond to partitions in S_{m-1}; one then rewrites the sum above a few times and expresses it in terms of a (modified!) free left annihilators acting on $\mathsf{H}_q^{\otimes n}$
- finally one uses estimates due to Bożejko on the uniform norms of the latter.

q-Araki-Woods – conclusions

Collecting the pieces above allows us to conclude what we need to land in the setup studied by Nelson.

Corollary

Let $\dim(\mathsf{H}_{\mathbb{R}}) < \infty$. Then $\Gamma_q(\mathsf{H}_{\mathbb{R}}, U_t)$ equipped with the canonical state φ is generated by a finite set $G = G^*$ of eigenoperators of the modular group of φ with finite free Fisher information.

Main result

Theorem

Let $(\mathsf{H}_\mathbb{R}, U_t)$ be given, with $\dim(\mathsf{H}_\mathbb{R}) \geqslant 2$. Then $\Gamma_q(\mathsf{H}_\mathbb{R}, U_t)$ is a factor of type

$$\begin{cases} \mathrm{III}_1 & \text{if } G = \mathbb{R}^\times_*, \\ \mathrm{III}_\lambda & \text{if } G = \lambda^\mathbb{Z}, 0 < \lambda < 1, \\ \mathrm{II}_1 & \text{if } G = \{1\}, \end{cases}$$

where $G < \mathbb{R}^\times_*$ is the closed subgroup generated by the spectrum of A. Moreover the centralizer $\Gamma_q(\mathsf{H}_\mathbb{R}, U_t)^\varphi$ with respect to the canonical state is irreducible (i.e. the respective relative commutant is trivial).

Proof.

- the case of infinite-dimensional $\mathsf{H}_\mathbb{R}$ can be deduced from earlier results;
- for $\dim(\mathsf{H}_\mathbb{R}) < \infty$ we may use the conjugate variables!

$\square$

Fullness and solidity

Definition

A factor M is called full if the inner automorphism group Inn(M) is closed in Out(M).

It is called solid if whenever $N \subset M$ is an expected subalgebra with with the relative commutant $N' \cap M$ diffuse, N must be amenable.

Given a non-principal ultrafilter ω we say that M is ω-solid if whenever $N \subset M$ is an expected subalgebra with with the relative commutant inside the ultraproduct (i.e. $N' \cap M^\omega$) diffuse, N must be amenable.

An important tool for deducing solidity is the Akemann-Ostrand property: i.e. the existence of nice (say exact) weak*-dense C^*-subalgebras $A \subset M$, $B \subset M'$ such that the product/quotient map $A \odot B \mapsto B(H)/K(H)$ is min-continuous.

Non-injectivity of q-Araki-Woods

Theorem

The factor $\Gamma_q(\mathsf{H}_{\mathbb{R}}, U_t)$ is not injective as soon as $\dim(\mathsf{H}_{\mathbb{R}}) \geqslant 2$.

Sketch of the proof:

- it suffices to find a non-injective expected subalgebra, so if there is a weakly mixing part, already Hiai's results suffice;
- otherwise we work with a two-dimensional U_t-invariant subspace of $\mathsf{H}_{\mathbb{R}}$: either U_t is there trivial, and non-injectivity follows from the work of Nou...
- or by what we said before based on work of Nelson we obtain a III_λ full factor, and fullness implies non-injectivity.

So we use fullness to deduce non-injectivity.

Fullness and solidity for q-Araki-Woods

Theorem

Let $(\mathsf{H}_{\mathbb{R}}, U_t)$ be given with $2 \leqslant \dim \mathsf{H}_{\mathbb{R}} < \infty$. Then $\Gamma_q(\mathsf{H}_{\mathbb{R}}, U_t)$ is solid and full.

Sketch of the proof:

- in the tracial case fullness follows from the work of Miyagawa and Speicher, and for the type III_λ, $0 < \lambda < 1$, from the work of Nelson;
- in general we can use the Akemann-Ostrand property with respect to the C^*-algebra contained in the Cuntz-Toeplitz algebra $T_q(\mathsf{H}) \subset B(\mathcal{F}_q(\mathsf{H}))$ (the latter is nuclear by the recent result of Kuzmin);
- then results of Houdayer and Raum imply first that $\Gamma_q(\mathsf{H}_{\mathbb{R}}, U_t)$ is ω-solid, and then, as by what we have already said the centralizer of $\Gamma_q(\mathsf{H}_{\mathbb{R}}, U_t)$ is a non-injective II_1 factor, we can deduce fullness.

So we use non-injectivity to deduce fullness.

What else is known about q-Araki-Woods von Neumann algebras?

- They have the Haagerup approximation property (Wasilewski, 2017)...
- and the weak*-CCAP (Avsec+Brannan+Wasilewski, 2017)

And what we would like to know

- Is $\Gamma_q(\mathsf{H}_\mathbb{R}, U_t)$ always full, even when $\dim \mathsf{H}_\mathbb{R} = \infty$?
- Can it ever happen that $\Gamma_q(\mathsf{H}_\mathbb{R}, U_t) \approx \Gamma_{q'}(\mathsf{K}_\mathbb{R}, V_t)$ if $\dim \mathsf{H}_\mathbb{R} < \infty$, $\dim \mathsf{K}_\mathbb{R} = \infty$, $q' \neq 0$?

Both are known in the tracial case: the first is due to Śniady ($\Gamma_q(\mathsf{H}_\mathbb{R})$ is full whenever $\dim(\mathsf{H}_\mathbb{R}) \geqslant 2$), and the second is stated below.

Proposition

If $\dim \mathsf{H}_\mathbb{R} < \infty$, $\dim \mathsf{K}_\mathbb{R} = \infty$, $q' \neq 0$ then

$$\Gamma_q(\mathsf{H}_\mathbb{R}) \not\approx \Gamma_{q'}(\mathsf{K}_\mathbb{R}).$$

This can be deduced, using the Akemann-Ostrand property, from the results of Caspers and Ozawa.

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