Heat flow, random matrices, and random polynomials

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INTRODUCTION:
Random matrices and heat flow
**Random matrices: Circular law**

- **Ginibre ensemble**: $N \times N$ matrix $Z$ with independent entries
- Each entry complex Gaussian of mean 0, variance $1/N$
- When $N$ is large, eigenvalues will be approximately uniformly distributed on the unit disk:
Define (random) **empirical eigenvalue measure** of $Z$ as

$$
\mu^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j}
$$

where $\{\lambda_1, \ldots, \lambda_N\}$ are eigenvalues of $Z$

**Theorem**

The random probability measure $\mu^N$ converges weakly almost surely to the uniform probability measure on unit disk
Random matrices: Semicircular law

- **Gaussian unitary ensemble**: $N \times N$ Hermitian matrix with independent entries on and above diagonal, with $X_{kj} = \overline{X_{jk}}$
- Entries on diagonal are real Gaussian with mean 0, variance $1/N$
- Entries off diagonal are complex Gaussian with mean 0, variance $1/N$
- Eigenvalues approx. semicircular shape on $[-2, 2]$: 

![Diagram showing semicircular distribution of eigenvalues](image-url)
Heat flow on polynomials: definition

- Heat operator on polynomial $p$ of degree $N$:

$$\exp\left\{\frac{\tau}{2N} \frac{d^2}{dz^2}\right\} p(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\tau}{2N}\right)^k \left(\frac{d}{dz}\right)^{2k} p(z), \quad \tau \in \mathbb{C},$$

- Series terminates for all $\tau \in \mathbb{C}$
- Will always put factor of $N = \deg(p)$ in denominator
- This is natural scaling of time variable
Zeros $z_j(\tau)$ satisfy

$$\frac{dz_j}{d\tau} = -\frac{1}{N} \sum_{k: k \neq j} \frac{1}{z_j(\tau) - z_k(\tau)}$$

Also

$$\frac{d^2 z_j}{d\tau^2} = -\frac{2}{N^2} \sum_{k: k \neq j} \frac{1}{(z_j(\tau) - z_k(\tau))^3}$$

Formula for second deriv. is rational Calogero–Moser system
Take Hermitian random matrix $X^N$ with e.v. distribution $\to \mu$.

Let $p^N$ be char. poly. of $X^N$.

Apply **backward** heat op. ($\tau = -t$) to get poly. $p^N_t$.

Pólya–Benz theorem [1934]: roots will remain real.

**Theorem (Kabluchko)**

*Empirical measure of zeros of $p^N_t$ approach free additive convolution:*

$$\mu \boxplus (\text{semi. circ. measure on } [-2\sqrt{t}, 2\sqrt{t}])$$

Result of Kabluchko using “finite free convolution” of Marcus, Spielman, Srivastava.
Free convolution: computes limiting e.v. distribution of sums of indep. Hermitian random matrices

Hence: zeros of $p^N_t$ resemble e.v. of $X^N + \sqrt{t} \text{GUE}$

Random matrix interpretation to backward heat op on polynomials with real roots: like adding a GUE!

Example: backward heat flow on char. poly. GUE gives semicirc. distrib. on $[-2\sqrt{1+t}, 2\sqrt{1+t}]$
Forward heat flow and random matrices

Question

What happens if we apply forward heat operator \((τ = t)\) to characteristic polynomial of GUE? Can we just replace \(t\) by \(-t\) in preceding result (shrinking semicircle)?

Let’s see!
Forward heat flow and random matrices
Apply *forward* heat op. \((\tau = t)\) to char. poly. of GUE

Zeros become **complex** even for small \(t\) if \(N\) is large

**Conjecture** (Hall–H0, '22)

*For* 0 < \(t\) < 2, *get uniform distrib. on ellipse with semi-axes* 2 − \(t\) and \(t\).

*E.g., with* \(t = 1\), *get uniform distribution on disk: semicircular becomes circular!*
Example: Semicircular to circular

- Forward heat op. for time $\tau = 1$ starting from char. poly. GUE
- Approx. uniform on unit disk—but not same distrib. as e.v. of Ginibre
Why does result for backward heat operator not extend?

- Method of Marcus–Spielman–Srivastava uses **expected char. poly.**
- *Expectation value* of (forward heat op.)\((\text{char. poly. GUE})\) is scaled Hermite polynomial
- Zeros of expected polynomial *will* have shrinking semicircular distribution
- But this does not tell you about zeros without expectation value—unless zeros are real
- Later: use expectation value of **absolute value squared** of char. poly.
Goal

Identify examples in which applying heat operator to characteristic polynomial of one random matrix model gives new polynomial whose zeros resemble the eigenvalues of a second random matrix model.

Goal

Develop general theory of how zeros of polynomials evolve under heat flow, apart from connections to random matrix theory.
PART 2

MODEL DEFORMATION PHENOMENON
The relationship between circular and semicircular laws

- Twice the real part of the eigenvalues in the circular law has the same bulk distribution as the eigenvalues in the semicircular law.
- Trivial from the formulas but: **why** is it true?
- Why are real parts of eigenvalues in one model related to the eigenvalues in different model?

**Circular-semicircular Challenge**

*Explain the relationship between limiting e.v. distributions of Ginibre ensemble and GUE without using the circular and semicircular laws.*
Generalizing: Hermitian plus elliptic model

- RMT: Let $X$ and $Y$ be independent GUE’s, set
  \[ Z = e^{i\theta}(aX + ibY) \]

- Free prob.: take $X$, $Y$ freely indep. semicircular elements
- Limiting e.v. distribution/Brown measure is uniform on ellipse:
Parameters for elliptic model

- Use parameters $s \in \mathbb{R}$ and $\tau \in \mathbb{C}$

\[
s = \mathbb{E} \left\{ \frac{1}{N} \text{trace}(Z^* Z) \right\} \quad \text{(variance)}
\]

\[
\tau = \mathbb{E} \left\{ \frac{1}{N} \text{trace}(Z^* Z) \right\} - \mathbb{E} \left\{ \frac{1}{N} \text{trace}(Z^2) \right\}
\]

- **Special cases**: $\tau = 0$ is Hermitian, $\tau = s$ is circular
- From Cauchy–Schwarz: $|\tau - s| \leq s$
Label elliptic element as $Z_{s,\tau}$; consider

$$X_0 + Z_{s,\tau}$$

where $X_0$ is Hermitian, indep. of $Z_{s,\tau}$

Let $\mu$ be limiting e.v. distribution of $X_0$

Additive case of work of Hall–Ho [2021]; results of Zhong [2021]
Theorem (Hall–Ho; Zhong)

Limiting e.v. distribution of $\mu_{s,\tau}$ of $X_0 + Z_{s,\tau}$ is supported on explicitly computable domain $\Omega_{s,\tau}$ and density of $\mu_{s,\tau}$ is constant (in $\Omega_{s,\tau}$) in the $i\tau$ direction.

- Example: $X_0$ is Bernoulli: $\mu$ is half sum of $\delta$-measures at $\pm 1$
- $s = 1, \tau = 1 + i/2$
“Model deformation” result: vary $\tau$ with $s$ fixed

- Fix $s$ and $X_0$, take $\tau_0$ and $\tau$

Theorem (Hall–Ho; Zhong)

There is a canonical map $\Phi_{s,\tau_0,\tau}$ such that push-forward of Brown measure of $X_0 + Z_{s,\tau_0}$ by $\Phi_{s,\tau_0,\tau}$ equals Brown measure of $X_0 + Z_{s,\tau}$.

- Bernoulli case with $s = 1$, $\tau_0 = 1$, and $\tau = 1 + i/2$
Map takes segments in $i\tau_0$-direction to segments in $i\tau$-direction.

$\Phi_{s,\tau_0,\tau}(z)$ is \textbf{linear in} $\tau$ for fixed $z$.

Bernoulli case with $s = 1$, $\tau_0 = 1$, and $\tau = 1 + i/2$. 
HOW to vary $\tau$ with $s$ fixed?

- Considering $Z_{s,\tau}$ with fixed $s$ and different values of $\tau$
- *Cannot* change $\tau$ with $s$ fixed by adding an indep. matrix $\Delta Z$
- $s$ is variance and variances add!
- *Can* decrease variance in Hermitian directions and increase variance in skew-Hermitian directions
- **Fiction**: Add an element of form $X_{-r} + iY_r$, semicircular with variances $-r$ and $r$
- Makes sense on element of form $W + \tilde{X}_t$, goes to $W + \tilde{X}_{t-r} + iY_r$
- Take $X_0 = 0$, with $s = 1$
- Take $\tau_0 = 1$ (circular) and $\tau = 0$ (semicircular)
- Map $\Phi_{s,\tau_0,\tau}$ gives circular-to-semicircular map:

\[ \Phi_{s,\tau_0,\tau}(z) = 2 \Re(z) \]

**Conclusion**

The map $z \mapsto 2 \Re(z)$ relating circular to semicircular laws is just one special case of a large family of maps with similar results.
PDE Method

- **PDE method** for proving these results

**Proposition (Hall–Ho, 2023)**

Log potential $S(z, s, \tau)$ of Brown measure $\mu_{s,\tau}$ satisfies a PDE w.r.t. $\tau$ with $s$ fixed:

$$\frac{\partial S}{\partial \tau} = \frac{1}{2} \left( \frac{\partial S}{\partial z} \right)^2 .$$

- Here $S$ is real-valued, $\tau$ and $z$ are complex variables
- Derivatives are complex partial derivatives (Cauchy–Riemann operators)
Multiplicative models

- Gaussian models: *sums* of i.i.d. matrices
- Also consider *products* of i.i.d. matrices close to identity
- Take

\[ B_{s,\tau} = \prod_{j=1}^{k} \left( I + \frac{i}{\sqrt{k}} Z_{s,\tau}^j - \frac{1}{2k} (s - \tau) I \right), \quad k \gg 1, \]

where \( Z_{s,\tau}^j \)'s are independent copies of \( Z_{s,\tau} \)
- Or: solve free SDE driven by elliptic Brownian motion:

\[ dB_{s,\tau}(t) = B_{s,\tau}(t) \left( i dZ_{s,\tau}(t) - \frac{1}{2} (s - \tau) dt \right) \]

then set \( t = 1 \)
Multiplicative models

- Then take $U$ unitary, indep. of $B_{s,\tau}$ and consider $UB_{s,\tau}$

- Limiting e.v. distribution of $UB_{s,\tau}$ computed in increasing generality by Driver–Hall–Kemp, Ho–Zhong, Hall–Ho

- **Example:** Law of $U$ at $\pm 1$ and $\pm i$ with $s = 1$, $\tau = 1 + i/2$
Relating models with different values of $\tau$

**Theorem (Hall–Ho, 2023)**

“Model deformation” holds in the multiplicative case.

- Fix $s$ and $U$, relate different values of $\tau$ with map
- Example: $s = 1$, $\tau_0 = 1$, $\tau = 1 + i/2$
HEAT FLOW CONJECTURE FOR RANDOM MATRICES
Idea of heat flow conjecture

Idea

Transformation between random matrices with different values of $\tau$ can be accomplished by applying heat operator to characteristic polynomial of one model.
Heat flow

- Fix $X_0$ and $s$, take $\tau_0$ and $\tau$
- Set $p_0^N = \text{char. poly. of } X_0 + Z_s, \tau_0$
- Set $p^N = \text{char. poly. of } X_0 + Z_s, \tau$
- Set
  \[
  q^N(z) = \exp \left\{ \frac{\tau - \tau_0}{2N} \frac{\partial^2}{\partial z^2} \right\} p_0^N(z)
  \]

Conjecture (Hall–Ho)

The empirical measure for zeros of $q^N$ converges weakly almost surely to the same limit as for zeros of $p^N$—namely, limiting e.v. distrib. of $X_0 + Z_s, \tau$. 
• $p^N$ is char. poly. of random matrix with parameter $\tau$

• $q^N$: start with char. poly. of random matrix with parameter $\tau_0$, apply heat flow for time $\tau - \tau_0$

• **Conjecture:** zeros of $q^N \approx$ zeros of $p^N$

• Heat flow for time $\tau - \tau_0$ **changes from** $\tau_0$ **to** $\tau$
Circular to semicircular case

- Take $X_0 = 0$, $s = 1$
- Take $\tau_0 = 1$ (circular) and $\tau = 0$ (semicircular)
- Roots of
  \[ q^N := \exp \left\{ -\frac{1}{2N} \frac{\partial^2}{\partial z^2} \right\} p_0^N \]
  approximate semicircular distribution on $[-2, 2]$
- Put $t$ in exponent with $0 \leq t \leq 1$
- Zeros move in approx. **straight lines** $z \mapsto z + t\bar{z}$
- Motion reflects that $\Phi_{s,\tau_0,\tau}(z)$ is linear in $\tau$ for fixed $z$
- Point starting at $z$ ends close to $2\Re(z)$
Semicircular to circular case \((\tau_0 = 0, \tau = 1)\)

- Points move in approx. straight lines
- Velocity in \(x\)-direction determined by initial \(x\)-value
- Velocity in \(y\)-direction random
- Points starting near given \(x\)-value end up on vertical line
Multiplicative case

- Similar results, with heat operator replaced by

\[
\exp \left\{ -\frac{\tau - \tau_0}{2N} \left( z^2 \frac{\partial^2}{\partial z^2} - (N - 2)z \frac{\partial}{\partial z} - N \right) \right\}
\]
GENERAL HEAT FLOW CONJECTURE
General conjecture

- Let $p^N_0$ be deg.-$N$ polynomials s.t. empirical measure of zeros converges to “nice” measure $\mu$
- Define
  \[
p^N(\tau, z) = \exp \left\{ \frac{\tau}{2N} \frac{d^2}{dz^2} \right\} p^N_0(z)\]

Conjecture

For sufficiently small $|\tau|$, the empirical measure of zeros of $p^N(\tau, z)$ converges to measure $\mu_\tau$ whose log potential $S(\tau, z)$ satisfies

\[
\frac{\partial S}{\partial \tau} = \frac{1}{2} \left( \frac{\partial S}{\partial z} \right)^2.
\]
• $S$ is real-valued but $\tau$ and $z$ are complex variables
• $\partial/\partial \tau$ and $\partial/\partial z$ are complex partial deriv. (Cauchy–Riemann ops.)
• Polynomials needn’t come from random matrices
• There is multiplicative version of conjecture
• Solution $S$ can degenerate; can’t expect $C^1$ solution for all $\tau$
• Expect straight-line motion for small $\tau$:

\[ z \mapsto z + \tau \frac{\partial S(0, z)}{\partial z} \]
Formal argument

- Define log potential of zeros \( \{ z^j(\tau) \}_{j=1}^N \) of \( p^N(\tau, z) \)

\[
S^N(\tau, z) = \frac{1}{N} \sum_{j=1}^N \log(|z - z_j(\tau)|^2)
\]

Proposition

*The log potential of the zeros \( S^N \) satisfies the PDE*

\[
\frac{\partial S^N}{\partial \tau} = \frac{1}{2} \left( \frac{\partial S^N}{\partial z} \right)^2 + \frac{1}{2N} \frac{\partial^2 S^N}{\partial z^2}
\]

*away from the zeros.*

- **But:** this is *not* viscosity approx. to PDE in conjecture
SUPPORTING THE CONJECTURES:
Rigorous results
First Rigorous Result: Second moments of char. poly.

- Fix $X_0$ and $s$, take $\tau_0$ and $\tau$
- Set $p_0^N = \text{char. poly. of } X_0 + Z_{s,\tau_0}$
- Set $p^N = \text{char. poly. of } X_0 + Z_{s,\tau}$
- Set
  
  $$q^N(z) = \exp \left\{ \frac{\tau - \tau_0}{2N} \frac{\partial^2}{\partial z^2} \right\} p_0^N(z)$$

- **Goal**: Show $p^N$ and $q^N$ have similar bulk distribution of zeros
Theorem (Hall–Ho)

For all \( z \in \mathbb{C} \), we have

\[
\mathbb{E} \left\{ |q^N(z)|^2 \right\} = \mathbb{E} \left\{ |p^N(z)|^2 \right\}
\]

Proof.

Both sides satisfy the PDE

\[
\frac{\partial u}{\partial \tau} = \frac{1}{2N} \frac{\partial^2 u}{\partial z^2}
\]

with equality at \( \tau = \tau_0 \).
Significance of the second moment

- If $p$ is a degree-$N$ polynomial,

  \[
  \text{empirical measure of zeros of } p = \frac{1}{4\pi N} \Delta \log(|p(z)|^2)
  \]

- Assume concentration—$|p^N(z)|^2 \approx \mathbb{E}\{|p^N(z)|^2\}$

- Then can freely insert a expectation:

  \[
  \text{empirical measure of zeros of } p^N \approx \frac{1}{4\pi N} \Delta \log(\mathbb{E}\{|p^N(z)|^2\})
  \]

- Conclusion: Hope to recover zeros of $p^N$ and $q^N$ from second moments—which are equal!
Second rigorous result: Polynomials with independent coefficients

- Work in progress with Ho, Jalowy, and Kabluchko
- Kabluchko and Zaporozhets have analyzed wide class of polynomials with independent coefficients
- Apply (backward) heat operator for time $t$
- First: extend KZ results by applying heat flow to the polynomials
- Second: Verify that results agree with general heat flow conjecture
Second rigorous result: Polynomials with independent coefficients

**Theorem (Hall–Ho–Jalowy–Kabluchko, 2023+)**

1. Log potential $S$ of limiting zero-distribution of heat-evolved KZ polynomials satisfies the PDE

$$\frac{\partial S}{\partial \tau} = \frac{1}{2} \left( \frac{\partial S}{\partial z} \right)^2$$

2. The limiting zero-distribution at time $\tau$ is push-forward of distribution at time 0 under an explicit transport map $T_t$
Consider Weyl polynomials $W_N$:

$$W_N(z) = \sum_{j=0}^{N} \xi_j \left(\sqrt{N}z\right)^j \sqrt{j!},$$

where $\{\xi_j\}$ are i.i.d. standard complex Gaussians.

Limiting distribution of zeros is uniform on unit disk (circular law).

Transport map is

$$T_t(z) = z + t\bar{z}$$

Limiting distribution of zeros of heat-evolved poly. is uniform on ellipse for $0 < t < 1$.

Limiting distribution is semicircular on $[-2, 2]$ for $t = 1$. 
Littlewood–Offord polynomials

- Start from Littlewood–Offord poly. with $\beta = 1/4$
- Distrib. at $t = 0$ is quadratic on unit disk
- Push forward by explicit map $T_t$ of disk to ellipse
Third rigorous result: Gaussian Analytic Function (GAF)

- GAF is “infinite Weyl polynomial” (without factor of $\sqrt{N}$)

**Definition**

The GAF is the random entire function given by

$$\sum_{j=0}^{\infty} \xi_j \frac{z^j}{\sqrt{j!}}$$

where $\{\xi_n\}$ are i.i.d. standard complex Gaussians.
Zeros of GAF

- Zeros of GAF form an interesting random set of points in the plane
- Zeros are invariant (in distrib.) under rotations and translations
GAF under heat flow

- Makes sense to apply $e^{\frac{\tau}{2} \frac{d^2}{dz^2}}$ to $G$, if $|\tau| < 1$

**Theorem (Hall–Ho–Jalowy–Kabluchko, 2023+)**

For all $\tau \in \mathbb{C}$ with $|\tau| < 1$ the function

$$(V_\tau G)(z) := (1 - |\tau|^2)^{1/4} e^{-\frac{\tau}{2} z^2} \left( e^{\frac{\tau}{2} \frac{d^2}{dz^2}} G \right) \left( \sqrt{1 - |\tau|^2} \ z \right)$$

has the same distribution as $G$.

- Hence: GAF remains invariant in distribution under heat flow, up to some simple transformations
Zeros of GAF under heat flow

- Constant and Gaussian factor don’t affect zeros

**Corollary**

*If* $z_j(\tau)$ *are zeros of* $e^{\tau \frac{d^2}{dz^2}} G$, *then*

$$\left\{ \frac{z_j(\tau)}{\sqrt{1 - |\tau|^2}} \right\}$$

*have same distribution as zeros of* $G$ *(for* $|\tau| < 1$).*

- So we understand how zeros of an “infinite-degree random polynomial” transform under heat flow!
- Exact result at level of individual zeros (not just bulk level)
Zeros $z_j(\tau)$ tend to move along straight lines: $z \mapsto z - \tau \bar{z}$
GAF: dynamics of individual zeros

- Let $G^a$ denote GAF $G$ conditioned to have a zero at $a \in \mathbb{C}$
- Let $z^a(\tau)$ denote the zero of $e^{\frac{\tau}{2} \frac{d^2}{dz^2}} G^a$ that starts at $a$

**Theorem (Hall–Ho–Jalowy–Kabluchko, 2023+)**

*We have the following equality in distribution:*

$$z^a(\tau) \overset{d}{=} a - \tau \bar{a} + z^0(\tau)$$

- $z^0(\tau)$ is a fixed random variable with distrib. indep. of $a$
- Result says that zero evolves in straight line, plus “order 1” error
GAF: dynamics of individual zeros

- Plots of zeros with straight-line motion subtracted off
- I.e., plot \( z_j(\tau) - (z_j(0) - \tau z_j(0)) \)
- All points then move with same scale
THANK YOU FOR YOUR ATTENTION
Driver–Hall–Kemp, The Brown measure of the free multiplicative Brownian motion, PTRF 2022

Ho–Zhong, Brown measures of free circular and multiplicative Brownian motions with self-adjoint and unitary initial conditions, JEMS 2023

Hall–Ho, The Brown measure of a family of free multiplicative Brownian motions, PTRF 2023

Zhong, Brown measure of the sum of an elliptic operator and a free random variable in a finite von Neumann algebra, arXiv:2108.09844


Hall–Ho–Jalowy–Kabluchko, two papers in preparation