Asymptotic expansions in Random Matrix Theory and application: the case of Haar unitary matrices

Félix Parraud

KTH Royal Institute of Technology

April 10, 2023

Based on the following work:

Asymptotic expansion of smooth functions in deterministic and iid Haar unitary matrices, and application to tensor products of matrices
Problem

Given a family $X^N = (X_1^N, \ldots, X_d^N)$ of self-adjoint random matrices, $P$ a noncommutative polynomial, how does the operator norm of $P(X^N)$ behaves asymptotically? i.e. can we compute $\lim_{N \to \infty} \|P(X^N)\|$?
Problem

Given a family $X^N = (X_1^N, \ldots, X_d^N)$ of self-adjoint random matrices, $P$ a noncommutative polynomial, how does the operator norm of $P(X^N)$ behaves asymptotically? I.e. can we compute $\lim_{N \to \infty} \|P(X^N)\|$?

A necessary assumption

There exists a family $x = (x_1, \ldots, x_d)$ of self-adjoint elements of a $C^*$-algebra $A$ endowed with a faithful trace $\tau$ such that almost surely, the family $X^N$ converges in distribution towards $x$. That is for any noncommutative polynomial $Q$,

$$\lim_{N \to \infty} \frac{1}{N} \text{Tr} \left( Q(X^N) \right) = \tau(Q(x)).$$
Original motivation

Problem

Given a family $X^N = (X_1^N, \ldots, X_d^N)$ of self-adjoint random matrices, $P$ a noncommutative polynomial, how does the operator norm of $P(X^N)$ behaves asymptotically? I.e. can we compute $\lim_{N \to \infty} \|P(X^N)\|$?

A necessary assumption

There exists a family $x = (x_1, \ldots, x_d)$ of self-adjoint elements of a $C^*$-algebra $A$ endowed with a faithful trace $\tau$ such that almost surely, the family $X^N$ converges in distribution towards $x$. That is for any noncommutative polynomial $Q$,

$$\lim_{N \to \infty} \frac{1}{N} \text{Tr} \left( Q(X^N) \right) = \tau(Q(x)).$$

Problem

Given a family $X^N = (X_1^N, \ldots, X_d^N)$ of random matrices, $P$ a noncommutative polynomial, can we prove that almost surely:

$$\lim_{N \to \infty} \|P(X^N)\| = \|P(x)\|?$$
We know that for any $k \in \mathbb{N}$,

$$
\left\| P \left( X^N \right) \right\| \geq \left( \frac{1}{N} \operatorname{Tr} \left( \left( P \left( X^N \right)^* P \left( X^N \right) \right)^k \right) \right)^{1/2k}.
$$

Consequently, thanks to the convergence in distribution,

$$
\liminf_{N \to \infty} \left\| P(X^N) \right\| \geq \tau \left( (P(x)^* P(x))^k \right)^{1/2k}.
$$

And since it is known that $\lim_{k \to \infty} \tau \left( (P(x)^* P(x))^k \right)^{1/2k} = \| P(x) \|$, we have

$$
\liminf_{N \to \infty} \left\| P(X^N) \right\| \geq \| P(x) \|.
$$
An upper bound on the limit

One has for any $k \in \mathbb{N}$ that:

$$\left\| P(X^N) \right\|^{2k} = \left\| \left( P(X^N)^* P(X^N) \right)^k \right\| \leq \text{Tr} \left( \left( P(X^N)^* P(X^N) \right)^k \right).$$

Thus heuristically,

$$\left\| P(X^N) \right\| \leq N^{1/2k} \left( \tau \left( (P(x)^* P(x))^k \right) \right)^{1/2k} \leq N^{1/2k} \left\| P(x) \right\|.$$

Thus one would like to take $k \gg \ln(N)$. However doing so make it way more complicated to control the error term.
A few notions of Free probability

**Definition**

Let $A = (a_1, \ldots, a_k)$ be a $k$-tuple of elements of a $C^*$-algebra. The joint $*$-distribution of the family $A$ is the linear form

$$\mu_A : P \mapsto \tau[P(A, A^*)]$$

on the set of polynomials in $2k$ noncommutative variables.

**Definition**

By convergence in distribution, for a sequence of families of variables $(A_N)_{N \geq 1} = (a_{1N}, \ldots, a_{kN})_{N \geq 1}$ in $C^*$-algebras $(\mathcal{A}_N, *, \tau_N, \|\cdot\|)$, we mean the pointwise convergence of the map

$$\mu_{A_N} : P \mapsto \tau_N[P(A_N, A_N^*)].$$
We say that $U^N$ is a Haar unitary random matrix of size $N$ if its law is the Haar measure on the group of unitary matrices of size $N$. 

Theorem (D. Voiculescu, 1991)

Let $U^N = (U^N_1, \ldots, U^N_d)$ be independent Haar unitary matrices, $u = (u_1, \ldots, u_d)$ a $d$-tuple of free Haar unitaries. Then almost surely $U^N$ converges in distribution towards $u$. That is almost surely for any noncommutative polynomial $P$, 

$$\lim_{N \to \infty} \frac{1}{N} \text{Tr} P(U^N, (U^N)^*) = \tau P(u, u^*)$$
Definition

We say that $U^N$ is a Haar unitary random matrix of size $N$ if its law is the Haar measure on the group of unitary matrices of size $N$.

Theorem (D. Voiculescu, 1991)

Let $U^N = (U_1^N, \ldots, U_d^N)$ be independent Haar unitary matrices, $u = (u_1, \ldots, u_d)$ a $d$-tuple of free Haar unitaries. Then almost surely $U^N$ converges in distribution towards $u$. That is almost surely for any noncommutative polynomial $P$,

$$\lim_{N \to \infty} \frac{1}{N} \text{Tr} \left( P(U^N, (U^N)^*) \right) = \tau\left( P(u, u^*) \right).$$
Let \((X_t)_{t \geq 0}\) be a Markov process associated with infinitesimal generator \(\mathcal{L}\), then it is known that:

- For \(f\) in the domain of \(\mathcal{L}\), \(\frac{d}{dt} \mathbb{E}[f(X_t)] = \mathbb{E}[\mathcal{L}f(X_t)]\).
- If \(Z\) is the invariant law of this Markov process, then \(X_t\) converges in law towards \(Z\).
Let \((X_t)_{t \geq 0}\) be a Markov process associated with infinitesimal generator \(L\), then it is known that:

- For \(f\) in the domain of \(L\), \(\frac{d}{dt} \mathbb{E}[f(X_t)] = \mathbb{E}[Lf(X_t)]\).
- If \(Z\) is the invariant law of this Markov process, then \(X_t\) converges in law towards \(Z\).

Thus,

\[
\mathbb{E}[f(Z)] - \mathbb{E}[f(X_0)] = \lim_{t \to \infty} \mathbb{E}[f(X_t)] - \mathbb{E}[f(X_0)] = \int_0^\infty \mathbb{E}[Lf(X_t)] \, dt.
\]

In particular if \(X_0 \sim Z\), then \(\mathbb{E}[f(X_t)]\) is constant and thus for any \(t\), \(\mathbb{E}[Lf(X_t)] = 0\).
Let \((x_t)_{t \geq 0}\) be a free Markov process associated with infinitesimal generator \(\mathcal{L}\), then it is known that:

- For any polynomial \(P\), \(\frac{d}{dt} \tau[P(x_t)] = \tau[\mathcal{L}P(x_t)]\).
- If \(z\) is the invariant distribution of this free Markov process, then \(x_t\) converges in distribution towards \(z\).
Let \((x_t)_{t \geq 0}\) be a free Markov process associated with infinitesimal generator \(\mathcal{L}\), then it is known that:

- For any polynomial \(P\), \(\frac{d}{dt} \tau[P(x_t)] = \tau[\mathcal{L}P(x_t)]\).
- If \(z\) is the invariant distribution of this free Markov process, then \(x_t\) converges in distribution towards \(z\).

Thus,

\[
\tau[P(x)] - \tau[P(x_0)] = \lim_{t \to \infty} \tau[P(x_t)] - \tau[P(x_0)] = \int_0^\infty \tau[\mathcal{L}P(x_t)] \, dt.
\]

In particular if \(x_0 \sim z\), then \(\tau[P(x_t)]\) is constant and thus for any \(t\), \(\tau[\mathcal{L}P(x_t)] = 0\).
Let us assume that there exists a free Markov process whose invariant distribution is the one of $u$. We set

- $(u^N_t)_{t \geq 0}$ such a Markov process started in $U^N$,
- $(u_t)_{t \geq 0}$ such a Markov process started in $u$.

Then after showing that this is well-defined, for any polynomial $Q$,

$$
\tau[Q(u)] - \mathbb{E}\left[\frac{1}{N} \text{Tr} \left[ Q(U^N) \right] \right] = \int_0^\infty \mathbb{E}\left[ \tau_N \left[ \mathcal{L}Q(u^N_t) \right] \right] dt.
$$
Let us assume that there exists a free Markov process whose invariant distribution is the one of \( u \). We set

- \((u^N_t)_{t \geq 0}\) such a Markov process started in \( U^N \),
- \((u_t)_{t \geq 0}\) such a Markov process started in \( u \).

Then after showing that this is well-defined, for any polynomial \( Q \),

\[
\tau[Q(u)] - \mathbb{E} \left[ \frac{1}{N} \text{Tr} \left[ Q(U^N) \right] \right] = \int_0^\infty \mathbb{E} \left[ \tau_N \left[ \mathcal{L} Q(u^N_t) \right] \right] \, dt.
\]

Heuristically since we know that \( U^N \) converges in distribution towards \( u \), then

\[
\lim_{N \to \infty} \mathbb{E} \left[ \tau_N \left[ \mathcal{L} Q(u^N_t) \right] \right] = \tau \left[ \mathcal{L} Q(u_t) \right] = 0.
\]
Theorem (P., 2023)

Let the following objects be given,
- \( U^N = (U_1^N, \ldots, U_d^N) \) independent Haar unitary matrices in \( \mathbb{M}_N(\mathbb{C}) \),
- \( P \) a self-adjoint polynomial,
- \( f \in C^{4k+7}(\mathbb{R}) \).

Then there exist deterministic constants \( (\alpha_i^P(f))_{i \in \mathbb{N}} \) such that,

\[
\mathbb{E} \left[ \frac{1}{N} \operatorname{Tr}_N \left( f(P(U^N)) \right) \right] = \sum_{0 \leq i \leq k} \frac{1}{N^{2i}} \alpha_i^P(f) + \mathcal{O}(N^{-2(k+1)}).
\]

Besides, if the support of \( f \) and the spectrum of \( P(u) \) are disjoint, then for any \( i \), \( \alpha_i^P(f) = 0 \).
Idea of the proof

We want to show the following formula:

$$\mathbb{E} \left[ \frac{1}{N} \text{Tr}_N \left( f(P(U^N)) \right) \right] = \sum_{0 \leq i \leq k} \frac{1}{N^{2i}} \alpha_i^p(f) + \mathcal{O}(N^{-2(k+1)}).$$
Idea of the proof

We want to show the following formula:

\[ \mathbb{E} \left[ \frac{1}{N} \operatorname{Tr}_N \left( f(P(U^N)) \right) \right] = \sum_{0 \leq i \leq k} \frac{1}{N^{2i}} \alpha_i^P(f) + O(N^{-2(k+1)}). \]

- We set \( v_s \) a family of free unitary Browian motions started in 1,

\[ \mathbb{E} \left[ \tau_N \left( Q(U^N, v_s) \right) \right] = \tau \left( Q(u, v_s) \right) - \int_0^\infty \mathbb{E} \left[ \tau_N \left( \mathcal{L}Q(u^N_t, v_s) \right) \right] dt. \]
Idea of the proof

We want to show the following formula:

$$\mathbb{E} \left[ \frac{1}{N} \text{Tr}_N \left( f(P(U^N)) \right) \right] = \sum_{0 \leq i \leq k} \frac{1}{N^{2i}} \alpha_i^P(f) + O(N^{-2(k+1)}).$$

- We set $v_s$ a family of free unitary Brownian motions started in 1,

$$\mathbb{E} \left[ \tau_N \left( Q(U^N, v_s) \right) \right] = \tau \left( Q(u, v_s) \right) - \int_0^\infty \mathbb{E} \left[ \tau_N \left( \mathcal{L}Q(u^N_t, v_s) \right) \right] dt.$$

- Then we show that there is a deterministic operator $T_{t,s}$ on the space of polynomials such that

$$\mathbb{E} \left[ \tau_N \left( \mathcal{L}Q(u^N_t, v_s) \right) \right] = \frac{1}{N^2} \mathbb{E} \left[ \tau_N \left( T_{t,s}(Q)(u^N_t, v_s) \right) \right].$$

- We can view the polynomial $T_{t,s}(Q)$ as a polynomial in $(U^N, u^N_t, v_s)$ and reiterate the process.
Corollary

We define,
- \( U^N = (U^N_1, \ldots, U^N_d) \) independent Haar unitary matrices of size \( N \),
- \( u = (u_1, \ldots, u_d) \) free Haar unitaries,
- \( P = \sum_i Q_i(U, U^*) \otimes Y^M_i \) with \( Q_i \) a non-commutative polynomial with \( Y^M_i \in M_M(\mathbb{C}) \).

If we assume that the family \( Y^M \) are uniformly bounded over \( M \) for the operator norm, then for any \( \delta > 0 \),

\[
P \left( \left\| P \left( U^N \right) \right\| \geq \left\| P\left(u\right)\right\| + \delta + \mathcal{O} \left( \left( \frac{M}{N} \right)^{1/2} \ln(NM)^{5/4} \right) \right) \leq e^{-\delta^2 N}.
\]

Thus, if \( M \ll N / \ln^{5/2}(N) \) and that a family \( Z^M \) converges strongly in distribution towards a family of non-commutative variable \( z \), then the family \( (U^N, \otimes I_M, I_N \otimes Z^M) \) also converges strongly towards \( (u \otimes 1, 1 \otimes z) \).
Idea of the proof: the moment method

- Given $n \in \mathbb{N}$, $Q = P^*P$ a self-adjoint polynomial, one has proved that for some operator $\Delta,$

$$
\mathbb{E} \left[ \text{Tr} \left( Q^n \left( U^N \right) \right) \right] = NM \, \tau(P(u)) + \frac{1}{N^2} \mathbb{E} \left[ \text{Tr} \left( (\Delta Q^n) \left( U^N \right) \right) \right].
$$
Idea of the proof: the moment method

- Given \( n \in \mathbb{N} \), \( Q = P^*P \) a self-adjoint polynomial, one has proved that for some operator \( \Delta \),

\[
\mathbb{E} \left[ \text{Tr} \left( Q^n \left( U^N \right) \right) \right] = NM \tau(P(u)) + \frac{1}{N^2} \mathbb{E} \left[ \text{Tr} \left( (\Delta Q^n) \left( U^N \right) \right) \right].
\]

- One can write

\[
\Delta Q^n(U^N) = \sum_{i+j+k+l=n-4} \alpha \left( Q^i(U^N), Q^j(U^N), Q^k(U^N), Q^l(U^N) \right),
\]

where

\[
\alpha(A_1 \otimes B_1, A_2 \otimes B_2, A_3 \otimes B_3, A_4 \otimes B_4) = A_2 A_1 A_4 A_3 \otimes B_1 B_2 B_3 B_4.
\]
Idea of the proof: the moment method

- Given $n \in \mathbb{N}$, $Q = P^* P$ a self-adjoint polynomial, one has proved that for some operator $\Delta$,

$$
\mathbb{E} \left[ \text{Tr} \left( Q^n \left( U^N \right) \right) \right] = NM \, \tau(P(u)) + \frac{1}{N^2} \mathbb{E} \left[ \text{Tr} \left( (\Delta Q^n) \left( U^N \right) \right) \right].
$$

- One can write

$$
\Delta Q^n(U^N) = \sum_{i+j+k+l=n-4} \alpha \left( Q^i \left( U^N \right), Q^j \left( U^N \right), Q^k \left( U^N \right), Q^l \left( U^N \right) \right),
$$

where

$$
\alpha(A_1 \otimes B_1, A_2 \otimes B_2, A_3 \otimes B_3, A_4 \otimes B_4) = A_2 A_1 A_4 A_3 \otimes B_1 B_2 B_3 B_4.
$$

- However

$$
\text{Tr}(B_1 B_2 B_3 B_4) = M^2 \mathbb{E} \left[ \text{Tr}(B_2 VB_1 WB_4 V^* B_3 W^*) \right].
$$
Idea of the proof: the moment method

- Given \( n \in \mathbb{N} \), \( Q = P^* P \) a self-adjoint polynomial, one has proved that for some operator \( \Delta \),
  \[
  E \left[ \text{Tr} \left( Q^n \left( U^N \right) \right) \right] = NM \, \tau(P(u)) + \frac{1}{N^2} E \left[ \text{Tr} \left( (\Delta Q^n) \left( U^N \right) \right) \right].
  \]

- One can write
  \[
  \Delta Q^n(U^N) = \sum_{i+j+k+l=n-4} \alpha \left( Q^i \left( U^N \right), Q^j \left( U^N \right), Q^k \left( U^N \right), Q^l \left( U^N \right) \right),
  \]
  where
  \[
  \alpha(A_1 \otimes B_1, A_2 \otimes B_2, A_3 \otimes B_3, A_4 \otimes B_4) = A_2 A_1 A_4 A_3 \otimes B_1 B_2 B_3 B_4.
  \]

- However
  \[
  \text{Tr}(B_1 B_2 B_3 B_4) = M^2 E \left[ \text{Tr}(B_2 V B_1 W B_4 V^* B_3 W^*) \right].
  \]

- Thus
  \[
  E \left[ \text{Tr} \left( Q^n \left( U^N \right) \right) \right] \leq NM \|P(u)\|^n + \frac{Cn^4 M^2}{N^2} E \left[ \text{Tr} \left( Q^{n-4} \left( U^N \right) \right) \right]
  \]
  \[
  \leq NM \|P(u)\|^n \times \frac{1}{1 - \frac{C' n^4 M^2}{N^2}}.
  \]
Idea of the proof: the moment method

- Given $n \in \mathbb{N}$, $Q = P^* P$ a self-adjoint polynomial, one has proved that for some operator $\Delta$,
  \[ \mathbb{E} \left[ \text{Tr} \left( Q^n \left( U^N \right) \right) \right] = NM \tau(P(u)) + \frac{1}{N^2} \mathbb{E} \left[ \text{Tr} \left( \Delta Q^n \left( U^N \right) \right) \right]. \]

- One can write
  \[ \Delta Q^n(U^N) = \sum_{i+j+k+l=n-4} \alpha \left( Q^i \left( U^N \right), Q^j \left( U^N \right), Q^k \left( U^N \right), Q^l \left( U^N \right) \right), \]
  where
  \[ \alpha(A_1 \otimes B_1, A_2 \otimes B_2, A_3 \otimes B_3, A_4 \otimes B_4) = A_2 A_1 A_4 A_3 \otimes B_1 B_2 B_3 B_4. \]

- However
  \[ \text{Tr}(B_1 B_2 B_3 B_4) = M^2 \mathbb{E} \left[ \text{Tr}(B_2 V B_1 W B_4 V^* B_3 W^*) \right]. \]

- Thus
  \[ \mathbb{E} \left[ \text{Tr} \left( Q^n \left( U^N \right) \right) \right] \leq NM \|P(u)\|^n + \frac{Cn^4 M^2}{N^2} \mathbb{E} \left[ \text{Tr} \left( Q^{n-4} \left( U^N \right) \right) \right] \]
  \[ \leq NM \|P(u)\|^n \times \frac{1}{1 - \frac{C'n^4 M^2}{N^2}}. \]

- Finally
  \[ \mathbb{E} \left[ \| Q \left( U^N \right) \| \right] \leq \mathbb{E} \left[ \text{Tr} \left( Q^n \left( U^N \right) \right) \right]^{1/n} \leq \|P(u)\| (1 + o(1)). \]
Problem

How do you compute the following quantity:

\[ X = \int_{U_N} U_{i_1,j_1} \cdots U_{i_d,j_d} U'_{i'_1,j'_1} \cdots U'_{i'_d,j'_d} dU, \]

where the integral is with respect to the Haar measure.
How do you compute the following quantity:

\[ X = \int_{\mathbb{U}_N} U_{i_1, j_1} \cdots U_{i_d, j_d} \overline{U}_{i_1', j_1'} \cdots \overline{U}_{i_d', j_d'} \, dU, \]

where the integral is with respect to the Haar measure.

Given a Haar unitary matrix \( U \) of size \( N \), one has

\[ X = \mathbb{E} \left[ \text{Tr}_N \left( UE_{j_1, i_2} U \cdots E_{j_{d-1}, i_d} U \overline{E}_{j_1', i_2'} U^* \overline{E}_{j_d'}, i_1 \right) \right]. \]

Thus one wants to compute

\[ N \mathbb{E} \left[ \frac{1}{N} \text{Tr} \left( P(U, U^*, Z^N) \right) \right], \]

with \( Z^N = (E_{i,j})_{i,j \in [1,N]} \).
However with $\tau_N = N^{-1} \text{Tr}$, in the process of proving the asymptotic expansion, we have proved that there exist an operator $\Delta$ such that

$$
E \left[ \tau_N \left( P \left( U^N, Z^N \right) \right) \right] = \tau(P(u, Z^N)) + \frac{1}{N^2} E \left[ \tau_N \left( (\Delta P) \left( U^N, Z^N \right) \right) \right],
$$

that is,

$$
E \left[ \tau_N \left( \left( \text{id} - \frac{1}{N^2} \Delta \right)(P) \left( U^N, Z^N \right) \right) \right] = \tau \left( P \left( u, Z^N \right) \right).
$$

Consequently,

$$
E \left[ \tau_N \left( P \left( U^N, Z^N \right) \right) \right] = \tau \left( \left( \text{id} - \frac{1}{N^2} \Delta \right)^{-1} (P) \left( u, Z^N \right) \right).
$$