Rectangular finite free probability theory

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Contents

Contents

Free probability summary

Theorem (Voiculescu)

For A_d and B_d $d \times d$ symmetric matrices whose eigenvalue distributions converge to μ_a and μ_b,and Q_d a random orthogonal matrix, then the eigenvalue distribution of $A_d + Q_d^T B_d Q_d$ is converging to $\mu_a \boxplus \mu_b$, the free sum measure.

Theorem (Voiculescu)

Define the Cauchy and R-transform of a Borel measure μ on $\mathbb{R}$ as

$$\mathcal{G}_\mu(x) = \int_{t\in\mathbb{R}} \frac{d\mu(t)}{x-t}, \quad \text{for } Im(x) > 0$$

$$\mathcal{R}_\mu(x) = \mathcal{G}_\mu^{-1}(x) - \frac{1}{x} = \mathcal{G}_\mu^{-1}(x) - \mathcal{G}_{\mu_0}^{-1}(x)$$

$$\mathcal{R}_{\mu_a \boxplus \mu_b}(x) = \mathcal{R}_{\mu_a}(x) + \mathcal{R}_{\mu_b}(x)$$

Finite free sum

Definition (following Marcus,Spielman,Srivastava)

For A and B $d \times d$ hermitian matrices, we define the additive convolution as

$$\chi_A \boxplus_d \chi_B := \mathbb{E}_{Q \in \mathcal{O}_d}[\chi_{A+Q^T BQ}]$$

Theorem (MSS)

The additive convolution of two hermitian matrices is real-rooted.If $p(x) = \sum_{i=0}^{d} a_i x^{d-i}$ *and* $q(x) = \sum_{i=0}^{d} b_i x^{d-i}$*:*

$$p \boxplus_d q = \frac{1}{d!} \sum_{k=0}^{d} D^k p(x) D^{d-k} p(0)$$

$$= \sum_{k=0}^{d} x^{d-k} \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} a_i b_j$$

Finite free linearization

To p of degree d, we associate $\mu_p := \frac{1}{d}\sum_{i=1}^{d}\delta_{\lambda_i(p)}$, and $\mathcal{R}_p := \mathcal{R}_{\mu_p}$

Theorem (from MSS)

For all $w > 0$ and real-rooted polynomials p and q of degree at most d,

$$\mathcal{R}_{p \boxplus_d q}(w) \leq \mathcal{R}_p(w) + \mathcal{R}_q(w)$$

with equality only when p or q has only one root up to multiplicity.

Theorem (from Marcus)

There is a polynomial of degree $d-1$ $\mathcal{R}_p^d$ whose coefficients, finite free cumulants, are functions of the coefficients of p such that (with μ_p fix)

$$\mathcal{R}_p^d(s) \to_{d\to\infty} \mathcal{R}_p(s)$$

$$\mathcal{R}_{p \boxplus_d q}^d(s) = \mathcal{R}_p^d(s) + \mathcal{R}_q^d(s)$$

Polynomials as independent random variables

- $\mathbb{E}[p \boxplus_d q] = \mathbb{E}[p] + \mathbb{E}[q]$
- $Var[p \boxplus_d q] = Var[p] + Var[q]$
- $p = (x-\mu)^d$ constant polynomial in dimension d
- $p = (x-\mu)^d \iff \mathrm{R}_p^d(s) = \mu$
- $p(x) = H_d((x-\mu)\sqrt{d-1}/\sigma) \iff \mathcal{R}_p^d(s) = \mu + s\sigma^2$ where H_d are the Hermite family = finite free Gaussians.

Asymptotic distributions

Proposition

(Law of large numbers)(Marcus) Let $p_1, p_2, \ldots$ *be a sequence of degree* d *real-rooted polynomials whose roots have fixed mean* μ*, and uniformly bounded variance. Write* $R_{1/N}(p)(x) := N^{-d}p(Nx)$*. Then,*

$$\lim_{N\to\infty} R_{1/N}([p_1 \boxplus_d p_2 \ldots \boxplus_d p_N])(x) = (x-\mu)^d$$

Proposition

(Central limit theorem)(Marcus) Consider as above $p_i(x) = \prod_j (x - r_{i,j})$ *such that* $\sum_j r_{i,j} = 0$, $\frac{1}{d}\sum_j r_{i,j}^2 = \sigma^2$*. Then*

$$\lim_{N\to\infty} R_{1/\sqrt{N}}([p_1 \boxplus_d p_2 \ldots \boxplus_d p_N])(x) \approx H_d\left((x-\mu)\sqrt{\frac{d-1}{\sigma^2}}\right)$$

Contents

Rectangular free probability

Theorem (Voiculescu and Benaych Georges)

Let, for all $d \geq 1$, Q_d and R_d be orthogonal Haar random $(q_1(d) \times q_1(d)$ and $q_2(d) \times q_2(d))$,A_d and B_d be independent rectangular $q_1(d) \times q_2(d)$ random matrices with $q_1(d) \geq q_2(d)$, and such that the symmetrizations of the singular law of A_d and B_d converge in probability to μ_a and μ_b respectively. Then the symmetrization of the singular law of $A_d + Q_d^T B_d R_d$ converges in probability to a symmetric probability measure on the real line, denoted by $\mu_a \boxplus^\lambda \mu_b$, which depends only on μ_a, μ_b, and $\lambda := \lim_{n\to\infty} q_2(d)/q_1(d)$. Notice that $\lambda \in [0, 1]$.

It gives a universal behavior for singular values of sums of large random rectangular matrices.

Adapted rectangular tools

Definition (from BG)

The λ-rectangular Cauchy transform for a symmetric compact measure μ (and x in a positive neighborhood of 0) is given by

$$H_\mu^\lambda(x) = \lambda\left[\mathcal{G}_\mu\left(\frac{1}{\sqrt{x}}\right)\right]^2 + (1-\lambda)\sqrt{x}\mathcal{G}_\mu\left(\frac{1}{\sqrt{x}}\right)$$

Definition (from BG)

For x small enough, let

$U^\lambda(x) := \frac{-\lambda-1+\left[(\lambda+1)^2+4\lambda x\right]^{1/2}}{2\lambda}$.The rectangular R-transform is given by

$$\mathcal{R}_\mu^\lambda(x) := U^\lambda\left(\frac{x}{[H_\mu^\lambda]^{-1}(x)} - 1\right)$$

Linearization property

Theorem (from BG)

The rectangular R-transform linearizes the rectangular additive convolution for symmetric measures μ_1 and μ_2:

$$\mathcal{R}^{\lambda}_{\mu_1 \boxplus_{\lambda} \mu_2}(x) = \mathcal{R}^{\lambda}_{\mu_1}(x) + \mathcal{R}^{\lambda}_{\mu_2}(x)$$

Can we define polynomial tools dealing with singular values of rectangular matrices by analogy?

Contents

From eigenvalues to singular values

Definition (rectangular singular free sum)

For $m \times d$ rectangular matrices A and B, $\lambda = d/m$, define

$$\chi_{A^TA} \boxplus_{d,\lambda} \chi_{B^TB} := \mathbb{E}_{R \in \mathcal{O}_m, Q \in \mathcal{O}_d} \left\{ \chi_{(A+QBR^T)(A+QBR^T)^T} \right\}$$
$$= \iint_{\mathcal{O}_m \times \mathcal{O}_d} \det \left[xI - (A + QBR^T)^T (A + QBR^T) \right] dRdQ$$

where the measures are Haar on the respective orthogonal groups.

Remark

Free probability: the symmetrization of the singular distribution of $A + QBR^T$ (roots of $\chi_{(A+QBR^T)(A+QBR^T)^T}$) is close to the Benaych-Georges' rectangular free sum $\mu_A \boxplus_{\frac{d}{m}} \mu_B$ when d, m are large

Polynomial expansion

Theorem (Algebraic form)

Consider two polynomials p and q with only real nonnegative roots (they can be written as $\chi_{A^T A}$ and $\chi_{B^T B}$ for some $m \times d$ matrices A and B). If we write $p(x) = \sum_{i=0}^{d} a_i x^{d-i}$ and $q(x) = \sum_{i=0}^{d} b_i x^{d-i}$ the following holds

$$p \boxplus_{d,\lambda} q = \sum_{k=0}^{d} x^{d-k} \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} \frac{(m-i)!(m-j)!}{m!(m-k)!} a_i b_j$$

Remark

This shows the bilinearity of the operation $\boxplus_{d,\lambda}$. We can extend the definition to polynomials of degree at most d through this formula.

Derivative form

Consider again polynomials p and q with nonnegative real roots.

Lemma (Derivative sum)

If we write $p(x,y) = y^{m-d}p(xy)$ *and* $q(x,y) = y^{m-d}q(xy)$, $\Delta_\lambda(p) := x\delta_x^2 + (m-d+1)\delta_x$, *then*

$$[p \boxplus_{d,\lambda} q](x) = \frac{(m-d)!}{d!m!}\sum_{k=0}^{d}[(\partial_x\partial_y)^{d-k}p](x,1)[(\partial_x\partial_y)^k q](0,1)$$

$$[p \boxplus_{d,\lambda} q](x) = \frac{(m-d)!}{d!m!}\sum_{k=0}^{d}\Delta_\lambda^k p(x)[\Delta_\lambda^{d-k}q(x)]|_{x=0}$$

Contents

R-transform inequality

Consider p and q polynomials of degree at most d with nonnegative roots, and $\mathbb{S}p(x) := p(x^2)$

Theorem (Marcus,G)

For $s > 0$,

$$\mathcal{R}^{\lambda}_{\mathbb{S}[p \boxplus_{d,\lambda} q]}(s) \leq \mathcal{R}^{\lambda}_{\mathbb{S}p}(s) + \mathcal{R}^{\lambda}_{\mathbb{S}q}(s)$$

with equality only when $p = x^d$ or $q = x^d$.

Remark

$$\mathcal{R}^{\lambda}_{\mu_{\mathbb{S}p} \boxplus_{\lambda} \mu_{\mathbb{S}q}}(s) = \mathcal{R}^{\lambda}_{\mathbb{S}p}(s) + \mathcal{R}^{\lambda}_{\mathbb{S}q}(s)$$

Polynomial version

Consider $V^n p(x) = x^n p(x)$.

Lemma

The following differential operator is real rooted:

$$W_\alpha^{m-d} p = [\mathbb{S}p][\mathbb{S}V^{m-d}p] - \alpha^2 [\mathbb{S}p]'[\mathbb{S}V^{m-d}p]' \qquad (1)$$

Theorem (Polynomial form of the inequality)

$$\Theta_\alpha^{m-d}(p \boxplus_{d,\lambda} q) \leq \Theta_\alpha^{m-d}(p) + \Theta_\alpha^{m-d}(q) - (m+d)\alpha$$

for all real numbers $\alpha > 0$, where

$$\Theta_\alpha^{m-d}(p) := \sqrt{(m-d)^2\alpha^2 + [\mathrm{maxroot}\left\{W_\alpha^{m-d} p\right\}]^2}.$$

Contents

Gegenbauer polynomials and convolution

Consider the Gegenbauer polynomials, $C_d^{(\alpha)}(x)$, the collection of polynomials orthogonal with respect to $w(x) = (1 - x^2)^{\alpha - 1/2}$ on the interval $[-1, 1]$. For all $\lambda, \mu > 0, m \geq d, d \geq 1$:

$$\binom{m}{d}[(x-\lambda)^d \boxplus_{d,\lambda} (x-\mu)^d] = (\lambda\mu)^{d/2} C_d^{m-d+1}\left(\frac{x-(\lambda+\mu)}{2\sqrt{\lambda\mu}}\right).$$

Monotonicity of Cauchy transforms

Theorem (Monotonicity with moving parameter)

Define for all $\theta > 0$*:*

$$\gamma_\theta^d := \operatorname{maxroot}\left\{C_d^{(1+\theta d)}(x)\right\}.$$

Then for $x > \max\left\{\gamma_\theta^d, \gamma_\theta^{(d+1)}\right\}$

$$\mathcal{G}_{C_d^{(1+\theta d)}}(x) \leq \mathcal{G}_{C_{d+1}^{(1+\theta[d+1])}}(x).$$

Corollary

The sequence $(\gamma_\theta^d)_d$ *is monotone increasing, and for* $\gamma_\theta = \frac{\sqrt{2\theta+1}}{\theta+1}$*,*

$$\lim_{d\to\infty} \gamma_\theta^d = \gamma_\theta$$

Contents

Rectangular finite R-transform

Fix a symmetric discrete measure $\mu_{\mathbb{S}p}$. We can build a polynomial of degree d $\mathcal{R}^{d,\lambda}_{\mathbb{S}p}(s)$ such that:

Theorem (Convergence)

$$\mathcal{R}^{d,\lambda}_{\mathbb{S}p}(s) \to_{d\to\infty} \mathcal{R}^{\lambda}_{\mu_{\mathbb{S}p}}(s)$$

Explictly, consider for a fix p and d, the limit of $\mathcal{R}^{dn,\lambda}_{\mathbb{S}p^n}(s)$ *in n.*

Theorem (Linearization)

$$\mathcal{R}^{d,\lambda}_{\mathbb{S}[p\boxplus_{d,\lambda}q]}(s) = \mathcal{R}^{d,\lambda}_{\mathbb{S}p}(s) + \mathcal{R}^{d,\lambda}_{\mathbb{S}q}(s)$$

It is the direct analogue of the free probability additivity property that defines the free R-rectangular transform. Rectangular finite free cumulants.

- $\mathbb{E}\left[\mathbb{S}[p \boxplus_{d,\lambda} q]\right] = 0, \quad Var\left[\mathbb{S}[p \boxplus_{d,\lambda} q]\right] = Var[\mathbb{S}p] + Var[\mathbb{S}q]$
- $\mathbb{S}p = x^{2d}$ constant polynomial in dimension d
- $p(x) = L_d{}^{(m-d)}(\frac{xm}{\sigma^2}) \iff \mathcal{R}^{d,\lambda}_{\mathbb{S}p}(s) = m\sigma^2 s$ where L_d are the Laguerre family = rectangular finite free Gaussians.

Proposition (Central limit theorem)

Let $p_1, p_2, \dots$ be a sequence of degree d with real nonnegative roots and same mean σ^2, with

$$p_i = \prod_j (x - r_{i,j}^2) \qquad \frac{1}{d}\sum_j r_{i,j}^2 = \sigma^2$$

Then

$$\lim_{N\to\infty} R_{1/\sqrt{N}}(\mathbb{S}[p_1 \boxplus_{d,\lambda} \dots \boxplus_{d,\lambda} p_N])(x) \approx L_d^{(m-d)}(\frac{x^2 m}{\sigma})$$

For $R_\alpha(p) = \prod_j (x - \alpha r_{i,j}^2)$.

Contents

Conclusion

We found a new bridge between algebra and analysis, roots of polynomials and probability distributions. It is just the beginning...

Contents

Defiintion of the extension

Definition

Consider $z \geq -1$, write $p(x) = \sum_{i=0}^{d} a_i x^{d-i}$ and $q(x) = \sum_{i=0}^{d} b_i x^{d-i}$,

$$p \boxplus_d^z q := \sum_{k=0}^{d} x^{d-k} \sum_{i+j=k} c_{i,j}(z)$$

$$c_{i,j}(z) := \frac{(d-i)!(d-j)!}{d!(d-k)!} \frac{\Gamma[d+z+1-i]\Gamma[d+z+1-j]}{\Gamma[d+z+1]\Gamma[d+z+1-k]} a_i b_j$$

Conjecture

$p \boxplus_d^z q$ is real rooted in x with nonnegative roots if p and q are.

Remark

$\mathbb{S}[p \boxplus_d^{-1/2} q] = \mathbb{S}p \boxplus_{2d} \mathbb{S}q$ *and* $\lim_{z\to\infty} p \boxplus_d^z q = p \boxplus_d q$

Comparing convolutions

Conjecture

There is continuous majorization (two polynomials majorizing mean that the vector of their ordered roots do), for $-1/2 < z_1 < z_2$ *:*

$$p \boxplus_d^{z_1} q \succeq p \boxplus_d^{z_2} q$$

Corollary

For all p, q *in* $\mathbb{P}^+_{\leq d}$*,and* z_1, z_2 *such that* $z_1 < z_2$ *we have:*

$$\mathrm{maxroot}\,\{U_\alpha \mathbb{S}[p \boxplus_d^{z_1} q]\} \leq \mathrm{maxroot}\,\{U_\alpha \mathbb{S}[p \boxplus_d^{z_2} q]\}$$

where $U_\alpha(p) := p - \alpha p'$

Contents

Bivariate convolution

$$p \boxplus_d q[x,z] := \binom{d+z}{d} p \boxplus_d^z q(x) \in \mathbb{R}[x,z]$$

Conjecture

For all l, $\partial_z^l \left(p \boxplus_d q[x,z]\right)$ is real-rooted in x. Also, $\partial_z \left(p \boxplus_d q[x,z]\right)$ and $\partial_x \left(p \boxplus_d q[x,z]\right)$ interlace. $p \boxplus_d^z q$ is real-rooted in z for x in some interval between the roots of p and q.

Orthogonal polynomials

$$(x-\lambda)^d \boxplus_d (x-\mu)^d[., z] \approx C_d^{z+1}\left(\frac{x-(\lambda+\mu)}{\sqrt{\lambda\mu}}\right)$$

Theorem

For $x \in [-1, 1]$, $C_d^z(x)$ is real-rooted in z. For $x \in [0, +\infty]$, $L_d^z(x)$ is real-rooted in z. Also, $\partial_z^l C_d^z(x)$ and $\partial_z^l L_d^z(x)$ are real-rooted polynomials in x and there is interlacing between derivatives in z and x.

Remark (Orthogonality support)

For an orthogonal family of polynomials $P_d^z(x)$, polynomial in the parameter z , orthogonal with respect to μ, then it seems that for $x \in Supp(\mu)$, $P_d^z(x)$ is real-rooted in z.

Contents

Finite free entropy and information

Definition (Voiculescu)

For a measure μ with no atoms,

$$h(\mu) := \int\int log|x-y|d\mu(x)d\mu(y)$$

Definition

For $p = \prod_{i=1}^{d}(x-\lambda_i)$ polynomial with distinct roots:

$$h(p) := \frac{1}{\binom{d}{2}} \sum_{i<j} log|\lambda_i - \lambda_j|$$

$$J_k(p) := \frac{1}{\binom{d}{2}} \sum_{i<j} \frac{1}{(\lambda_i - \lambda_j)^{2k}}$$

Dilation monotonicity

For $p = \prod_{i=1}^{d}(x - \lambda_i)$, define $p_t := \prod_{i=1}^{d}(x - t\lambda_i)$.

Theorem

For $t > s > 0$

$$h(p \boxplus_d q_t) \geq h(p \boxplus_d q_s) \geq h(p)$$

Conjecture

$$h(p \boxplus_d^z q_t) \geq h(p \boxplus_d^z q_s) \geq h(p)$$

Conjecture

$h(p_{\sqrt{t}} \boxplus_d q_{\sqrt{1-t}})$ is concave in t and $J_k(p_{\sqrt{t}} \boxplus_d q_{\sqrt{1-t}})$ are convex. Equivalent of $f(\sqrt{t}X + \sqrt{1-t}Y)$ for independent/free random variables X,Y. Rectangular version?

Inequalities

Conjecture (Power entropy inequalities)

For p, q real rooted polynomials, we have

$$e^{2h(p \boxplus_d q)} \geq e^{2h(p)} + e^{2h(q)}$$

with equality only for p,q Hermite polynomials.

Rectangular version?

Similarly, we could derive Stam's inequalities.

Conjecture

For $p := \prod(x - \lambda_i)$ *with d distinct real numbers* λ_i*, denote by* $S_i^k(p) := \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)^k}$ *we have*

$$Var(p) \sum_i S_i^1(p) S_i^3(p) \geq K(d) \sum_i S_i^2(p)$$