Central Limit Theorems for Star-Generators of the Infinite Symmetric Group

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- Revised Version of Slides -
Introduction & Motivation
Infinite symmetric group, star-generators, characters

General Notation

- \( \mathbb{N} = \{1, 2, 3, \ldots, n, \ldots\} \), \([n] = \{1, 2, \ldots, n\}\)
- \(S_\infty\) is the group of all finite permutations on \(\mathbb{N}\):
  \[
  S_\infty = \left\{ \sigma: \mathbb{N} \to \mathbb{N} \mid \sigma \text{ is bijective with } \sigma(n) \neq n \right. \\
  \text{for only finitely many } n \in \mathbb{N} \left. \right\}
  \]
- The transpositions
  \[
  \gamma_1 = (1, 2), \quad \gamma_2 = (1, 3), \quad \ldots \quad \gamma_n = (1, n + 1), \quad \ldots
  \]
  are called star-generators of \(S_\infty\); and \(\gamma_0 = e\) denotes the identity.
- A character \(\chi: S_\infty \to \mathbb{C}\) is a positive definite function which is constant on conjugacy classes and normalized, i.e. \(\chi(e) = 1\).
A simple observation . . .

Proposition (‘exchangeability’ of star-generators [Gohm & K ’10])

Let $\chi$ be a character on $S_\infty$. Then one has

$$\chi(\gamma k_1 \gamma k_2 \cdots \gamma k_n) = \chi(\gamma \sigma(k_1) \gamma \sigma(k_2) \cdots \gamma \sigma(k_n))$$

for any $n \in \mathbb{N}$, $k_1, k_2, \ldots, k_n \in \mathbb{N}$, and all $\sigma \in S_\infty$.

Proof.

$$\gamma \sigma(k) = (1, \sigma(k) + 1) = (\tilde{\sigma}(1), \tilde{\sigma}(k + 1)) = \tilde{\sigma}(1, k + 1) \tilde{\sigma}^{-1}$$

for $\tilde{\sigma} \in S_\infty$ with $\tilde{\sigma}(1) = 1$ and $\tilde{\sigma}(k + 1) = \sigma(k) + 1$.

- $(\mathbb{C}[S_\infty], \text{tr}_\chi)$ is a tracial *-algebraic probability space.
- The noncommutative random variables $(\gamma_n)_{n=1}^\infty$ are exchangeable.
- $(\gamma_n)_{n=1}^\infty$ should be identically distributed and ‘conditional independent’ (in a certain sense in noncommutative probability).
A simple observation . . .

**Proposition (‘exchangeability’ of star-generators [Gohm & K ’10])**

Let \( \chi \) be a character on \( S_\infty \). Then one has

\[
\chi(\gamma_{k_1} \gamma_{k_2} \cdots \gamma_{k_n}) = \chi(\gamma_{\sigma(k_1)} \gamma_{\sigma(k_2)} \cdots \gamma_{\sigma(k_n)})
\]

for any \( n \in \mathbb{N}, k_1, k_2, \ldots, k_n \in \mathbb{N}, \) and all \( \sigma \in S_\infty \).

**Proof.**

\[
\gamma_{\sigma(k)} = (1, \sigma(k) + 1) = (\tilde{\sigma}(1), \tilde{\sigma}(k + 1)) = \tilde{\sigma}(1, k + 1) \tilde{\sigma}^{-1}
\]

for \( \tilde{\sigma} \in S_\infty \) with \( \tilde{\sigma}(1) = 1 \) and \( \tilde{\sigma}(k + 1) = \sigma(k) + 1 \). □

- (\( \mathbb{C}[S_\infty], \text{tr}_\chi \)) is a **tracial *-algebraic probability space**.
- The **noncommutative random variables** \( (\gamma_n)_{n=1}^\infty \) are exchangeable.
- \( (\gamma_n)_{n=1}^\infty \) **is identically distributed** and ‘conditionally CS independent’ (see Gohm & K ’10 for example).
Classical De Finetti Theorem

An infinite sequence of random variables $X \equiv (X_1, X_2, \ldots)$ is

- **exchangeable** if

  $$
  \mathbb{E}(X_{i_1} \cdots X_{i_n}) = \mathbb{E}(X_{\sigma(i_1)} \cdots X_{\sigma(i_n)})
  $$

  for all $n \in \mathbb{N}$, $i_1, \ldots, i_n \in \mathbb{N}$ and finite permutations $\sigma$ on $\mathbb{N}$.

**Theorem (De Finetti)**

The following are equivalent:

(a) $X$ is exchangeable

(b) $X$ is conditionally i.i.d.
A Noncommutative De Finetti Theorem

Theorem (Gohm & K '09, K '10)

Let the pair \((\mathcal{M}, \varphi)\) denote a von Neumann algebra \(\mathcal{M}\) equipped with a faithful normal (tracial) state \(\varphi\). Then an exchangeable sequence \((x_n)_{n \geq 1} \subset \mathcal{M}\) is identically distributed and \textbf{CS-independent} over its tail algebra

\[
\mathcal{T} := \bigcap_{n \geq 1} \text{vN}\{x_n, x_{n+1}, \ldots\}
\]

i.e.

\[
E(xy) = E(x)E(y)
\]

for any \(x \in \text{vN}\{x_i \mid i \in I\}\) and \(y \in \text{vN}\{x_i \mid j \in J\}\), where \(I, J \subset \mathbb{N}\) are disjoint. Here \(E\) denotes the \(\varphi\)-preserving normal conditional expectation from \(\mathcal{M}\) onto \(\mathcal{T}\).
Thoma’s Theorem as a Quantum de Finetti Theorem

**Theorem (Thoma ’64, Kerov & Vershik ’81, Okounkov ’97)**

An extreme character of $S_\infty$ has the form:

$$
\chi(\sigma) = \prod_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} a_i^k + (-1)^{k-1} \sum_{j=1}^{\infty} b_j^k \right)^{m_k(\sigma)}
$$

Here $m_k(\sigma)$ denotes the number of $k$-cycles in the permutation $\sigma$ and the double sequence $(a_1, a_2, \ldots; b_1, b_2, \ldots)$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq 0, \quad b_1 \geq b_2 \geq \cdots \geq 0, \quad \sum_{i=1}^{\infty} a_i + \sum_{j=1}^{\infty} b_j \leq 1.$$

- **Gohm & K ’10:** first operator algebraic proof!
- **Key idea:** exchangeability of the sequence of star-generators
Algebraic CLTs

Theorem (von Waldenfels, Speicher, ...)

Let \((\mathcal{A}, \varphi)\) be a \(*\)-algebraic probability space and suppose that the sequence \((x_n = x_n^*)_{n=1}^{\infty} \subset \mathcal{A}\) satisfies:

1. exchangeability
2. singleton vanishing property, i.e. for all \(n \in \mathbb{N}\),
   \[
   \varphi(x_{\ell(1)}x_{\ell(2)} \cdots x_{\ell(n)}) = 0
   \]
   for any \(\ell: \{1, 2, \ldots, n\} \to \mathbb{N}\) with \(\#\ell^{-1}(\{m\}) = 1\) for some \(m \in \mathbb{N}\).

Let \(S_n = \frac{1}{\sqrt{n}}(x_1 + x_2 + \ldots + x_n)\). Then one has

\[
\lim_{n \to \infty} \varphi(S_n^k) = \sum_{\pi \in \mathcal{P}_2([k])} \varphi_\pi
\]

where \(\varphi_\pi = \varphi(x_{\ell(1)} \cdots x_{\ell(k)})\) with \(\pi = \ker(\ell)\).
Central Limit Theorems for Star-Generators

- left regular character (Biane ’95)
- block characters (K-Nica ’21)
- beyond block characters (Campbell-K-Nica ’22)
CLT for the character $\varphi_{\text{reg}}$ of $S_{\infty}$

\[(0, 0, \ldots; 0, 0 \ldots) \leftrightarrow \varphi_{\text{reg}}(\sigma) := \begin{cases} 1 & \text{if } \sigma = e \\ 0 & \text{otherwise} \end{cases} \quad (\sigma \in S_{\infty})\]

**Theorem (Biane’95)**

Let

\[s_n = \frac{1}{\sqrt{n}} (\gamma_1 + \gamma_2 + \cdots + \gamma_n) .\]

Then there exists $\mu_{\text{reg}} \in M_1(\mathbb{R})$ such that

\[\lim_{n \to \infty} \varphi_{\text{reg}}(s_n^k) = \int_{\mathbb{R}} t^k \mu_{\text{reg}}(dt) \quad (k \in \mathbb{N}).\]

Moreover, $\mu_{\text{reg}}$ equals Wigner’s semicircle law.
CLT for a block character $\varphi_d$

\[
\left( \frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d}, 0, 0, \ldots; 0, 0 \ldots \right) \quad \longleftrightarrow \quad \varphi_d(\sigma) := \left( \frac{1}{d} \right)^{\|\sigma\|} \quad (\sigma \in S_\infty)
\]

Here $\|\sigma\| = \min\{m \mid \sigma = \tau_1\tau_2 \cdots \tau_m \text{ for transpositions } \tau_1, \tau_2, \ldots, \tau_m\}$

**Theorem (K-Nica '21, Campbell-K-Nica '22)**

Let

\[
s_n = \frac{1}{\sqrt{n}} \left( \gamma_1 + \gamma_2 + \cdots + \gamma_n - \frac{n}{d} \right).
\]

Then there exists a central limit law $\mu_d \in M_1(\mathbb{R})$ such that

\[
\lim_{n \to \infty} \varphi_d(s_n^k) = \int_{\mathbb{R}} t^k \mu_d(dt) \quad (k \in \mathbb{N}).
\]

Moreover, $\mu_d$ equals the average empirical distribution $\nu_d$ of a traceless random $d \times d$ GUE matrix with variance 1.
Traceless GUE Matrix

Consider a collection of $d^2$ independent centred Gaussian random variables

$$\{X_{i,j} \mid 1 \leq i \leq j \leq d\} \cup \{Y_{i,j} \mid 1 \leq i < j < d\}$$

with $\text{Var}(X_{ii}) = \frac{1}{d}$ for $1 \leq i \leq d$

and $\text{Var}(X_{ij}) = \text{Var}(Y_{ij}) = \frac{1}{2d}$ for $1 \leq i < j \leq d$.

Traceless Random $d \times d$ GUE Matrix of Variance 1

$$M = \begin{bmatrix}
X_{11} & X_{12} + iY_{12} & \cdots & X_{1d} + iY_{1d} \\
X_{12} - iY_{12} & X_{22} & \cdots & X_{2d} + iY_{2d} \\
\vdots & \vdots & \ddots & \vdots \\
X_{1d} - iY_{1d} & X_{2d} - iY_{2d} & \cdots & X_{dd}
\end{bmatrix} - \frac{X_{11} + \ldots + X_{dd}}{d} I_d$$

with average empirical distribution $\nu_d$ (uniquely determined by $\nu_d(M^k) = \mathbb{E}(\text{tr}_d(M^k))$ for all $k \in \mathbb{N}$).
The character $\varphi_w$

For the Thoma double sequence $\underline{w} := (w_1, w_2, \ldots, w_d, 0, 0, \ldots; 0, 0 \ldots)$, summing up to 1

$$\varphi_w(\sigma) := \prod_{k=2}^{\infty} \left( \sum_{i=1}^{d} w_i^k \right)^{m_k(\sigma)}$$

$(\sigma \in S_\infty)$

Postponing some notation ....

**Theorem (Abstract CLT for Thoma character $\varphi_w$)**

Let

$$s_n = \frac{1}{\sqrt{n}} \left( U(\gamma_1) + U(\gamma_2) + \cdots + U(\gamma_n) - nA_0 \right).$$

Then there exists a central limit law $\mu_w \in M_1(\mathbb{R})$ such that

$$\lim_{n \to \infty} \varphi_w(s_n^k) = \int_{\mathbb{R}} t^k \mu_w(dt) \quad (k \in \mathbb{N}).$$
...providing needed notation:

Let $U : S_\infty \to B(\mathcal{H})$ denote the GNS representation of $\varphi_w$.

**Proposition (Law of Large Numbers)**

Let $\underline{w} = (w_1, w_2, \ldots, w_d)$ be given. The operator

$$A_0 := \text{SOT-lim}_{n \to \infty} \frac{1}{n} (U(\gamma_1) + \ldots + U(\gamma_n)) \in B(\mathcal{H})$$

is a selfadjoint contraction with $\text{Spectrum}(A_0) = \bigcup_{i=1}^{d} \{w_i\}$.

**Remark**

- $A_0 = \text{WOT-lim}_{n \to \infty} U(\gamma_n)$ (‘limit-2-cycle $(1, \infty)$’)
- $\mathcal{T} = \text{vN}\{A_0\}$ (tail algebra of the sequence $(U(\gamma_n))_{n=1}^{\infty}$)
- $\mathcal{T}$ is trivial $\iff$ $\text{Spectrum}(A_0) = \{1/d\}$
CCR-analogue of complex Gaussian random variable

Let \((\mathcal{A}, \varphi)\) be a \(*\)-probability space and \(0 < \omega_{(1,*)}, \omega_{(*,1)} < \infty\) be fixed. An element \(a \in \mathcal{A}\) is said to be a centred CCR-complex-Gaussian element if

\[
a^*a = aa^* + (\omega_{(*,1)} - \omega_{(1,*)})1_{\mathcal{A}}
\]

and, for \(p, q \in \mathbb{N}_0\),

\[
\varphi(a^p(a^*)^q) = \begin{cases} 
0 & \text{if } p \neq q; \\
 p! \omega_{(1,*)}^p & \text{if } p = q.
\end{cases}
\]

**Remark:** If \(\omega_{(*,1)} = \omega_{(1,*)}\), then one recovers the usual notion of a complex Gaussian random variables.
Traceless CCR-GUE Matrix

A selfadjoint matrix $M = [a_{i,j}]_{1 \leq i,j \leq d} \in M_d(\mathcal{A})$ is called a traceless CCR-GUE matrix with parameters $w_1, w_2, \ldots, w_d$ if there exist commuting independent unital *-subalgebras $\{\mathcal{A}_o\} \cup \{\mathcal{A}_{i,j} \mid 1 \leq i < j \leq d\} \subset \mathcal{A}$ s.t:

1. For $1 \leq i < j \leq d$, the element $a_{i,j} \in \mathcal{A}_{i,j}$ is centred CCR-Gaussian (with parameters $w_j$ and $w_i$) and $a_{j,i} := a_{i,j}^*$ such that

$$a_{j,i}a_{i,j} = a_{i,j}a_{j,i} + (w_j - w_i)1_{\mathcal{A}}$$

2. $\mathcal{A}_o$ is commutative and the selfadjoint elements $a_{1,1}, \ldots, a_{d,d} \in \mathcal{A}_o$ form a centred Gaussian family with covariance matrix

$$\begin{bmatrix}
w_1 - w_1^2 & -w_1w_2 & \cdots & -w_1w_d \\
-w_1w_2 & w_2 - w_2^2 & \cdots & -w_2w_d \\
\vdots & \vdots & \ddots & \vdots \\
-w_1w_d & -w_2w_d & \cdots & w_d - w_d^2
\end{bmatrix}$$
Main Result

Theorem (Campbell-K-Nica ’22)

Let $(\mathcal{A}, \varphi)$ be a *-probability space and let $M = [a_{i,j}]_{1 \leq i,j \leq d}$ be a traceless CCR-GUE matrix with parameters $w_1, w_2, \ldots, w_d$. Consider the linear functional $\varphi_w : M_d(\mathcal{A}) \to \mathbb{C}$ defined by

$$\varphi_w(X) = \sum_{i=1}^{d} w_i \varphi(x_{i,i}) \quad \text{for} \quad X = [x_{i,j}]_{i,j=1}^{d} \in M_d(\mathcal{A}).$$

Then the law of $M$ in the *-probability space $(M_d(\mathcal{A}), \varphi_w)$ is equal to the central limit law $\mu_w$. 
Rough Strategy for Proof of Main Result

(1) Determine a moment formula for

\[ \varphi_w \left( (U(\gamma_{\ell(1)}) - A_0) \cdots (U(\gamma_{\ell(k)}) - A_0) \right) \]

for \( \ell : [k] \to \mathbb{N} \) with \( \ker(\ell) \in \mathcal{P}_2([k]) \).

(2a) Provide a Wick’s lemma for a CCR-complex Gaussian element.

(2b) Use this to get a formula for the joint moments of the entries of a CCR-GUE matrix.

(3) Show that the formula from (1) and that from (2b) coincide.

For details see:

Thank you for your attention!
References


