Central Limit Theorems for Star-Generators of the Infinite Symmetric Group

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- Revised Version of Slides -

Introduction & Motivation

Infinite symmetric group, star-generators, characters

General Notation

- $\mathbb{N} = \{1, 2, 3, \ldots, n, \ldots\}, \qquad [n] = \{1, 2, \ldots, n\}$
- S_∞ is the **group of all finite permutations** on $\mathbb{N}$:

$$S_\infty = \left\{ \sigma \colon \mathbb{N} \to \mathbb{N} \;\middle|\; \begin{array}{l} \sigma \text{ is bijective with } \sigma(n) \neq n \\ \text{for only finitely many } n \in \mathbb{N} \end{array} \right\}$$

- The transpositions

$$\gamma_1 = (1, 2), \quad \gamma_2 = (1, 3), \quad \ldots \quad \gamma_n = (1, n+1), \quad \ldots$$

  are called **star-generators** of S_∞; and $\gamma_0 = e$ denotes the identity.
- A **character** $\chi \colon S_\infty \to \mathbb{C}$ is a positive definite function which is constant on conjugacy classes and normalized, i.e. $\chi(e) = 1$.

A simple observation . . .

Proposition ('exchangeability' of star-generators [Gohm & K '10])

Let χ be a character on S_∞. Then one has

$$\chi(\gamma_{k_1}\gamma_{k_2}\cdots\gamma_{k_n}) = \chi(\gamma_{\sigma(k_1)}\gamma_{\sigma(k_2)}\cdots\gamma_{\sigma(k_n)})$$

for any $n \in \mathbb{N}$, $k_1, k_2, \ldots, k_n \in \mathbb{N}$, and all $\sigma \in S_\infty$.

Proof.

$\gamma_{\sigma(k)} = (1, \sigma(k)+1) = (\widetilde{\sigma}(1), \widetilde{\sigma}(k+1)) = \widetilde{\sigma}\,(1, k+1)\,\widetilde{\sigma}^{-1}$
for $\widetilde{\sigma} \in S_\infty$ with $\widetilde{\sigma}(1) = 1$ and $\widetilde{\sigma}(k+1) = \sigma(k)+1$. □

- $(\mathbb{C}[S_\infty], \mathrm{tr}_\chi)$ is a **tracial *-algebraic probability space**.
- The **noncommutative random variables** $(\gamma_n)_{n=1}^\infty$ are exchangeable.
- $(\gamma_n)_{n=1}^\infty$ should be **identically distributed** and **'conditional independent'** (in a certain sense in noncommutative probability).

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Classical De Finetti Theorem

An infinite sequence of random variables $X \equiv (X_1, X_2, \dots)$ is

- **exchangeable** if
$$\mathbb{E}(X_{i_1} \cdots X_{i_n}) = \mathbb{E}(X_{\sigma(i_1)} \cdots X_{\sigma(i_n)})$$
for all $n \in \mathbb{N}$, $i_1, \dots, i_n \in \mathbb{N}$ and finite permutations σ on $\mathbb{N}$.

Theorem (De Finetti)

The following are equivalent:

(a) X *is exchangeable*

(b) X *is conditionally i.i.d.*

Consequence: A concrete model of the joint distribution of an infinite exchangeable sequence X is available in terms of convex combinations of infinite product measures.

A Noncommutative De Finetti Theorem

Theorem (Gohm & K '09, K '10)

Let the pair $(\mathcal{M}, \varphi)$ *denote a von Neumann algebra* $\mathcal{M}$ *equipped with a faithful normal (tracial) state* φ*. Then an* **exchangeable sequence** $(x_n)_{n\geq 1} \subset \mathcal{M}$ *is identically distributed and* **CS-independent** *over its* **tail algebra**

$$\mathcal{T} := \bigcap_{n\geq 1} \mathrm{vN}\{x_n, x_{n+1}, \ldots\}$$

i.e.

$$E(xy) = E(x)E(y)$$

for any $x \in \mathrm{vN}\{x_i \mid i \in I\}$ *and* $y \in \mathrm{vN}\{x_i \mid j \in J\}$*, where* $I, J \subset \mathbb{N}$ *are disjoint. Here* E *denotes the* φ*-preserving normal conditional expectation from* $\mathcal{M}$ *onto* $\mathcal{T}$*.*

Remark: The affix '**CS**-' spells out as 'commuting squares'. In general, this geometric notion of noncommutative independence is without universality properties as they are present for tensor or free independence.

Thoma's Theorem as a Quantum de Finetti Theorem

Theorem (Thoma '64, Kerov & Vershik '81, Okounkov '97)

An extreme character of S_∞ has the form:

$$\chi(\sigma) = \prod_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} a_i^k + (-1)^{k-1} \sum_{j=1}^{\infty} b_j^k \right)^{m_k(\sigma)}$$

Here $m_k(\sigma)$ denotes the number of k-cycles in the permutation σ and the double sequence $(a_1, a_2, \ldots; b_1, b_2, \ldots)$ satisfies

$$a_1 \geq a_2 \geq \cdots \geq 0, \qquad b_1 \geq b_2 \geq \cdots \geq 0, \qquad \sum_{i=1}^{\infty} a_i + \sum_{j=1}^{\infty} b_j \leq 1.$$

- Gohm & K '10: **first operator algebraic proof**!
- **Key idea:** exchangeability of the sequence of star-generators

Algebraic CLTs

Theorem (von Waldenfels, Speicher, ...)

Let $(\mathcal{A}, \varphi)$ *be a *-algebraic probability space and suppose that the sequence* $(x_n = x_n^*)_{n=1}^{\infty} \subset \mathcal{A}$ *satisfies:*

1. *exchangeability*
2. *singleton vanishing property, i.e. for all* $n \in \mathbb{N}$,

$$\varphi(x_{\ell(1)} x_{\ell(2)} \cdots x_{\ell(n)}) = 0$$

for any $\ell\colon \{1, 2, \ldots, n\} \to \mathbb{N}$ *with* $\#\ell^{-1}(\{m\}) = 1$ *for some* $m \in \mathbb{N}$.

Let $S_n = \frac{1}{\sqrt{n}}(x_1 + x_2 + \ldots + x_n)$. *Then one has, for any* $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \varphi(S_n^k) = \sum_{\pi \in \mathcal{P}_2([k])} \varphi_\pi$$

where $\varphi_\pi = \varphi(x_{\ell(1)} \cdots x_{\ell(k)})$ *with* $\pi = \ker(\ell)$.

Central Limit Theorems for Star-Generators

- left regular character (Biane '95)
- block characters (K-Nica '21)
- beyond block characters (Campbell-K-Nica '22)

CLT for the character φ_{reg} of S_∞

$$(0,0,\ldots;0,0\ldots) \quad \longleftrightarrow \quad \varphi_{\text{reg}}(\sigma) := \begin{cases} 1 & \text{if } \sigma = e \\ 0 & \text{otherwise} \end{cases} \qquad (\sigma \in S_\infty)$$

Theorem (Biane'95)

Let

$$s_n = \frac{1}{\sqrt{n}}\left(\gamma_1 + \gamma_2 + \cdots + \gamma_n\right).$$

Then there exists $\mu_{reg} \in M_1(\mathbb{R})$ *such that*

$$\lim_{n\to\infty} \varphi_{reg}(s_n^k) = \int_{\mathbb{R}} t^k \mu_{reg}(dt) \qquad (k \in \mathbb{N}).$$

Moreover, μ_{reg} *equals Wigner's semicircle law.*

CLT for a block character φ_d

$$\underbrace{(\frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d}}_{d \text{ occurrences}}, 0, 0, \ldots; 0, 0 \ldots) \quad \longleftrightarrow \quad \varphi_d(\sigma) := (1/d)^{\|\sigma\|} \qquad (\sigma \in S_\infty)$$

Here $\|\sigma\| = \min\{m \mid \sigma = \tau_1 \tau_2 \cdots \tau_m$ for transpositions $\tau_1, \tau_2, \ldots, \tau_m\}$

Theorem (K-Nica '21, Campbell-K-Nica '22)

Let

$$s_n = \frac{1}{\sqrt{n}} \left(\gamma_1 + \gamma_2 + \cdots + \gamma_n - \frac{n}{d} \right).$$

Then there exists a central limit law $\mu_d \in M_1(\mathbb{R})$ such that

$$\lim_{n \to \infty} \varphi_d(s_n^k) = \int_{\mathbb{R}} t^k \mu_d(dt) \qquad (k \in \mathbb{N}).$$

Moreover, μ_d equals the average empirical distribution ν_d of a traceless random $d \times d$ GUE matrix with variance 1.

Traceless GUE Matrix

Consider a collection of d^2 independent centred Gaussian random variables

$$\{X_{i,j} \mid 1 \leq i \leq j \leq d\} \cup \{Y_{i,j} \mid 1 \leq i < j < d\}$$

with $\mathrm{Var}(X_{ii}) = \frac{1}{d}$ for $1 \leq i \leq d$
and $\mathrm{Var}(X_{ij}) = \mathrm{Var}(Y_{ij}) = \frac{1}{2d}$ for $1 \leq i < j \leq d$.

Traceless Random $d \times d$ GUE Matrix of Variance 1

$$M = \begin{bmatrix} X_{11} & X_{12} + \mathrm{i}\,Y_{12} & \cdots & X_{1d} + \mathrm{i}\,Y_{1d} \\ X_{12} - \mathrm{i}\,Y_{12} & X_{22} & \cdots & X_{2d} + \mathrm{i}\,Y_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ X_{1d} - \mathrm{i}\,Y_{1d} & X_{2d} - \mathrm{i}\,Y_{2d} & \cdots & X_{dd} \end{bmatrix} - \frac{X_{11} + \ldots + X_{dd}}{d} I_d$$

with average empirical distribution ν_d (uniquely determined by $\nu_d(M^k) = \mathbb{E}(\mathrm{tr}_d(M^k))$ for all $k \in \mathbb{N}$).

The character $\varphi_{\underline{w}}$

For the Thoma double sequence $\underline{w} := (\underbrace{w_1, w_2, \ldots, w_d}_{\text{summing up to } 1}, 0, 0, \ldots; 0, 0 \ldots)$,

$$\varphi_{\underline{w}}(\sigma) := \prod_{k=2}^{\infty} \left(\sum_{i=1}^{d} w_i^k \right)^{m_k(\sigma)} \qquad (\sigma \in S_\infty)$$

Postponing some notation ....

Theorem (Abstract CLT for Thoma character $\varphi_{\underline{w}}$)

Let

$$s_n = \frac{1}{\sqrt{n}} \big(U(\gamma_1) + U(\gamma_2) + \cdots + U(\gamma_n) - nA_0 \big).$$

Then there exists a central limit law $\mu_{\underline{w}} \in M_1(\mathbb{R})$ such that

$$\lim_{n \to \infty} \varphi_{\underline{w}}(s_n^k) = \int_{\mathbb{R}} t^k \mu_{\underline{w}}(dt) \qquad (k \in \mathbb{N}).$$

...providing needed notation:

Let $U\colon S_\infty \to B(\mathcal{H})$ denote the GNS representation of $\varphi_{\underline{w}}$.

Proposition (Law of Large Numbers)

Let $\underline{w} = (w_1, w_2, \ldots, w_d)$ *be given. The operator*

$$A_0 := \mathrm{SOT}\text{-}\lim_{n\to\infty} \frac{1}{n}\big(U(\gamma_1) + \ldots + U(\gamma_n)\big) \in B(\mathcal{H})$$

is a selfadjoint contraction with $\mathrm{Spectrum}(A_0) = \bigcup_{i=1}^{d}\{w_i\}$.

Remark

- $A_0 = \mathrm{WOT}\text{-}\lim_{n\to\infty} U(\gamma_n)$ *('limit-2-cycle* $(1,\infty)$*')*
- $\mathcal{T} = \mathrm{vN}\{A_0\}$ *(tail algebra of the sequence* $(U(\gamma_n))_{n=1}^{\infty}$*)*
- $\mathcal{T}$ *is trivial* $\iff$ $\mathrm{Spectrum}(A_0) = \{1/d\}$

CCR-analogue of complex Gaussian random variable

Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space and $0 < \omega_{(1,*)}, \omega_{(*,1)} < \infty$ be fixed. An element $a \in \mathcal{A}$ is said to be a **centred CCR-complex-Gaussian element** if

$$a^*a = aa^* + (\omega_{(*,1)} - \omega_{(1,*)})1_{\mathcal{A}}$$

and, for $p, q \in \mathbb{N}_0$,

$$\varphi\big(a^p(a^*)^q\big) = \begin{cases} 0 & \text{if } p \neq q; \\ p!\,\omega_{(1,*)}^p & \text{if } p = q. \end{cases}$$

Remark: One recovers the usual notion of a complex Gaussian random variables for $\omega_{(*,1)} = \omega_{(1,*)}$.

Traceless CCR-GUE Matrix

A selfadjoint matrix $M = [a_{i,j}]_{1\leq i,j\leq d} \in M_d(\mathcal{A})$ is called a **traceless CCR-GUE matrix with parameters** $w_1, w_2, \ldots, w_d$ if there exist commuting independent unital *-subalgebras $\{\mathcal{A}_o\} \cup \{\mathcal{A}_{i,j} \mid 1 \leq i < j \leq d\} \subset \mathcal{A}$ such that:

1. For $1 \leq i < j \leq d$, the element $a_{i,j} \in \mathcal{A}_{i,j}$ is centred CCR-Gaussian (with parameters w_j and w_i) and $a_{j,i} := a_{i,j}^*$ such that
$$a_{j,i}a_{i,j} = a_{i,j}a_{j,i} + (w_j - w_i)1_{\mathcal{A}}.$$
2. $\mathcal{A}_o$ is commutative and the selfadjoint elements $a_{1,1}, \ldots, a_{d,d} \in \mathcal{A}_o$ form a centred Gaussian family with covariance matrix
$$\begin{bmatrix} w_1 - w_1^2 & -w_1w_2 & \cdots & -w_1w_d \\ -w_1w_2 & w_2 - w_2^2 & \cdots & -w_2w_d \\ \vdots & \vdots & \ddots & \vdots \\ -w_1w_d & -w_2w_d & \cdots & w_d - w_d^2 \end{bmatrix}.$$

Main Result - Identification of the Central Limit Law $\mu_{\underline{W}}$

Theorem (Campbell-K-Nica '22)

Let $(\mathcal{A}, \varphi)$ *be a *-algebraic probability space and let* $M = [a_{i,j}]_{1\leq i,j\leq d}$ *be a traceless CCR-GUE matrix with parameters* $w_1, w_2, \ldots, w_d$*. Consider the linear functional* $\varphi_{\underline{w}} \colon M_d(\mathcal{A}) \to \mathbb{C}$ *defined by*

$$\varphi_{\underline{w}}(X) = \sum_{i=1}^{d} w_i \varphi(x_{i,i}) \qquad \text{for } X = [x_{i,j}]_{i,j=1}^{d} \in M_d(\mathcal{A}).$$

Then the law of M *in the *-algebraic probability space* $(M_d(\mathcal{A}), \varphi_{\underline{w}})$ *equals the central limit law* $\mu_{\underline{w}}$*.*

Rough Strategy for Proof of Main Result

(1) Determine a moment formula for

$$\varphi_{\underline{w}}\Big(\big(U(\gamma_{\ell(1)}) - A_0\big)\cdots\big(U(\gamma_{\ell(k)}) - A_0\big)\Big)$$

for $\ell\colon [k] \to \mathbb{N}$ with $\ker(\ell) \in \mathcal{P}_2([k])$.

(2a) Provide a Wick's lemma for a CCR-complex Gaussian element.

(2b) Use this to get a formula for the joint moments of the entries of a CCR-GUE matrix.

(3) Show that the formula from (1) and that from (2b) coincide.

For details see:

J. Campbell, C. Köstler, A. Nica; A central limit theorem for star-generators of S_∞, which relates to traceless CCR-GUE matrices. *International Journal of Mathematics*, Volume 33, Issue 09, August 2022.

Thank you for your attention!

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