Weighted Sums in Free Probability Theory

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Overview

Weighted sums in classical probability

Weighted sums in free probability

Outline of the proofs

Berry-Esseen type estimates in the free central limit theorem

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The classical Berry-Esseen theorem

Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random variables with mean zero, unit variance and finite third absolute moment β_3. The classical Berry-Esseen theorem asserts

$$\Delta(\mu_n, \gamma) \le \frac{c\beta_3}{\sqrt{n}}, \qquad c > 0,$$

where

- μ_n is the distribution of the normalized sum $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$,
- γ is the standard normal distribution,
- $\Delta(\cdot, \cdot)$ denotes the Kolmogorov distance, i.e.

$$\Delta(\nu_1, \nu_2) := \sup_{x \in \mathbb{R}} \left| \nu_1((-\infty, x]) - \nu_2((-\infty, x]) \right|$$

  for probability measures (=pm's) on $\mathbb{R}$.

We know: The rate of convergence of order $n^{-1/2}$ is sharp!

Weighted sums in classical probability

Let us consider *weighted sums*, i.e. sums of the form

$$S_\theta = \theta_1 X_1 + \cdots + \theta_n X_n, \qquad \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{S}^{n-1}$$

for $X, X_1, X_2, \ldots$ i.i.d. as before.

Theorem (Klartag, Sodin, 2011)

Assume that X has mean zero, unit variance and finite fourth moment m_4 and denote the distribution of S_θ by μ_θ. Choose $\rho \in (0,1)$. Then, there exists a set $\mathcal{F} \subset \mathbb{S}^{n-1}$ with $\sigma_{n-1}(\mathcal{F}) \geq 1-\rho$ such that for all $\theta \in \mathcal{F}$ one has

$$\Delta(\mu_\theta, \gamma) \leq \frac{C_\rho m_4}{n}, \qquad C_\rho > 0.$$

Here, σ_{n-1} denotes the uniform probability measure on $\mathbb{S}^{n-1}$.

We conclude: The random choice of the weights has an improving effect on the rate of convergence compared to the standard normalization via $n^{-\frac{1}{2}}$.

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The (non-id) free Berry-Esseen theorem

Before we talk about weighted sums in free probability theory, let us recall the free analogue of the Berry-Esseen theorem.

Theorem (Chistyakov, Götze, 2008)

Let ω denote Wigner's semicircle distribution. Let $\nu_1, \dots, \nu_n$ be pm's with mean zero, variances $\sigma_i^2 > 0$ and finite third absolute moments $\beta_3(\nu_i)$. For $B_n := \left(\sum_{i=1}^n \sigma_i^2\right)^{1/2}$ and $\nu_{\boxplus n} := D_{B_n^{-1}}\nu_1 \boxplus \cdots \boxplus D_{B_n^{-1}}\nu_n$, we have

$$\Delta(\nu_{\boxplus n}, \omega) \leq c\sqrt{\frac{\sum_{i=1}^n \beta_3(\nu_i)}{B_n^3}}, \qquad c > 0.$$

In the special case that $\nu_1, \dots, \nu_n$ have the same distribution with $\sigma_1^2 = 1$, we have $B_n = \sqrt{n}$ and $\Delta(\nu_{\boxplus n}, \omega) \leq \frac{c\beta_3(\nu_1)}{\sqrt{n}}$ for $c > 0$.

Weighted sums in free probability - Unbounded case

Theorem 1 (N., 2023, unbounded case)

Let μ be a pm with mean zero, unit variance and finite fourth moment $m_4(\mu)$. For $i \in [n]$ and $\theta \in \mathbb{S}^{n-1}$, let $\mu_i := D_{\theta_i}\mu$ and define $\mu_\theta := \mu_1 \boxplus \cdots \boxplus \mu_n$. Choose $\rho \in (0,1)$. Then, there exists a set $\mathcal{F} \subset \mathbb{S}^{n-1}$ with $\sigma_{n-1}(\mathcal{F}) \geq 1-\rho$ such that for all $\theta \in \mathcal{F}$ we have

$$\Delta(\mu_\theta, \omega) \leq C_{\rho,\mu}\sqrt{\frac{\log n}{n}}, \qquad C_{\rho,\mu} > 0.$$

- not comparable to Klartag-Sodin
- still improves the known results in free probability:
  - by free analogue of Berry-Esseen: $\Delta(\mu_\theta, \omega) \leq c\sqrt{\sum_{i=1}^n |\theta_i|^3}$
  - by Hölder: $\sum_{i=1}^n |\theta_i|^3 \geq n^{-1/2}$
  - a priori rate for $\Delta(\mu_\theta, \omega)$ larger than $n^{-1/4}$

Weighted sums in free probability - Bounded case

We can get rid of the logarithmic factor in the bounded case.

Theorem 2 (N., 2023, bounded case)

In the setting of Theorem 1, assume that μ has compact support in $[-L, L]$ for $L > 0$ and let $\rho \in (0,1)$. Then, there exists a set $\mathcal{F} \subset \mathbb{S}^{n-1}$ with $\sigma_{n-1}(\mathcal{F}) \geq 1 - \rho$ such that for all $\theta \in \mathcal{F}$ we have

$$\Delta(\mu_\theta, \omega) \leq \frac{C_{\rho,\mu}}{\sqrt{n}}, \qquad C_{\rho,\mu} > 0.$$

Note: For most vectors θ the convolution μ_θ exhibits the same rate of convergence as μ_{θ^*} for $\theta^* = (n^{-1/2}, \dots, n^{-1/2})$, but this is still not comparable to Klartag and Sodin's result.

Weighted sums in free probability - Free analogue of K-S

If we replace the Kolmogorov distance Δ by

$$\Delta_\varepsilon(\nu_1, \nu_2) := \sup_{x \in [-2+\varepsilon, 2-\varepsilon]} |\nu_1((-2+\varepsilon, x]) - \nu_2((-2+\varepsilon, x])|, \qquad \varepsilon > 0,$$

we get the free analogue of the Klartag-Sodin result.

Theorem 3 (N., 2023, Free analogue of Klartag-Sodin)

Let μ be as in Theorem 2 and let $\rho, \varepsilon \in (0,1)$. Then, there exists a set $\mathcal{F} \subset \mathbb{S}^{n-1}$ with $\sigma_{n-1}(\mathcal{F}) \geq 1 - \rho$ such that for all $\theta \in \mathcal{F}$ we have

$$\Delta_\varepsilon(\mu_\theta, \omega) \leq C_{\varepsilon,\rho,\mu} \frac{\log n}{n}, \qquad C_{\varepsilon,\rho,\mu} > 0.$$

Note: For the usual normalization with $\theta^* = (n^{-1/2}, \dots, n^{-1/2})$ we have the sharp rate $\Delta_\varepsilon(\mu_{\theta^*}, \omega) \lesssim \frac{1}{\sqrt{n}}$.

Weighted sums in free probability - Superconvergence

The randomization of the weights has an improving effect in the context of superconvergence, too.

Theorem 4 (N., 2023, Superconvergence)

Let μ be as in Theorem 2 and let $\rho \in (0,1)$. Then, there exists a set $\mathcal{F} \subset \mathbb{S}^{n-1}$ with $\sigma_{n-1}(\mathcal{F}) \geq 1-\rho$ such that for all $\theta \in \mathcal{F}$ we have

$$\operatorname{supp} \mu_\theta \subset \left(-2 - \frac{C_{\rho,\mu}}{n}, 2 + \frac{C_{\rho,\mu}}{n}\right), \qquad C_{\rho,\mu} > 0.$$

Note: This improves upon the standard rate $n^{-1/2}$ established by Kargin for the usual normalization via $\theta^* = (n^{-1/2}, \ldots, n^{-1/2})$.

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Berry-Esseen type estimates in the free central limit theorem

How to prove Berry-Esseen type estimates?

- useful tool: Bai's inequality relating the Kolmogorov distance to Cauchy transforms
- Cauchy transform G_ν of a pm ν given by $G_\nu(z) := \int_{\mathbb{R}} \frac{\nu(dx)}{z-x}$, $z \in \mathbb{C}^+$

Theorem (Bai's inequality - Version by Götze, Tikhomirov, 2001)

Let ν be a pm on $\mathbb{R}$ with second moment $m_2(\nu) = 1$. Let $a \in (0,1)$ and ε, τ be positive numbers such that $\frac{1}{\pi}\int_{|u|\leq\tau} \frac{1}{u^2+1} du = \frac{3}{4}$ and $\varepsilon > 2a\tau$ hold. Then, there exist constants $D_1, D_2, D_3 > 0$ such that

$$\Delta(\nu,\omega) \leq D_1 a + D_2 \varepsilon^{3/2} + D_3 \int_{-\infty}^{\infty} |G_\nu(u+i) - G_\omega(u+i)|\, du$$

$$+ D_3 \sup_{|u|\leq 2-\frac{\varepsilon}{2}} \int_a^1 |G_\nu(u+iv) - G_\omega(u+iv)| dv.$$

How to prove Berry-Esseen type estimates?

We have several methods to control the difference of Cauchy transforms. Some of them are:

- via subordination functions (Chistyakov, Götze)
- via R- and K-transforms (Kargin)
- via operator theoretic approaches such as Lindeberg exchange method (Austern, Banna, Mai, Speicher)

Proof in the unbounded case - Subordination

We will need the concept of subordination functions.

Subordination (Voiculescu, Biane, Bercovici, Belinschi, Chistyakov, Götze)

Let $\nu_1, \dots, \nu_n$ be pm's on $\mathbb{R}$ and define $\nu_{\boxplus n} := \nu_1 \boxplus \cdots \boxplus \nu_n$. There exist unique holomorphic functions $Z_1, \dots, Z_n : \mathbb{C}^+ \to \mathbb{C}^+$ such that for any $z \in \mathbb{C}^+$ we have:

$$Z_1(z) + Z_2(z) + \cdots + Z_n(z) - z = \frac{n-1}{G_{\nu_1}(Z_1(z))},$$

$$G_{\nu_1}(Z_1(z)) = \cdots = G_{\nu_n}(Z_n(z)) = G_{\nu_{\boxplus n}}(z).$$

The subordination functions $Z_1, \dots, Z_n$ satisfy $\Im Z_i(z) \geq \Im z$ for all $z \in \mathbb{C}^+, i \in [n]$.

Proof in the unbounded case - Overview

Reminder:

Let μ be a pm with mean zero, unit variance and $m_4(\mu) < \infty$. For $\theta \in \mathbb{S}^{n-1}$, set $\mu_i := D_{\theta_i}\mu$, $\mu_\theta := \mu_1 \boxplus \cdots \boxplus \mu_n$. Let $Z_1, \ldots, Z_n$ denote the subordination functions with respect to μ_θ.

Theorem 1 (Unbounded case)

Let $\rho \in (0,1)$. Then, there exists a set $\mathcal{F} \subset \mathbb{S}^{n-1}$ with $\sigma_{n-1}(\mathcal{F}) \geq 1 - \rho$ such that for all $\theta \in \mathcal{F}$ we have $\Delta(\mu_\theta, \omega) \lesssim \sqrt{\frac{\log n}{n}}$.

Overview:

1. Apply Bai's inequality.
2. Define $\mathcal{F} \subset \mathbb{S}^{n-1}$ and fix $\theta \in \mathcal{F}$. Assume $\theta_1^2 = \min_{i \in [n]} \theta_i^2$.
3. Derive and solve cubic functional equation for Z_1.
4. Derive and solve quadratic functional equation for Z_1.

Step 1: Apply Bai's inequality

- Let G_θ denote the Cauchy transform of μ_θ for $\theta \in \mathbb{S}^{n-1}$. By Bai's inequality, we have to bound

with quadratic functional eq.

$$\int_{-\infty}^{\infty} |G_\theta(u+i) - G_\omega(u+i)| du$$

and

with cubic functional eq.

$$\sup_{|u| \leq 2 - \frac{\varepsilon_n}{2}} \int_{a_n}^{1} |G_\theta(u+iv) - G_\omega(u+iv)| dv$$

for appropriate choices of a_n and ε_n.

- We will use

$$|G_\theta(z) - G_\omega(z)| \leq \left| \frac{1}{Z_1(z)} - G_\omega(z) \right| + \left| G_\theta(z) - \frac{1}{Z_1(z)} \right|$$

in order to handle both integrals.

- This means: We need to know $Z_1(z)$!

Step 2: Choice of $\mathcal{F}$

- We choose $\mathcal{F} \subset \mathbb{S}^{n-1}$ in such a way that

$$\max_{i \in [n]} |\theta_i| \lesssim \sqrt{\frac{\log n}{n}}$$

and

$$\sum_{i=1}^{n} |\theta_i|^k \lesssim \frac{1}{n^{\frac{k-2}{2}}}$$

hold for all $\theta \in \mathcal{F}$ and all $k \in \mathbb{N}$ with $2 < k \leq 7$.

- By choosing the implicit constants above appropriately, we can achieve $\sigma_{n-1}(\mathcal{F}) \geq 1 - \rho$ for fixed $\rho \in (0, 1)$ as requested.
- From now on, fix $\theta \in \mathcal{F}$ and assume $\theta_1^2 = \min_{i \in [n]} \theta_i^2$.

Step 3: Cubic functional equation for Z_1

3.1 Derivation of the cubic functional equation

Let G_i be the Cauchy transform of $\mu_i = D_{\theta_i}\mu$.

- From subordination, we have

$$Z_1(z) - z = \sum_{i=2}^{n} \frac{1}{G_i(Z_i(z))} - Z_i(z), \qquad z \in \mathbb{C}^+.$$

- After manipulations:

$$Z_1^3(z) - zZ_1^2(z) + (1-\theta_1^2)Z_1(z) - r(z) = 0$$

  for $z \in \mathbb{C}^+$ for some appropriately defined $r(z)$.

- Goal:

$$Z_1^3(z) - zZ_1^2(z) + Z_1(z) \approx 0.$$

  This implies $\frac{1}{Z_1(z)} \approx G_\omega(z)$.

Step 3: Cubic functional equation for Z_1

3.2 Key steps in the derivation of the bound for the error $r(z)$

- For any $z \in \mathbb{C}^+$, we can expand

$$Z_i(z)G_i(Z_i(z)) = 1 + \frac{1}{Z_i(z)} \int_{\mathbb{R}} \frac{u^2}{Z_i(z) - u} \mu_i(du).$$

- Thus, for any $z \in \mathbb{C}^+$ with $\Im z \gtrsim \sqrt{\frac{\log n}{n}}$, we have

$$|Z_i(z)G_i(Z_i(z)) - 1| \leq \frac{\theta_i^2}{\Im z |Z_i(z)|} \lesssim \frac{\frac{\log n}{n}}{(\Im z)^2} < 1.$$

  → starting point for all our estimates!

  → determines the final rate of convergence

- In the end: $r(z)$ is of order $\frac{1}{\sqrt{n}}$ for $\Im z \gtrsim \sqrt{\frac{\log n}{n}}$.

3.3 Solving the cubic functional equation

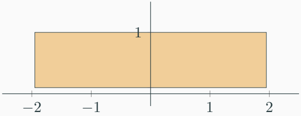
- Recall: We wanted to bound

$$\sup_{|u|\le 2-\frac{\varepsilon_n}{2}} \int_{a_n}^{1} \left| G_\theta(u+iv) - \frac{1}{Z_1(u+iv)} \right| + \left| \frac{1}{Z_1(u+iv)} - G_\omega(u+iv) \right| dv.$$

  for appropriate ε_n and a_n by use of the cubic functional equation.

- From now on, let $\varepsilon_n, a_n \approx \sqrt{\frac{\log n}{n}}$ and consider the cubic functional equation only in the set B given by

$$B := \left\{ z \in \mathbb{C}^+ : |\Re z| \le 2 - \varepsilon_n, 1 \ge \Im z \ge a_n \right\}$$



Step 3: Cubic functional equation for Z_1

3.3 Solving the cubic functional equation

In the end:

$$Z_1(z) = \frac{1}{2}\left(z - r_1(z) + \sqrt{z^2 - 4 + r_2(z)}\right), \qquad z \in B,$$

for some error terms $r_1(z)$ and $r_2(z)$ with $|r_i(z)| \lesssim \frac{1}{\sqrt{n}}$, $i = 1, 2$.

Note:

- All calculations rely on the bound $|r(z)| \lesssim \frac{1}{\sqrt{n}}$ holding for $\Im z \gtrsim a_n$.
- The formula for $1/Z_1$ looks very similar to the formula for the Cauchy transform G_ω.

Step 3: Cubic functional equation for Z_1

3.4 Evaluating the integral

- Integration yields

$$\sup_{|u| \leq 2 - \frac{\varepsilon_n}{2}} \int_{a_n}^{1} \left| \frac{1}{Z_1(u+iv)} - G_\omega(u+iv) \right| dv \lesssim \frac{1}{\sqrt{n}}.$$

- In total:

$$\sup_{|u| \leq 2 - \frac{\varepsilon_n}{2}} \int_{a_n}^{1} \left| G_\theta(u+iv) - G_\omega(u+iv) \right| dv \lesssim \frac{1}{\sqrt{n}} + \frac{\log n}{n}.$$

Step 4: Quadratic functional equation for Z_1

Proceeding similarly to the cubic functional equation, we can derive and solve a quadratic functional equation for Z_1.

We arrive at:

$$\int_{-\infty}^{\infty} |G_\theta(u+i) - G_\omega(u+i)| du \lesssim \frac{1}{\sqrt{n}} + \frac{\log n}{n}.$$

The integral above does not decay faster than $n^{-1/2}$ by our approach.

Proof in the unbounded case - Final conclusion

Using Bai's inequality, we obtain

$$\Delta(\mu_\theta, \omega) \lesssim \text{"contribution from the integrals"} + a_n + \varepsilon_n^{3/2}$$
$$\lesssim \frac{1}{\sqrt{n}} + \sqrt{\frac{\log n}{n}} + \sqrt{\frac{\log n}{n}}^{3/2} \lesssim \sqrt{\frac{\log n}{n}}.$$

We note: The lower bound on $\Im z$, i.e. a_n, is responsible for the logarithmic factor.

We can get rid of that factor if we assume that μ has compact support.

We will combine the concept of subordination with K-transforms.

K-transforms

Let ν be a pm with $\operatorname{supp}\nu \subset [-M, M]$. Then, the functional inverse $K_\nu(z) := G_\nu^{-1}(z)$ is well-defined and analytic in $0 < |z| < (6M)^{-1}$ with

$$G_\nu(K_\nu(z)) = z \text{ for } 0 < |z| < (6M)^{-1}, \ \ K_\nu(G_\nu(z)) = z \text{ for } |z| > 7M.$$

and Laurent series expansion given by

$$K_\nu(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \kappa_m(\nu) z^{m-1}, \qquad 0 < |z| < (6M)^{-1}.$$

Here, $\kappa_m(\nu)$ denotes the m-th free cumulant of ν.

Proof in the bounded case - Overview

Reminder:

Theorem 2 (Bounded case)

Assume that $\operatorname{supp}\mu \subset [-L, L]$ holds for some $L > 0$. Let $\rho \in (0, 1)$. Then, there exists a set $\mathcal{F} \subset \mathbb{S}^{n-1}$ with $\sigma_{n-1}(\mathcal{F}) \geq 1 - \rho$ such that for all $\theta \in \mathcal{F}$ we have $\Delta(\mu_\theta, \omega) \lesssim \frac{1}{\sqrt{n}}$.

Overview:

1. Apply Bai's inequality.
2. Choose $\mathcal{F} \subset \mathbb{S}^{n-1}$ and fix $\theta \in \mathcal{F}$. Assume $\theta_1^2 = \min_{i \in [n]} \theta_i^2$.
3. Derive lower bound for $|Z_i|$.
4. Derive and solve cubic functional equation for Z_1.
5. Derive and solve quadratic functional equation for Z_1.

We will just consider Step 3.

Step 3: Lower bound for $|Z_i|$

Assume that we have done the first two steps and that $\mathcal{F} \subset \mathbb{S}^{n-1}$ is defined similarly to the unbounded case. Now, fix $\theta \in \mathcal{F}$ and assume that

$$\Im z \gtrsim \frac{(\log n)^{3/2}}{n}$$

holds. Then:

- By integration by parts, we have

$$|G_\theta(z)| \leq |G_\omega(z)| + |G_\omega(z) - G_\theta(z)| \leq 1 + \frac{\pi\Delta(\mu_\theta, \omega)}{\Im z}$$
$$\lesssim 1 + \sqrt{\frac{\log n}{n}} \frac{1}{\Im z} \lesssim \frac{\sqrt{n}}{\log n} \lesssim \frac{1}{6L|\theta_i|}, \qquad i \in [n].$$

- We have $\operatorname{supp}\mu_i \subset [-|\theta_i|L, |\theta_i|L]$.
  $\Rightarrow$ K-transform K_i of μ_i is analytic in $0 < |z| < (6|\theta_i|L)^{-1}$.
  $\Rightarrow$ $K_i(G_\theta(z))$ is analytic for all z with $\Im z \gtrsim \frac{(\log n)^{3/2}}{n}$
- with identity theorem: $Z_i(z) = K_i(G_i(Z_i(z))) = K_i(G_\theta(z))$ for z as above

Step 3: Lower bound for $|Z_i|$

- Finally, for $z \in \mathbb{C}^+$ with $\Im z \gtrsim \frac{(\log n)^{3/2}}{n}$ and any $i \in [n]$, we get:

$$\begin{aligned} |Z_i(z)| &= |K_i(G_\theta(z))| \\ &\geq \left|\frac{1}{G_\theta(z)}\right| - \left|\theta_i^2 G_\theta(z)\right| - \left|K_i(G_\theta(z)) - \frac{1}{G_\theta(z)} - \theta_i^2 G_\theta(z)\right| \\ &\gtrsim \frac{\log n}{\sqrt{n}}. \end{aligned}$$

Step 3: Lower bound for $|Z_i|$

How does the lower bound on $|Z_i|$ help to improve the rate of convergence?

- In unbounded case: starting point was the inequality

$$|Z_i(z)G_i(Z_i(z)) - 1| \lesssim \frac{\frac{\log n}{n}}{\underbrace{\Im z |Z_i(z)|}_{\geq (\Im z)^2}} < 1, \qquad \Im z \gtrsim \sqrt{\frac{\log n}{n}}$$

- In bounded case:

$$|Z_i(z)G_i(Z_i(z)) - 1| \leq \frac{\frac{\log n}{n}}{\Im z |Z_i(z)|} \lesssim \frac{\frac{\log n}{n}}{\Im z \cdot \frac{\log n}{\sqrt{n}}} < 1, \qquad \Im z \gtrsim \frac{1}{\sqrt{n}}$$

- Choose $a_n \approx \frac{1}{\sqrt{n}}$, $\varepsilon_n \approx \sqrt{\frac{\log n}{n}}$ and repeat the calculations for the cubic and quadratic functional equation with small modifications.

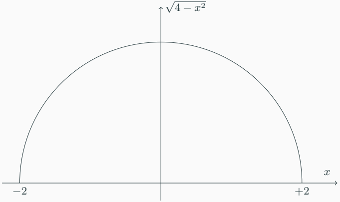
Free analogue of Klartag-Sodin

Theorem 3 (Free analogue of Klartag-Sodin)

Assume that $\operatorname{supp}\mu \subset [-L, L]$ holds for some $L > 0$. Let $\rho, \varepsilon \in (0,1)$. Then, there exists a set $\mathcal{F} \subset \mathbb{S}^{n-1}$ with $\sigma_{n-1}(\mathcal{F}) \geq 1 - \rho$ such that for all $\theta \in \mathcal{F}$ we have

$$\Delta_\varepsilon(\mu_\theta, \omega) = \sup_{x \in [-2+\varepsilon, 2-\varepsilon]} |\mu_\theta((-2+\varepsilon, x]) - \omega((-2+\varepsilon, x])| \lesssim \frac{\log n}{n}.$$

Note that Δ_ε guarantees that we stay away from the points $-2, 2$.



Free analogue of Klartag-Sodin - Overview

Overview of the proof:

1. Choose $\mathcal{F} \subset \mathbb{S}^{n-1}$. Fix $\theta \in \mathcal{F}$ with $\theta_1^2 = \min_{i\in[n]} \theta_i^2$.
   add new "condition"

2. Bound $|G_\theta|$ by constant.
   not with integration by parts

3. Establish constant lower bound on $|Z_i|$.

4. Relate Δ_ε to Cauchy transforms.
   replaces Bai's inequality

5. Derive and solve (only) cubic functional equation for Z_1.
   possible due to the restriction to $[-2+\varepsilon, 2-\varepsilon]$ in the definition of Δ_ε

Step 1: Choice of $\mathcal{F}$

- In the proofs before, we mainly used that

$$\max_{i\in[n]} |\theta_i| \lesssim \sqrt{\frac{\log n}{n}}, \qquad \sum_{i=1}^{n} |\theta_i|^k \lesssim \frac{1}{n^{\frac{k-2}{2}}}$$

  hold with high probability with respect to σ_{n-1} for $k \in \mathbb{N}, 2 < k \leq 7$.

- Now, we additionally use that

$$\left|\sum_{i=1}^{n} \theta_i^3\right| \lesssim \frac{1}{n}$$

  holds with high probability with respect to σ_{n-1}.

Step 2 + 3: Bounds on $|G_\theta|$ and $|Z_i|$

Let $d_L(\cdot,\cdot)$ denote the Lévy distance between two pm's and let $\omega_{1/2}$ have semicircular distribution with variance $\frac{1}{2}$.

Theorem (Bao, Erdős, Schnelli, 2016)

Let $\mathcal{I} \subset (-2, 2)$ be a compact non-empty interval and fix $\eta \in (0, \infty)$. Then, there exist constants $b = b(\omega_{1/2}, \mathcal{I}, \eta) > 0$ and $Z = Z(\omega_{1/2}, \mathcal{I}, \eta) < \infty$ such that whenever two pm's ν_1 and ν_2 on $\mathbb{R}$ satisfy

$$d_L(\omega_{1/2}, \nu_1) + d_L(\omega_{1/2}, \nu_2) \le b,$$

we have

$$\max_{x\in\mathcal{I}, y\in[0,\eta]} |G_\omega(x+iy) - G_{\nu_1\boxplus\nu_2}(x+iy)| \le Z\left(d_L(\omega_{1/2}, \nu_1) + d_L(\omega_{1/2}, \nu_2)\right).$$

Step 2 + 3: Bounds on $|G_\theta|$ and $|Z_i|$

- Define

$$\mu_\theta^1 := \mu_1 \boxplus \cdots \boxplus \mu_{M_n}, \qquad \mu_\theta^2 = \mu_{M_n+1} \boxplus \cdots \boxplus \mu_n$$

  with M_n chosen such that the variances of μ_θ^1 and μ_θ^2 are $\approx \frac{1}{2}$.
- Then, we can prove $d_L(\mu_\theta^i, \omega_{1/2}) \lesssim n^{-1/4}$, $i = 1, 2$.
- For sufficiently large n, we obtain

$$\max_{\substack{x \in [-2+\varepsilon, 2-\varepsilon], \\ y \in [0,4]}} |G_\omega(x+iy) - G_\theta(x+iy)| \lesssim \frac{1}{n^{1/4}}.$$

- For all $z \in \mathbb{C}^+$ with $\Re z \in [-2+\varepsilon, 2-\varepsilon]$, $\Im z \in (0, 4]$ and large n, we have

$$|G_\theta(z)| \leq C_\varepsilon, \qquad C_\varepsilon > 0.$$

- In particular: $|Z_i(z)| \geq (2C_\varepsilon)^{-1}$ for z, n as above and any $i \in [n]$.

Step 4: Relate Δ_ε to Cauchy transforms

- With Stieltjes-Perron inversion formula:

$$\Delta_\varepsilon(\mu_\theta, \omega) = \sup_{x\in[-2+\varepsilon, 2-\varepsilon]} \left| \lim_{\delta\downarrow 0} \frac{1}{\pi} \int_{-2+\varepsilon}^{x} \Im\left(G_\theta(u+i\delta) - G_\omega(u+i\delta)\right) du \right|$$

- Apply Cauchy's integral theorem:

$$\begin{aligned}\Delta_\varepsilon(\mu_\theta, \omega) \leq & \frac{1}{\pi}\int_{-2+\varepsilon}^{2-\varepsilon} |G_\theta(u+i) - G_\omega(u+i)|\, du \\ & + \sup_{x\in[-2+\varepsilon, 2-\varepsilon]} \frac{2}{\pi}\int_{a_n}^{1} |G_\theta(x+iv) - G_\omega(x+iv)|\, dv + I_n\end{aligned}$$

with $a_n := (\log n)^2 n^{-1}$ and I_n given by

$$I_n := \sup_{x\in[-2+\varepsilon, 2-\varepsilon]} \frac{2}{\pi}\int_{0}^{a_n} |G_\theta(x+iv) - G_\omega(x+iv)|\, dv$$

- For sufficiently large n, we know:

$$|I_n| \lesssim \frac{a_n}{n^{1/4}} \lesssim \frac{1}{n}$$

Step 5: Cubic functional equation for Z_1

It remains to derive and solve a cubic (not a quadratic!) functional equation for Z_1. After that, we apply the results to

$$\int_{-2+\varepsilon}^{2-\varepsilon} |G_\theta(u+i) - G_\omega(u+i)| \, du$$

and

$$\sup_{x \in [-2+\varepsilon, 2-\varepsilon]} \int_{a_n}^{1} |G_\theta(x+iv) - G_\omega(x+iv)| \, dv$$

leading to the desired rate of convergence.

Overview

Weighted sums in classical probability

Weighted sums in free probability

Outline of the proofs

Berry-Esseen type estimates in the free central limit theorem

Berry-Esseen type estimates in the free central limit theorem

Our approach generalizes to free additive convolutions of not necessarily equally distributed compactly supported probability measures.

Theorem 5 (N., 2023)

Let $\mu_1, \dots, \mu_n$ be probability measures on $\mathbb{R}$ with $\operatorname{supp} \mu_i \subset [-M_i, M_i]$ for $M_i > 0$. Assume that each μ_i has mean zero and variance $\sigma_i^2 \in (0, \infty)$. Define

$$B_n := \left(\sum_{i=1}^{n} \sigma_i^2 \right)^{1/2}, \qquad L_n := \frac{\sum_{i=1}^{n} M_i^3}{B_n^3}$$

and let $\mu_{\boxplus n} := D_{B_n^{-1}} \mu_1 \boxplus \dots \boxplus D_{B_n^{-1}} \mu_n$. Then, we have

$$\Delta(\mu_{\boxplus n}, \omega) \leq c L_n, \qquad c > 0.$$

Note: This improves upon the square root in the known rate $\left(B_n^{-3} \sum_{i=1}^{n} \beta_3(\mu_i) \right)^{1/2}$ at the cost of an increase in the nominator.

Thank you for your attention!

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