Around the Exact Formula for the Quasicentral Modulus

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Perturbations of Selfadjoint Operators

finite dimension

$T - T'$ small, all eigenvalues may change

infinite dimension, big difference

depending on what $T - T'$ "small" means

if "small" $\sim \in$ some ideal of $B(\mathcal{H})$

there may be conserved parts

$B(\mathcal{H})$ has non-trivial ideals.

Normed Ideals $(\mathcal{J}, |\ |_{\mathcal{J}})$

$$|AXB|_{\mathcal{J}} \leq \|A\| \, |X|_{\mathcal{J}} \, \|B\|$$

$$s_k(T) = \min_{\dim \mathcal{H}/\mathcal{X} = k} \|T|\mathcal{X}\|$$

$$|X|_{\mathcal{J}} = \Phi_{\mathcal{J}}(s_1(X), s_2(X), \ldots)$$

Schatten-v. Neumann $(\mathcal{C}_p, |\ |_p)$

$$|X|_p = \Big(\sum_k (s_k(X))^p\Big)^{1/p}$$

Lorentz $(p,1)$, $(\mathcal{C}_p^-, |\ |_p^-)$

$$|X|_p^- = \sum_k k^{-1+1/p} s_k(X)$$

$$\mathcal{C}_p^- \subset \mathcal{C}_p, \quad \mathcal{C}_1^- = \mathcal{C}_1$$

n-tuples of Commuting Selfadjoint Operators

(4)

$\tau = (T_1, \ldots, T_n)$, $\tau' = (T_1', \cdots, T_n')$, $T_j - T_j' \in \mathcal{J}$, $1 \le j \le n$

$\mathcal{J} = \mathcal{C}_n^- \implies \tau_{ac} = U \tau'_{ac} U^*$, U unitary

$\mathcal{J} \not\supseteq \mathcal{C}_n^- \implies \tau, \tau'$ can be mutually singular

$n=1$ Kato–Rosenblum, Weyl-v. Neumann-Kuroda

$n \geq 2$ V. different technique
adaptation of Voiculescu Thm. to normed ideals and use of quasicentral modulus

Kuroda $n \geq 2$ analogue Bercovici–V. short note

The Quasicentral Modulus

$k_{\mathcal{J}}(\tau)$, $\tau = (T_1, \dots, T_n)$ not necessarily commuting bdd. selfadjoint

$k_{\mathcal{J}}(\tau) =$ smallest $C \in [0, \infty]$ for which $\exists A_m \uparrow I$
$0 \le A_m \le I$, finite rank, $\max\limits_{1 \le j \le n} \| [A_m, T_j] \|_{\mathcal{J}} \xrightarrow[m \to \infty]{} C$

$\mathcal{J} = \mathcal{C}_p$, $k_p(\tau)$, $\mathcal{J} = \mathcal{C}_p^-$, $k_p^-(\tau)$

General properties:

1°. $p \longrightarrow k_p^-(\tau)$

decreasing function of $p \in [1, \infty]$.

2°. there is $p_0 \in [1,\infty]$ so that

$$p \in [1, p_0) \Longrightarrow k_p^-(\tau) = \infty$$

$$p \in (p_0, \infty] \Longrightarrow k_p^-(\tau) = 0$$

3°. $p > 1 \Longrightarrow k_p(\tau) \in \{0, \infty\}$

4°. $\tau - \tau' \in \mathcal{J} \Longrightarrow k_{\mathcal{J}}(\tau) = k_{\mathcal{J}}(\tau')$

assuming finite rank ops. dense in $\mathcal{J}$

5°.

$$k_{\mathcal{J}}(\tau) > 0 \Longleftrightarrow \left| \begin{array}{l} \exists\, Y_j = Y_j^* \in \mathcal{J}^{\text{dual}}, \ 1 \leq j \leq n \\ \underbrace{\operatorname{Tr} i \sum_j [T_j, Y_j]}_{\in} > 0 \\ \mathcal{C}_1 + \mathcal{B}(\mathcal{H})_+ \end{array} \right.$$

Exact Formula for n-tuples τ of commuting selfadjoint operators

$$\left(k_n^-(\tau)\right)^n = \gamma_n \int_{\mathbb{R}^n} \underbrace{m(s)}_{\text{multiplicity function}} \, d\overbrace{\lambda(s)}^{n\text{-dim Lebesgue}} \qquad (\mathrm{V.})$$

$0 < \gamma_n < \infty$, $\quad \gamma_1 = \frac{1}{\pi}$

Technical use in Connes' Spectral Characterization of Manifolds.

2 Recent Generalizations: Hybrid, Fractal

Hybrid Perturbations (V.)

8

$\tau = (T_j)_{1\le j\le n}$, $\tau' = (T_j')_{1\le j\le n}$

n-tuples of commuting selfadjoint

$T_j - T_j' \in \mathcal{J}_j$, $1 \le j \le n$

ideal varies with index j

general n-tuple

$k^-_{\mathcal{J}_1,\dots,\mathcal{J}_n}(\tau)$ = smallest $C \in [0,\infty]$, $\exists A_m$

$0 \le A_m \le I$, finite rank

$$\max_{1\le j\le n} |[A_m, T_j]|_{\mathcal{J}_j} \xrightarrow[m\to\infty]{} C$$

$k_{p_1,\dots,p_n}(\tau)$, $k^-_{p_1,\dots,p_n}(\tau)$.

$\frac{1}{p_1}+\cdots+\frac{1}{p_n}=1$, $1<p_j$, $1\le j\le n$

τ, τ' n-tuples of commuting selfadjoint

$T_j - T'_j \in \mathcal{C}^-_{p_j}$, $1\le j\le n \Longrightarrow \tau_{ac} = \underset{\text{unitary}}{U}\, \tau'_{ac}\, U^*$

ac w.r.t. Lebesgue

threshold hybrid setting

τ n-tuple of commuting selfadjoint

p_j as above

<u>exact formula</u>

$$\left(k^-_{p_1,\dots,p_n}(\tau)\right)^n = \gamma_{p_1\dots p_n} \int_{\mathbb{R}^n} m(s)\,d\lambda(s)$$

$$0<\gamma_{p_1,\dots,p_n}<\infty.$$

<u>singular integrals with mixed homogeneity</u>

<u>for the proof of</u> $\sigma_{p_1,\dots,p_n} > 0$ <u>if</u> $\frac{1}{p_1} + \dots + \frac{1}{p_n} = 1$

$p_j > 1,\ 1 \le j \le n$

Y_j operator in $L^2([-1,1]^n; d\lambda)$ with kernel

$$K_j(x,y) = \operatorname{sign}(x_j - y_j)\,|x_j - y_j|^{p_j - 1}\Big(\sum_{1\le k\le n} |x_k - y_k|^{p_k}\Big)^{-1}$$

<u>Fact</u> $Y_j \in (\mathcal{C}^-_{p_j})^{\text{dual}},\ 1 \le j \le n$

(Use : $\sum_{1\le k\le n} [\underbrace{M_{x_k}}_{\text{multiplication by } x_k}, Y_k] = \text{rank one} \ge 0$)

Beyond Integer Dimension

same abstract machinery works reducing problems to proving results about $k_{\mathcal{J}}(\tau)$

hard part: $k_{\mathcal{J}}(\tau) > 0$ results

Guy David – V. μ Radon meas. supp $\mu \Subset \mathbb{R}^n$, $p > 1$

$\mu(B(x,r)) \leq C r^p$, $\forall x \in \mathbb{R}^n$, $r \leq 1$

τ_μ n-tuple multiplication ops. by coordinate functns. in $L^2(\mathbb{R}^n, d\mu)$, then:

$k^-_p(\tau_\mu) > 0$

[essence: $\dfrac{x_j - y_j}{|x-y|^2}$ kernels of ops. in $(\mathcal{C}^-_p)^{dual}$ in $L^2(\mathbb{R}^n, d\mu)$]

Spectra on Fractals

replace $\mathbb{R}^n$ by fractal set $X \subseteq \mathbb{R}^n$

$\tau = (T_j)_{1 \le j \le n}$, $\tau' = (T_j')_{1 \le j \le n}$

n-tuples of commuting selfadjoint ops.

$\sigma(\tau) \subset X$, $\sigma(\tau') \subset X$

$T_j - T_j' \in \mathcal{J} \quad 1 \le j \le n$

Lebesgue meas. on $\mathbb{R}^n$ $\rightsquigarrow$ unique self-similar meas on certain self-similar X (Hutchinson meas. . .)

τ_{ac}, τ'_{ac} w.r.t. self-similar meas.

Hausdorff p-measure in examples considered

Cantor-like X (v.)

(13)

$F_i(x) = \lambda(x - b(i)) + b(i)$, $1 \le i \le N$, $x \in \mathbb{R}^n$

$0 < \lambda < 1$, $b(i) \in \mathbb{R}^n$

$$X = \bigcup_{1 \le i \le N} F_i X$$

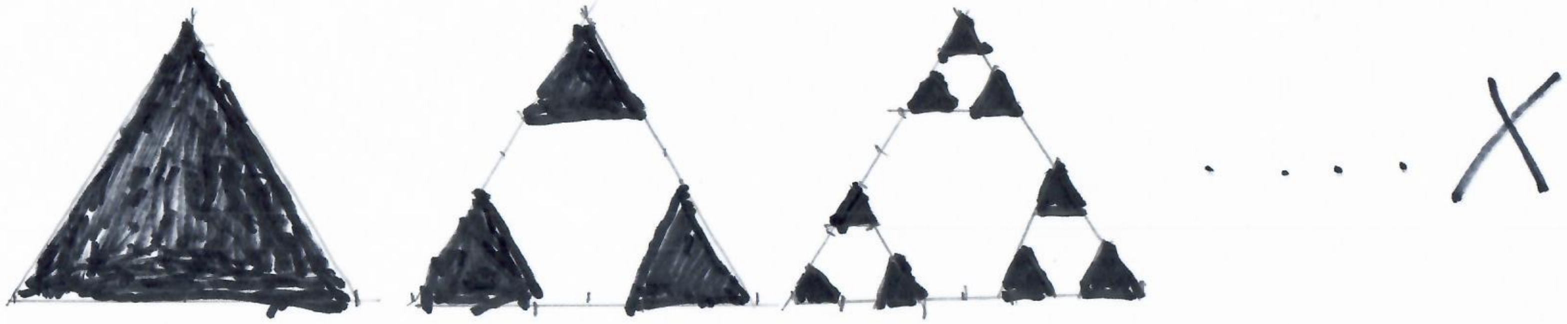
additionally

$$i_1 \ne i_2 \implies F_{i_1} X \cap F_{i_2} X = \emptyset$$

Dimension of X (Hausdorff.., box..)

$$p = \frac{\log N}{\log(1/\lambda)}$$

p-Hausdorff meas. on X (Hutchison meas. ...)



τ n-tuple of commuting selfadjoint

$\sigma(\tau) \subset X$

exact formula

$$\left(k_p^-(\tau)\right)^p = \gamma_X \int_X m(s)\, d\mathcal{H}_p(s)$$

$0 < \gamma_X < \infty$ $\quad p$ - Hausdorff dim of X

$\mathcal{H}_p$ - Hausdorff measure

important ingredient

- general ampliation homogeneity

$$k_p^-(\tau \otimes I_n) = n^{1/p} k_p^-(\tau)$$

$1 \leq p \leq \infty$, general τ.

- variant $$\tilde{k}_p^-(\tau \otimes I_n) = n^{1/p}\, \tilde{k}_p^-(\tau)$$

here $\tilde{k}_\gamma(\tau)$ defined by replacing

$$\max_{1\leq j\leq n} |[\cdots]|_\gamma \rightsquigarrow \left|\begin{pmatrix} [\ ] \\ \vdots \\ [\ ] \end{pmatrix}\right|_\gamma$$

Ikeda - Izumi improvements (beyond totally disconnected X)

1°. Exact result

Assume only

$F_i(x) = \lambda(x - b(i)) + b(i)$, $1 \le i \le N$, $x \in \mathbb{R}^n$

$0 < \lambda < 1$, $b(i) \in \mathbb{R}^n$

$$X = \bigcup_{1 \le i \le N} F_i X$$

and $p =$ Hausdorff dim $X > 1$ and

Open set condition: $\exists\, \emptyset \neq V \underset{\text{open bdd.}}{\subset} \mathbb{R}^n$

$$\coprod_{1 \le j \le n} F_j V \subset V.$$

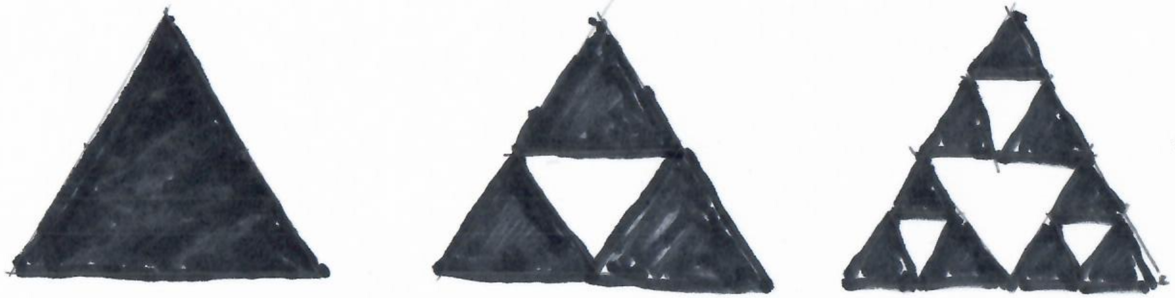
Under above assumptions

$$(\bar{k}_p(\tau))^p = \gamma_X \int_X m(x)\, d\mathcal{H}_p(x)$$

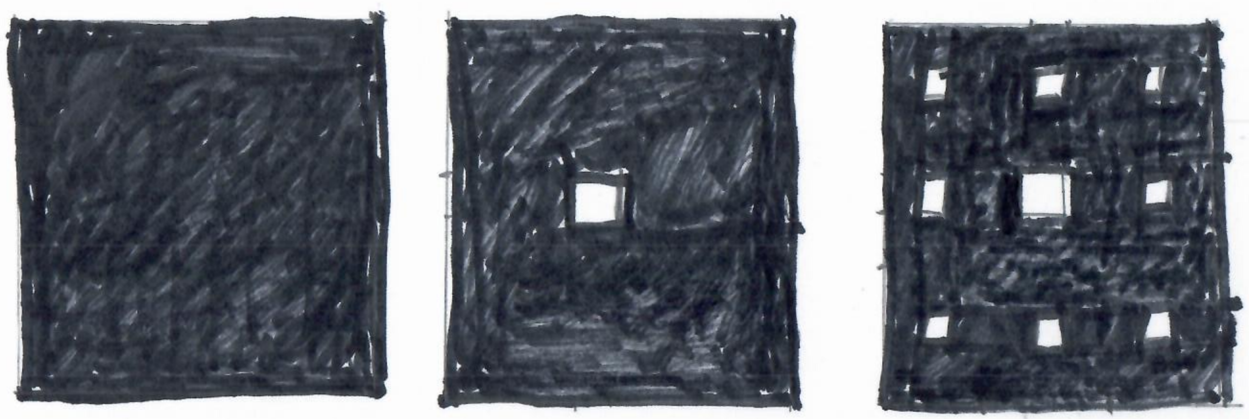
if $\sigma(\tau) \subset X$.

In particular: X can be the Sierpinski gasket or the Sierpinski carpet.

Similar result for $\widetilde{\bar{k}_p}(\tau)$.



.... Sierpinski Gasket



.... Sierpinski Carpet

2°. Upper and lower bounds

$$F_j(x) = \lambda_j U_j (x - b(j)) + b(j), \quad 1 \le j \le N, \quad x \in \mathbb{R}^n$$

$$0 < \lambda_j < 1, \quad U_j \in \mathcal{O}(n), \quad b(j) \in \mathbb{R}^n$$

$$p = \text{Hausdorff dim } X > 1, \quad \sum_{1 \le j \le N} \lambda_j^p = 1.$$

and Open set condition:

$$\exists\, \phi \ne \underset{\text{open bdd.}}{V} \subset \mathbb{R}^n$$

$$\coprod_{1 \le j \le n} F_j V \subset V.$$

Under above assumptions there are constants C_1, C_2 determined by $F_1, \dots, F_N$ so that if $\sigma(\tau) \subset X$

$$C_1 \int_X m(x)\, d\mathcal{H}_p(x) \le \left(k_p^-(\tau)\right)^p \le C_2 \int_X m(x)\, d\mathcal{H}_p(x)$$

Similar result for $\tilde{k}_p^-(\tau)$.

Ref - 1

References

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