Conditional Free Entropy: $\chi \leq \chi^*$

(Part 2!)

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Definitions 0

For $i = 1, \ldots, m$, define the partial non-commutative derivatives ∂_i as linear maps:

$$\partial_i : \mathbb{C}\langle x_1, \ldots, x_m \rangle \to \mathbb{C}\langle x_1, \ldots, x_m \rangle \otimes \mathbb{C}\langle x_1, \ldots x_m \rangle \text{ by}$$

$$\partial_i 1 = 0, \qquad \partial_i x_j = \delta_{ij} 1 \otimes 1 \quad (\text{for } j = 1, \ldots, m),$$

and extend by the Leibniz rule, so on monomials ∂_i is given by

$$\partial_i(x_{i(1)} \cdots x_{i(\ell)}) = \sum_{k=1}^{\ell} \delta_{i,i(k)} x_{i(1)} \cdots x_{i(k-1)} \otimes x_{i(k+1)} \cdots x_{i(\ell)}.$$

Also extend ∂_i for $i = 1, \ldots, m$ to $\mathbb{C}\langle x_1, \ldots, x_m, y_1, y_2, \ldots \rangle$.

Definitions I

An element $\xi = (\xi_1, \dots, \xi_m) \in L^1(W^*(\mathcal{A}\langle \mathbf{X} \rangle))^m$ is the free score function of $\mathbf{X}$ with respect to $\mathcal{A}$ (also called the conjugate variable of $\mathbf{X}$ w.r.t. $\mathcal{A}$), if for all $1 \leq i \leq m$,

$$\tau(\xi_i p(\mathbf{X})) = \tau \otimes \tau(\partial_i p(\mathbf{X})) \text{ for all } p(x) \in \mathcal{A}\langle x_1, \dots, x_m \rangle .$$

If the conjugate variable $\xi \in L^2(\mathcal{M})^m$, then we define the relative free Fisher information of $\mathbf{X}$ with respect to $\mathcal{A}$ to be

$$\Phi^*(\mathbf{X} \mid \mathcal{A}) = \|\xi\|_2^2 = \sum_{1 \leq j \leq m} \|\xi_j\|_2^2 .$$

Otherwise, we define $\Phi^*(\mathbf{X} \mid \mathcal{A}) = \infty$. When $\mathbf{Y}$ generates $\mathcal{A}$, we also write $\Phi^*(\mathbf{X} \mid \mathbf{Y})$.

Definitions II

Define the relative non-microstates free entropy of $\mathbf{X}$ with respect to $\mathcal{A} = W^*(\mathbf{Y})$ as

$$\chi^*(\mathbf{X} \mid \mathbf{Y}) = \frac{1}{2}\int_0^\infty \left(\frac{m}{1+t} - \Phi^*(\mathbf{X} + t^{1/2}\mathbf{S} \mid \mathbf{Y})\right) dt + \frac{m}{2}\log 2\pi e,$$

where $\mathbf{S} = (S_1, \ldots, S_m)$ is an m-tuple of freely independent standard semicircular variables which are also free from $\mathcal{A}\langle \mathbf{X} \rangle$.

Free Fisher Information

Theorem (Lemma)

Let $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ *be tuple from* $(\mathcal{M}, \tau)_{sa}$. *Then*

$$\Phi^*(\mathbf{X} \mid \mathcal{A}) = \sup_{\|f(\mathbf{X})\|_2 \leq 1} \{|\tau \otimes \tau(\partial f(\mathbf{X}))|^2 \ : \ f(x) \in \mathcal{A}\langle x_1, \ldots, x_m\rangle^m\}.$$

Idea of Proof:

$$\begin{aligned}\Phi^*(\mathbf{X} \mid \mathcal{A}) = \|\xi\|_2^2 &= \sup\{|\langle \xi, f(\mathbf{X})\rangle|^2 \ : \ \|f(\mathbf{X})\|_2 \leq 1\} \\ &= \sup\{|\tau \otimes \tau(\partial f(\mathbf{X}))|^2 \ : \ \|f(\mathbf{X})\|_2 \leq 1\}.\end{aligned}$$

Free & Classical Fisher Info.: Relationship

Usual assumptions apply:
$\mathbf{X} \in \mathcal{M}_{sa}^{m}$,
$\mathbf{Y}^{(n)} \rightsquigarrow \mathbf{Y}$ a fixed sequence of microstates.

Theorem (Jekel, P.)

Suppose that $\mathbf{X}^{(n)} \in M_n(\mathbb{C})_{sa}^{m}$ is a sequence of random matrix tuples with finite moments such that the law of $(\mathbf{X}^{(n)}, \mathbf{Y}^{(n)})$ converges in probability to the law of $(\mathbf{X}, \mathbf{Y})$ and for each k, we have $\lim_{n\to\mathcal{U}} E\mathrm{tr}_n((X_j^{(n)})^{2k}) < \infty$ and $\lim_{n\to\mathcal{U}} E\mathrm{tr}_n((Y_j^{(n)})^{2k}) < \infty$.
Then,

$$\Phi^*(\mathbf{X} \mid \mathbf{Y}) \leq \lim_{n\to\mathcal{U}} \frac{1}{n^4} \mathcal{I}(\mathbf{X}^{(n)}).$$

Main Result

Theorem (Jekel, P.)

Let $\mathbf{X} = (X_1, \ldots, X_m)$ be an m-tuple of self-adjoint elements in $(\mathcal{M}, \tau)$, and fix a tuple of generators $\mathbf{Y}$ for a von Neumann subalgebra $\mathcal{A} \subseteq \mathcal{M}$. Fix a sequence $\mathbf{Y}^{(n)} \in M_n(\mathbb{C})_{sa}^{\mathbb{N}}$ such that $\mathbf{Y}^{(n)} \rightsquigarrow \mathbf{Y}$. Then for any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$:

$$\chi^{\mathcal{U}}(\mathbf{X} \mid \mathbf{Y}^{(n)} \rightsquigarrow \mathbf{Y}) \leq \chi^*(\mathbf{X} \mid \mathbf{Y}).$$

Hence, for $\chi^{\mathcal{U}}(\mathbf{X} \mid \mathcal{A}) := \sup_{\mathbf{Y}^{(n)} \rightsquigarrow \mathbf{Y}} \chi^{\mathcal{U}}(\mathbf{X} \mid \mathbf{Y}^{(n)} \rightsquigarrow \mathbf{Y})$,

$$\chi^{\mathcal{U}}(\mathbf{X} \mid \mathcal{A}) \leq \chi^*(\mathbf{X} \mid \mathcal{A}).$$

Proof Sketch

1. Fix uniformly bounded $\mathbf{X}^{(n)}$ so that $(\mathbf{X}^{(n)}, \mathbf{Y}^{(n)}) \rightsquigarrow (\mathbf{X}, \mathbf{Y})$. Recall:

$$\chi^{\mathcal{U}}(\mathbf{X} \mid \mathbf{Y}^{(n)} \rightsquigarrow \mathbf{Y}) = \lim_{n \to \mathcal{U}} h^{(n)}(\mathbf{X}^{(n)}), \quad \text{and}$$

$$\chi^*(\mathbf{X} \mid \mathbf{Y}) = \frac{1}{2} \int_0^\infty \left(\frac{m}{1+t} - \Phi^*(\mathbf{X} + t^{1/2}\mathbf{S} \mid \mathbf{Y}) \right) dt + \frac{m}{2} \log 2\pi e.$$

2. Relate the integral of Φ^* with some quantity involving $h^{(n)}(\mathbf{X}^{(n)})$.

3. Get an upper bound for $h^{(n)}(\mathbf{X}^{(n)})$ involving $\int_0^t \Phi^*(\mathbf{X} + u^{1/2}\mathbf{S} \mid \mathbf{Y})\, du$ (with some appropriate normalizations so that the upper bound converges as $t \to \infty$).

Proof Step 2

Set $\mathbf{S}^{(n)} \rightsquigarrow \mathbf{S}$, an m-tuple of freely independent semicircular random variables, also freely independent from $\mathbf{X}$ and $\mathbf{Y}$.

Apply the relationship between free and classical Fisher information:

$$\Phi^*(\mathbf{X} + u^{1/2}\mathbf{S} \mid \mathbf{Y}) \leq \lim_{n \to \mathcal{U}} \frac{1}{n^4} \mathcal{I}(\mathbf{X}^{(n)} + u^{1/2}\mathbf{S}^{(n)}).$$

Take $\frac{1}{2}\int_0^t (\cdot)$ on both sides and use relation between classical entropy and Fisher information, obtain:

$$\frac{1}{2}\int_0^t \Phi^*(\mathbf{X} + u^{1/2}\mathbf{S} \mid \mathbf{Y})\, du \leq \lim_{n \to \mathcal{U}} \frac{1}{n^2} \left(h(\mathbf{X}^{(n)} + t^{1/2}\mathbf{S}^{(n)}) - h(\mathbf{X}^{(n)}) \right).$$

Step 3

From RHS of Step 2: $\frac{1}{n^2}\left(h(\mathbf{X}^{(n)}+t^{1/2}\mathbf{S}^{(n)})-h(\mathbf{X}^{(n)})\right)$

Add 0 and apply subadditivity of entropy to the first term:

$$\frac{1}{n^2}\left(h(\mathbf{X}^{(n)}+t^{1/2}\mathbf{S}^{(n)})-h(\mathbf{X}^{(n)})\right)$$
$$\leq \frac{m}{2}\log\left(E\left\|\mathbf{X}^{(n)}\right\|_2^2/m+t\right)+\frac{m}{2}\log(2\pi e)-\left(\frac{1}{n^2}h(\mathbf{X}^{(n)})+m\log n\right).$$

Isolate $h^{(n)}(\mathbf{X}^{(n)})$:

$$h^{(n)}(\mathbf{X}^{(n)}) \leq \frac{m}{2}\log\left(E\left\|\mathbf{X}^{(n)}\right\|_2^2/m+t\right)+\frac{m}{2}\log(2\pi e)$$
$$-\frac{1}{n^2}\left(h(\mathbf{X}^{(n)}+t^{1/2}\mathbf{S}^{(n)})-h(\mathbf{X}^{(n)})\right)$$

Combine Steps 2 and 3

Step 2:

$$\frac{1}{2}\int_0^t \Phi^*(\mathbf{X} + u^{1/2}\mathbf{S} \mid \mathbf{Y})\, du \le \lim_{n\to\mathcal{U}} \frac{1}{n^2}\left(h(\mathbf{X}^{(n)} + t^{1/2}\mathbf{S}^{(n)}) - h(\mathbf{X}^{(n)})\right)$$

Step 3:

$$h^{(n)}(\mathbf{X}^{(n)}) \le \frac{m}{2}\log\left(E\left\|\mathbf{X}^{(n)}\right\|_2^2/m + t\right) + \frac{m}{2}\log(2\pi e) - \frac{1}{n^2}\left(h(\mathbf{X}^{(n)} + t^{1/2}\mathbf{S}^{(n)}) - h(\mathbf{X}^{(n)})\right)$$

$$\lim_{n\to\mathcal{U}} h^{(n)}(\mathbf{X}^{(n)}) \le \lim_{n\to\mathcal{U}}\left(\frac{m}{2}\log\left(E\left\|\mathbf{X}^{(n)}\right\|_2^2/m + t\right)\right) + \frac{m}{2}\log(2\pi e) - \frac{1}{2}\int_0^t \Phi^*(\mathbf{X} + u^{1/2}\mathbf{S} \mid \mathbf{Y})\, du$$

Finishing Up

$$\chi^{\mathcal{U}}(\mathbf{X} \mid \mathbf{Y}^{(n)} \rightsquigarrow \mathbf{Y}) = \lim_{n\to\mathcal{U}} h^{(n)}(\mathbf{X}^{(n)})$$

$$\leq \lim_{n\to\mathcal{U}} \left(\frac{m}{2} \log \left(\frac{E\left\|\mathbf{X}^{(n)}\right\|_2^2}{m} + t \right) \right) + \frac{m}{2}\log(2\pi e) - \frac{1}{2}\int_0^t \Phi^*(\mathbf{X} + u^{1/2}\mathbf{S} \mid \mathbf{Y})\, du$$

$$= \frac{1}{2}\int_0^t \left(\frac{m}{1+u} - \Phi^*(\mathbf{X} + u^{1/2}\mathbf{S} \mid \mathbf{Y}) \right) du$$

$$+ \lim_{n\to\mathcal{U}} \left(\frac{m}{2} \left(\log \left(\frac{E\left\|\mathbf{X}^{(n)}\right\|_2^2}{m} + t \right) - \log(1+t) \right) \right) + \frac{m}{2}\log(2\pi e)$$

$$\xrightarrow{t\to\infty} \frac{1}{2}\int_0^\infty \left(\frac{m}{1+u} - \Phi^*(\mathbf{X} + u^{1/2}\mathbf{S} \mid \mathbf{Y}) \right) du + \frac{m}{2}\log(2\pi e)$$

$$= \chi^*(\mathbf{X} \mid \mathbf{Y}).$$

Finishing Up

$$\chi^{\mathcal{U}}(\mathbf{X} \mid \mathbf{Y}^{(n)} \rightsquigarrow \mathbf{Y}) = \lim_{n\to\mathcal{U}} h^{(n)}(\mathbf{X}^{(n)})$$

$$\leq \lim_{n\to\mathcal{U}} \left(\frac{m}{2}\log\left(\frac{E\left\|\mathbf{X}^{(n)}\right\|_2^2}{m}+t\right)\right)+\frac{m}{2}\log(2\pi e)-\frac{1}{2}\int_0^t \Phi^*(\mathbf{X}+u^{1/2}\mathbf{S} \mid \mathbf{Y})\,du$$

$$= \frac{1}{2}\int_0^t \left(\frac{m}{1+u} - \Phi^*(X+u^{1/2}\mathbf{S} \mid \mathbf{Y})\right)du$$

$$+ \lim_{n\to\mathcal{U}} \left(\frac{m}{2}\left(\log\left(\frac{E\left\|\mathbf{X}^{(n)}\right\|_2^2}{m}+t\right) - \log(1+t)\right)\right) + \frac{m}{2}\log(2\pi e)$$

$$\xrightarrow{t\to\infty} \frac{1}{2}\int_0^\infty \left(\frac{m}{1+u} - \Phi^*(\mathbf{X}+u^{1/2}\mathbf{S} \mid \mathbf{Y})\right)du + \frac{m}{2}\log(2\pi e)$$

$$= \chi^*(\mathbf{X} \mid \mathbf{Y}).$$

Finishing Up

$$\chi^{\mathcal{U}}(\mathbf{X} \mid \mathbf{Y}^{(n)} \rightsquigarrow \mathbf{Y}) = \lim_{n\to\mathcal{U}} h^{(n)}(\mathbf{X}^{(n)})$$

$$\leq \lim_{n\to\mathcal{U}} \left(\frac{m}{2} \log \left(\frac{E \left\| \mathbf{X}^{(n)} \right\|_2^2}{m} + t \right) \right) + \frac{m}{2} \log(2\pi e) - \frac{1}{2} \int_0^t \Phi^*(\mathbf{X} + u^{1/2}\mathbf{S} \mid \mathbf{Y})\, du$$

$$= \frac{1}{2} \int_0^t \left(\frac{m}{1+u} - \Phi^*(X + u^{1/2}\mathbf{S} \mid \mathbf{Y}) \right) du$$

$$+ \lim_{n\to\mathcal{U}} \left(\frac{m}{2} \left(\log \left(\frac{E \left\| \mathbf{X}^{(n)} \right\|_2^2}{m} + t \right) - \log(1+t) \right) \right) + \frac{m}{2} \log(2\pi e)$$

$$\xrightarrow{t\to\infty} \frac{1}{2} \int_0^\infty \left(\frac{m}{1+u} - \Phi^*(\mathbf{X} + u^{1/2}\mathbf{S} \mid \mathbf{Y}) \right) du + \frac{m}{2} \log(2\pi e)$$

$$= \chi^*(\mathbf{X} \mid \mathbf{Y}).$$

Conclusion

Main Result

An elementary proof of $\chi^{\mathcal{U}}(\mathbf{X} \mid \mathcal{A}) \leq \chi^*(\mathbf{X} \mid \mathcal{A})$

Other Things to Take Away

Relationships between classical and free conditional entropy and Fisher information:

1. $\chi^{\mathcal{U}}(\mathbf{X} \mid \mathbf{Y}^{(n)} \rightsquigarrow \mathbf{Y})$ is the supremum of $\lim_{n \to \mathcal{U}} h^{(n)}(\mathbf{X}^{(n)})$ over sufficiently nice microstates $\mathbf{X}^{(n)}$ satisfying $(\mathbf{X}^{(n)}, \mathbf{Y}^{(n)}) \rightsquigarrow (\mathbf{X}, \mathbf{Y})$.
2. $\Phi^*(\mathbf{X} \mid \mathbf{Y}) \leq \lim_{n \to \mathcal{U}} n^{-4} \mathcal{I}(\mathbf{X}^{(n)})$ over sufficiently nice $\mathbf{X}^{(n)}$ with $\operatorname{law}(\mathbf{X}^{(n)}, \mathbf{Y}^{(n)}) \to \operatorname{law}(\mathbf{X}, \mathbf{Y})$ in probability.