Methods of holomorphic dynamics in the study of branching processes

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based on a joint work (still in progress) with

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Outline of the talk

- Simplest example: Galton–Watson processes
- Continuous-time inhomogeneous processes and evolution families of holomorphic self-maps
- Loewner’s Parameteric Representation method in Conformal Mapping
- Modern Loewner Theory due to F. Bracci, M.D. Contreras, and S. Díaz-Madrigal
- REMARK: evolution families in Quantum Probabilities
- CB-processes and evolution families of Bernstein functions
- Differentiability problem
- Probabilistic interpretation of the Denjoy–Wolff point
- Spatial embedding
Galton–Watson process

is a Markov chain \((X_n)\), where \(X_n\) is the number of identical "particles" in the \(n\)-th generation, \(X_0 = 1\).

Each particle splits giving rise to \(k \in \mathbb{N}_0\) offsprings with probability \(p(k)\) independently from others (and prehistory).

Probability generating function

\[
F(z) := \mathbb{E}[z^{X_1}] = \sum_{k=0}^{+\infty} p(k)z^k
\]

is a holomorphic self-map of \(\mathbb{D} := \{z : |z| < 1\}\) with a boundary fixed point at 1 (except for the degenerate case \(p(0) = 1, p(k) = 0, k \in \mathbb{N}\)).

Relation to Dynamics:

If \(p(k)\)'s do not change in time, then the probability distribution of \(X_n\) is given by the Taylor coefficients of the \(n\)-th iterate of the function \(F\),

\[
F^n := F \circ \ldots \circ F : \mathbb{D} \rightarrow \mathbb{D}.
\]

\(n\) times
To a Galton–Watson process with continuous time one associates the family

\[(F_t)_{t \in \mathbb{R}_0} \subset \text{Hol}(\mathbb{D}, \mathbb{D}), \quad F_t(z) := \mathbb{E}[z^{X_t}]\]

(i) \(F_0 = \text{id}_{\mathbb{D}}\);
(ii) \(F_t \circ F_s = F_{t+s}\) for any \(t, s \in \mathbb{R}_0\);

Under a mild continuity assumption on the transition probabilities \(p_{s,t}(k)\):
(iii) as \(t \to 0^+\), \(F_t \to \text{id}_{\mathbb{D}}\) pointwise and hence locally uniformly in \(\mathbb{D}\).

A family \((F_t)_{t \in \mathbb{R}_0} \subset \text{Hol}(\mathbb{D}, \mathbb{D})\) satisfying (i) – (iii) is usually called a one-parameter semigroup in \(\mathbb{D}\).

**Time-inhomogeneous case**

Prob’ty generating f’ns: \((F_{s,t})_{t \geq s \geq 0} \subset \text{Hol}(\mathbb{D}, \mathbb{D}), \quad F_{s,t}(z) := \mathbb{E}[z^{X_t} \mid X_s = 1]\),
(i) \(F_{s,s} = \text{id}_{\mathbb{D}}\) for any \(s \geq 0\);
(ii) \(F_{s,u} \circ F_{u,t} = F_{s,t}\) whenever \(0 \leq s \leq u \leq t\);

Again, under a mild continuity assumption on \(p_{s,t}(k)\)’s:
(iii) \((s, t) \in \mathbb{R}_0 : s \leq t\} =: \Delta \ni (s, t) \mapsto F_{s,t} \in \text{Hol}(\mathbb{D}, \mathbb{D})\) is continuous.
A family \((F_{s,t})_{(s,t) \in \Delta} \subset \text{Hol}(\mathbb{D}, \mathbb{D})\) is said to be a \textit{topological reverse evolution family} if the above 3 conditions hold:

(i) \(F_{s,s} = \text{id}_\mathbb{D}\);  
(ii) \(F_{s,u} \circ F_{u,t} = F_{s,t}\) whenever \(0 \leq s \leq u \leq t\);  
(iii) \((s, t) \mapsto F_{s,t}\) is continuous.

Special feature of the homogeneous case (E. Berkson, H. Porta, 1978)

If \((F_t)\) is a one-parameter semigroup, then

\[
\frac{dF_t(z)}{dt} = G(F_t(z)), \quad t \geq 0, \quad F_0(z) = z, 
\tag{\text{*}}
\]

for a suitable \(G \in \text{Hol}(\mathbb{D}, \mathbb{C})\) called the \textit{infinitesimal generator} of \((F_t)\).

The infinitesimal generators form a convex cone \(\text{Gen}(\mathbb{D}) \subset \text{Hol}(\mathbb{D}, \mathbb{C})\).

Inhomogeneous extension of (\text{*}) \ [from the Dynamics viewpoint]

\[
\frac{dF_{s,t}(z)}{dt} = G(F_{s,t}(z), t), \quad t \geq s \geq 0, \quad F_{s,s}(z) = z; 
\]

\(G(\cdot, t) \in \text{Gen}(\mathbb{D})\) for a.e. \(t \geq 0\). \[(\text{Generalized) Loewner–Kufarev ODE}\]
The Loewner–Kufarev ODE generates

*absolutely continuous evolution families:*

\[(F_{s,t}) \subset \text{Hol}(\mathbb{D}, \mathbb{D})\]

(i) \(F_{s,s} = \text{id}_\mathbb{D}\);  
(ii) \(F_{u,t} \circ F_{s,u} = F_{s,t}\) whenever \(0 \leq s \leq u \leq t\);  
(iii) stronger than continuity of \((s, t) \mapsto F_{s,t}\).

\[\begin{align*}
\text{One-parameter semigroups:} & \quad \text{E. Schröder, 1871; G. Koenigs, 1884; E. Berkson, H. Porta, 1978:} \\
G \in \text{Gen}(\mathbb{D}) & \iff G(z) = (\tau - z)(1 - \bar{\tau}z)p(z), \tau \in \overline{\mathbb{D}}, \text{Re } p \geq 0.
\end{align*}\]

\[\begin{align*}
\text{Evolution families:} & \quad \text{Ch. Loewner, Math. Ann. (1923)} \\
\mathcal{S} & := \left\{ f : \mathbb{D} \to \mathbb{C} \text{ injective holomorphic with } f(z) = z + a_2z^2 + \ldots \right\}
\end{align*}\]

A dense subclass \(\mathcal{S}_{sl}\) is formed by *slit mappings*, i.e. by those \(f \in \mathcal{S}\) for which \(\Gamma := \overline{\mathbb{C}} \setminus f(\mathbb{D})\) is a Jordan arc with one end-point at \(\infty\).
Consider a slit mapping \( f \in S_{s1}, \Gamma := \mathbb{C} \setminus f(\mathbb{D}). \)

- Choose a parametrization \( \gamma : [0, +\infty] \to \Gamma, \gamma(+\infty) = \infty. \)
- Consider the domains \( \Omega_t := \mathbb{C} \setminus \gamma([t, +\infty]), t \geq 0. \)
- By Riem’s Mapping Th’m \( \forall t \geq 0 \exists! \text{ conformal mapping} \)
  \[ f_t : \mathbb{D} \rightarrow \Omega_t, \quad f_t(0) = 0, \ f'_t(0) > 0. \]

- Note that \( f_0 = f \)
- and that \( \Omega_s = f_s(\mathbb{D}) \subset f_t(\mathbb{D}) = \Omega_t \) whenever \( 0 \leq s \leq t. \)
- Reparameterizing \( \Gamma: \forall t \geq 0 \quad f'_t(0) = e^t. \)

THEN: \( \varphi_{s,t} := f_t^{-1} \circ f_s \in \text{Hol}(\mathbb{D}, \mathbb{D}), \ 0 \leq s \leq t, \)
are \( C^1 \) in \((s, t)\) and form an evolution family \( (\varphi_{s,t}). \)
**Theorem (Ch. Loewner, 1923)** \( f, (f_t), (\varphi_{s,t}) \) as above

\[ \exists! \text{ continuous function } \xi : [0, +\infty) \to \partial \mathbb{D} \text{ s.t. for any } z \in \mathbb{D} \text{ and } s \geq 0, \]

\[ w = w_{z,s}(t) := \varphi_{s,t}(z) \text{ solves the IVP for the } \text{Loewner ODE} \]

\[ \frac{dw}{dt} = -w \frac{1 + \xi(t)w}{1 - \xi(t)w}, \quad t \geq s; \quad w(s) = z. \]

Moreover,

\[ f_s = \lim_{t \to +\infty} e^t \varphi_{s,t} \quad \text{for any } s \geq 0. \quad \text{NB: } f = f_0 \]

**NB:** a sort of converse is also true \( \mathfrak{Z} \) **parametric representation**

of \( S', \ S_{s1} \subset S' \subset S. \)

**Extension to the whole class** \( S: \) the **classical Loewner–Kufarev ODE**

P. P. Kufarev, 1943; Ch. Pommerenke, 1965; V. Ja. Gutljanskii, 1970:

\[ \frac{dw}{dt} = -w(t) p(w(t), t), \quad t \geq s; \quad w(s) = z; \]

\( \text{Re } p \geq 0, \ p(0, t) = 1, \text{ measurable in } t, \text{ not } C^1 \) — Carathéodory’s ODE!
Introduction - 7 - Chordal Loewner ODE

Hydrodynamic normalization:

\[ f_t(\zeta) = \zeta - \frac{c(t)}{\zeta} + o(1/\zeta) \quad \text{(HD)} \]

as \( H \ni \zeta \to \infty, \ c(t) \geq 0; \)

The evolution family

\[ \varphi_{s,t} := H^{-1} \circ (f_t^{-1} \circ f_s) \circ H, \]

\( \mathbb{D} \ni z \mapsto \zeta = H(z) := i \frac{1+z}{1-z}, \)

satisfies the chordal L. – K. ODE

\[ \frac{dw}{dt} = (1 - w)^2 p(w, t). \]

Berkson – Porta:

\[ G(z) = (\tau - z)(1 - \overline{\tau}z) p(z), \quad \text{Re } p \geq 0, \ \tau \in \overline{\mathbb{D}}. \]

The r. h. s.’s \( G_{\text{cla}}(w, t) = -w p(w, t) \) and \( G_{\text{cho}} = (1 - w)^2 p(w, t) \)

are \( t \)-dependent infinitesimal generators, with \( \tau := 0 \) and \( \tau := 1. \)
What is the meaning of $\tau$ in B.–P.’s $G(z) = (\tau - z)(1 - \bar{\tau}z)p(z)$?

**Theorem (Denjoy and Wolff)**

\[ \forall \varphi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\text{id}_\mathbb{D}\}, \exists! \tau \in \mathbb{D} \text{, called the Denjoy – Wolff point of } \varphi, \text{ s.t.:} \]

- if $\tau \in \mathbb{D}$, then $\varphi(\tau) = \tau$, and $\varphi^n \overset{n \to +\infty}{\longrightarrow} \tau$ l.u. in $\mathbb{D}$ if $\varphi \notin \text{Aut}(\mathbb{D})$;
- if $\tau \in \partial\mathbb{D}$, then $\angle \lim_{z \to \tau} \varphi(z) = \tau$ and $\varphi^n \overset{n \to +\infty}{\longrightarrow} \tau$ l.u. in $\mathbb{D}$.

**Convention:** for $\varphi = \text{id}_\mathbb{D}$, every $\tau \in \mathbb{D}$ is its DW-point.

- The point $\tau$ in B.–P.’s formula is the DW-point of all $\phi_t$’s in the one-parameter semigroup $(\phi_t) \sim G$.
- $\tau = 0$ is the DW-point of evolution families $(\varphi_{s,t})$ in Loewner’s classical construction.
- $\tau = 1$ is the DW-point in the chordal version of Loewner’s construction.
In 2012, F. Bracci, M.D. Contreras, and S. Díaz-Madrigal proposed

generalized L.–K. ODE: \[
\frac{dw}{dt} = G(w(t), t), \quad t \geq s; \quad w(s) = z \in \mathbb{D},
\]

with \( G(z, t) := (\tau(t) - z)(1 - \overline{\tau(t)}z) p(z, t), \quad \text{Re} \ p \geq 0, \quad \tau : [0, +\infty) \to \mathbb{D}. \)

More precisely, they assumed that \( G \) is a Herglotz vector field:

a function \( G : \mathbb{D} \times [0, +\infty) \to \mathbb{C} \) is called a **Herglotz vector field**, if:

**HVF1**: for a.e. \( t \geq 0 \), \( G(\cdot, t) \in \text{Gen}(\mathbb{D}) \);

**HVF2**: for each \( z \in \mathbb{D} \), \( G(z, \cdot) \) is measurable on \([0, +\infty)\);

**HVF3**: for each compact \( K \subset \mathbb{D} \),

\[ \exists \text{ a loc. integrable } M : [0, +\infty) \to [0, +\infty) \text{ s.t.} \]

\[ \max_K |G(\cdot, t)| \leq M(t) \quad \text{for a.e. } t \geq 0. \]
F. Bracci, M.D. Contreras, and S. Díaz-Madrigal proved that the generalized Loewner–Kufarev ODE establishes a one-to-one correspondence between Herglotz vector fields and \textit{abs. continuous evolution families}:

\[
(\varphi_{s,t})_{t \geq s \geq 0} \subset \text{Hol}(D, \bar{D}) \quad \text{is an absolutely continuous evolution family if:}
\]

- \textbf{EF1:} \( \varphi_{s,s} = \text{id}_D \) for all \( s \geq 0 \);
- \textbf{EF2:} \( \varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}, \ 0 \leq s \leq u \leq t \);
- \textbf{EF3:} for every \( z \in D \), \exists \ a locally integrable \( k_z : [0, +\infty) \rightarrow [0, +\infty) \) s.t.

\[
|\varphi_{s,t}(z) - \varphi_{s,u}(z)| \leq \int_u^t k_z(v) \, dv \quad \text{whenever } 0 \leq s \leq u \leq t.
\]

- If \textbf{EF2} is replaced by \( \varphi_{s,t} = \varphi_{s,u} \circ \varphi_{u,t} \), then we get the definition of an (absolutely continuous) \textit{reverse} evolution family.

- If \textbf{EF3} is replaced by (joint) continuity in \((s, t)\), then we talk about \textit{topological} (reverse) evolution families.
Modern Loewner Theory - 11 -

Herglotz vector fields

VS evolution families – bis

EFs ↔ reverse EFs

(φs,t) is an (AC or topological) reverse evolution family iff

∀ T > 0, (∅T): ∅T = φuT(t),uT(s), where uT(t) := max {T − t, 0},

is a evolution family (AC or topological, resp.)

Theorem (Bracci, Contreras, Díaz-Madrigal)

• Let G be a Herglotz vector field. Then ∀ z ∈ ℍ ∀ t > 0, the IVP

  \[ \frac{dw}{ds} = - G\left(w(s), s\right) \] a.e. s ∈ [0, t]; \quad w\big|_{s=t} = z. \quad (*)

has a unique solution w = wz,t : [0, t] → ℍ,

and (φs,t), φs,t(z) := wz,t(s), is an AC reverse evolution family.

• Conversely, any AC reverse evolution family (φs,t) is generated in the above sense by a corresponding Herglotz vector field G

  [unique up to a null-set on the t-axis].
A self-map $\varphi \in \text{Hol}(\mathbb{D}, \mathbb{D})$ is called \textit{parabolic} if its DW-point $\tau \in \partial \mathbb{D}$ and
\[
\varphi'(\tau) := \angle \lim_{z \to \tau} \frac{\varphi(z) - \tau}{z - \tau} = 1.
\]

For a self-map $\Phi \in \text{Hol}(\mathbb{H}, \mathbb{H})$, $\mathbb{H} := \{\zeta : \text{Im} \zeta > 0\}$, this translates to
\[
\angle \lim_{\zeta \to \infty} \Phi(\zeta)/\zeta = 1 \iff \frac{1}{\Phi(\zeta) - x} = \int_{\mathbb{R}} \frac{k(x; dy)}{\zeta - y}, \; x \in \mathbb{R}, \quad (*)
\]
for some (uniquely determined) Borel probability measures $k(x; \cdot)$ on $\mathbb{R}$.

\begin{itemize}
\item \textbf{R.O. Bauer, 2004:} Every reverse EF of parabolic self-maps of $\Phi_{s,t} : \mathbb{H} \to \mathbb{H}$
\quad $\longrightarrow$ a family $(k_{s,t})$ of transition kernels of a Markov process.
\item Relation $\Phi_{s,u} \circ \Phi_{u,t} = \Phi_{s,t}$ is $\iff$ to Chapman – Kolmogorov.
\end{itemize}

\begin{itemize}
\item \textbf{U. Franz, T. Hasebe, S. Schleißinger, 2020 [100+ page paper]:}
\quad a complete characterization and a 1-to-1 correspondence with SAIPs.
\end{itemize}
Continuous-state branching processes (CB-processes for short) are Markov stochastic processes analogous to Galton–Watson processes but with the state space $[0, +\infty]$. Their transition kernels

$$k_{s,t} : [0, +\infty] \times \mathcal{B}([0, +\infty]) \to [0, 1], \quad 0 \leq s \leq t,$$

satisfy the branching property:

$$k_{s,t}(x; \cdot) \ast k_{s,t}(y; \cdot) = k(x + y; \cdot), \quad x, y \geq 0.$$

Branching property $\iff$ the Laplace transform of $k_{s,t}(x, \cdot)$ is of the form

$$\mathcal{L}[k_{s,t}(x; \cdot)](\lambda) := \int_0^{+\infty} e^{-\lambda \xi} k_{s,t}(x; d\xi) = \exp\left(-\varphi_{s,t}(\lambda) x\right), \quad x, \lambda \in (0, +\infty),$$

where $\varphi_{s,t}$, referred to as the Laplace exponent, is a Bernstein function, i.e. non-negative $C^\infty$-function in $(0, +\infty)$ with $(-1)^{n-1} \varphi_{s,t}^{(n)}(x) \geq 0, n \in \mathbb{N}$.

- Every Bernstein function $\not\equiv 0$ is a restriction of a holomorphic self-map $\varphi : \mathbb{H}_r \to \mathbb{H}_r := \{z \in \mathbb{C} : \Re z > 0\};$

- $\mathcal{B} \mathcal{F} := \{\text{Bernstein functions } \varphi \not\equiv 0\}$ is closed w.r.t. $\cdot \circ \cdot$ and also topologically closed in $\text{Hol}(\mathbb{H}_r, \mathbb{H}_r)$. 
CB-processes - 14 -

time homogeneous and inhomogeneous

Time-homogeneous case: \( k_{s,t} = k_{0,t-s}, \, \varphi_{s,t} = \varphi_{0,t-s} \)

The Laplace exponents \( \phi_t := \varphi_{0,t-s} \) form a one-parameter semigroup in \( \mathcal{B}\mathcal{F} \).

▶ M. Jiřina, 1958  
▶ M.L. Silverstein, 1968: \( \text{Gen}[\mathcal{B}\mathcal{F}] \)

\[
G(\zeta) = q + a\zeta - b\zeta^2 + \int_0^{+\infty} \left( 1 - e^{-x\zeta} - x\zeta 1_{(0,1)}(x) \right) \pi(dx), \quad \zeta \in \mathbb{H}_r,
\]

where \( a \in \mathbb{R}, \, q, b \geq 0 \) and \( \pi \) is a Borel non-negative measure on \((0, +\infty)\)

satisfying

\[
\int_0^{+\infty} \min\{x^2, 1\} \pi(dx) < +\infty \quad (*)
\]

The correspondence between \( \phi \in \text{Gen}[\mathcal{B}\mathcal{F}] \) and quadruples \( (q, a, b, \pi) \) is 1-to-1.

Inhomogeneous case ("varying environments")

▶ V. Bansaye, F. Simatos, 2015

▶ R. Fang, Z. Li, 2022: constructing CB-processes via an integral eq’n for the Laplace exponents \( \varphi_{s,t} \). But no Complex Analysis so far!
Inhomogeneous case; complex-analytic tools


\[
\mathcal{L}[k_{s,t}(x; \cdot)](\lambda) = \exp \left( - \varphi_{s,t}(\lambda) x \right), \quad x, \lambda \in (0, +\infty). \quad (*)
\]

The Chapman–Kolmogorov equation:

\[
k_{s,t}(x; \cdot) = \int_{[0, +\infty]} k_{s,u}(x; dy) \, k_{u,t}(y; \cdot), \quad 0 \leq s \leq u \leq t,
\]

\[\iff\] the composition rule \( \varphi_{s,t} = \varphi_{s,u} \circ \varphi_{u,t} \).

**Theorem (T. Hasebe, J.L. Pérez, P.G., 2022)**

The Laplace transform \( (*) \) establishes a 1-to-1 correspondence between topological reverse EFs \( (\varphi_{s,t}) \subset \mathcal{B} \) and families \( (k_{s,t}) \) of transition kernels of CB-processes with

\[\Delta \times [0, +\infty) \ni (s, t, x) \mapsto k_{s,t}(x; \cdot) \in P([0, +\infty]) \text{ is weakly continuous.}\]
CB-processes  - 16 -  with absolutely continuous REFs

Homogeneous case: continuity $\Rightarrow$ differentiability in $t$.

Inhomogeneous case: “AC” is stronger than “topological”.

Assume that $(\varphi_{s,t})$ is an AC reverse evolution family in $H_r$.

Problem

Characterize Herglotz vector fields $G : H_r \times [0, +\infty) \to \mathbb{C}$ whose REFs $(\varphi_{s,t}) \subset BF$.

General Theorem (T. Hasebe, J.L. Pérez, P.G., 2022)

Let $\mathcal{S} \subset \text{Hol}(D,D)$, $D \in \{\mathbb{D}, H, H_r\}$.

Denote by $\text{Gen}[\mathcal{S}]$ the set of all inf. gen’tors $G$ such that $(\varphi^G_t) \subset \mathcal{S}$.

Suppose:

(i) $\mathcal{S}$ is closed w.r.t. $\cdot \circ \cdot$ and $\text{id}_D \in S$;

(ii) $\mathcal{S}$ is (topologically) closed in $\text{Hol}(D,D)$.

Then: $\text{Gen}[\mathcal{S}]$ is closed cone in $\text{Hol}(D,\mathbb{C})$. Moreover,

$(\varphi^{G}_{s,t}) \subset \mathcal{S} \iff G(\cdot, t) \in \text{Gen}[\mathcal{S}]$ for a.e. $t \geq 0$. 
Recall Silverstein’s representation formula for $G \in \text{Gen}[\mathbb{B} \mathbb{F}]$:

$$G(\zeta) = q + a\zeta - b\zeta^2 + \int_0^{+\infty} \left(1 - e^{-x\zeta} - x\zeta 1_{(0,1)}(x)\right) \pi(dx), \quad \zeta \in \mathbb{H}_r,$$

where $a \in \mathbb{R}$, $q, b \geq 0$, and $\int_0^{+\infty} \min\{x^2, 1\} \pi(dx) < +\infty$.

T. Hasebe, J.L. Pérez, P.G.: (less technical) Complex-analytic proof.

A family $(q_t, a_t, b_t, \pi_t)_{t \geq 0}$ is said to be a Lévy family if:

(a) $a_t \in \mathbb{R}$, $q_t, b_t \geq 0$, and $\pi_t$ are a non-negative Borel measures on $(0, +\infty)$;

(b) $t \mapsto \pi_t(B)$ is measurable for any Borel set $B \subset (0, +\infty)$;

(c) $t \mapsto q_t$, $t \mapsto a_t$, $t \mapsto b_t$, $t \mapsto \int_0^{+\infty} \min\{x^2, 1\} \pi_t(dx)$ are in $L^1_{\text{loc}}$.

**Theorem (T. Hasebe, J.L. Pérez, P.G., 2022)**

$G : \mathbb{H}_r \times [0, +\infty) \to \mathbb{C}$ is a Herglotz vector field whose REF $(\varphi_{s,t}) \subset \mathbb{B} \mathbb{F}$ iff $G$ admits the representation

$$G(\zeta, t) = q_t + a_t\zeta - b_t\zeta^2 + \int_0^{+\infty} \left(1 - e^{-x\zeta} - x\zeta 1_{(0,1)}(x)\right) \pi_t(dx)$$

for all $\zeta \in \mathbb{H}_r$ and a.e. $t \geq 0$, where $(q_t, a_t, b_t, \pi_t)_{t \geq 0}$ is a Lévy family.
A CB-process \((Z_t)\) is \textit{conservative}, i.e. \(Z_t < +\infty\) a.s., iff 
\[ \zeta = 0 \] is a boundary fixed point of \(\varphi_{s,t}'s\), i.e. 
\[ \angle \lim_{\zeta \to 0} \varphi_{s,t}(\zeta) = 0. \]

In the homogeneous case: conservative \iff \[ \int_0^1 \frac{dx}{G(x)} = \infty. \]

**Explanation:** \(G\) is related to the \textit{Koenings map} of \((\phi_t)\), 
\[ h : \mathbb{H}_r \overset{\text{onto}}{\longrightarrow} \Omega \subset \mathbb{C} \text{ conformal, } h \circ \phi_t \circ h^{-1} = \begin{cases} 
\text{id}_\Omega + t, & \text{if } \tau \in \partial \mathbb{H}_r, \\
e^{-\lambda t} \text{id}_\Omega, & \text{if } \tau \in \mathbb{H}_r, \end{cases} \]
where \(\lambda := G'(\tau)\).

M.D. Contreras, S. Días-Madrigal, Ch. Pommerenke, 2004:
\[ \sigma \text{ is a boundary f. pt. } \iff \angle \lim_\sigma h = \infty. \]

No characterization of boundary f. pt.'s in the inhomogeneous case!
a CB-process \((Z_t)\) is \textit{conservative}, i.e. \(Z_t < +\infty\) a.s., \iff \(\zeta = 0\) is a boundary fixed point of \(\varphi_{s,t}\)'s, i.e. \(\angle \lim_{\zeta \to 0} \varphi_{s,t}(\zeta) = 0\).

\textbf{Expectation:} \(\mathbb{E}[Z_t \mid Z_s = x] = x \varphi'_{s,t}(0)\) [for the conservative case]

The angular derivative

\[
\varphi'(\sigma) := \angle \lim_{\zeta \to \sigma} \frac{\varphi(\zeta) - \sigma}{\zeta - \sigma} = \liminf_{\zeta \to \sigma} \frac{\text{Re } \varphi(\zeta)}{\text{Re } \zeta}
\]

A boundary fixed point \(\sigma\) is said to be \textit{regular} if \(\varphi'(\sigma) \neq \infty\).

M.D. Contreras, S. Días-Madrigal, Ch. Pommerenke, 2006:

\(\sigma\) is a boundary regular f. pt. (BRFP) of \((\varphi_t)\) \iff \(\lambda := \angle \lim_{\zeta \to \sigma} \frac{G(\zeta)}{\zeta - \sigma} \neq \infty\).
A CB-process \((Z_t)\) is **conservative**, i.e. \(Z_t < +\infty\) a.s., iff 
\[ \zeta = 0 \text{ is a boundary fixed point of } \varphi_{s,t} \text{'s, i.e. } \angle \lim_{\zeta \to 0} \varphi_{s,t}(\zeta) = 0. \]

**Expectation:** 
\[ \mathbb{E}[Z_t | Z_s = x] = x \varphi'_{s,t}(0) \quad \text{[for the conservative case]} \]

The angular derivative 
\[ \varphi'(\sigma) := \angle \lim_{\zeta \to \sigma} \frac{\varphi(\zeta) - \sigma}{\zeta - \sigma} = \liminf_{\zeta \to \sigma} \frac{\text{Re } \varphi(\zeta)}{\text{Re } \zeta} \]

A boundary fixed point \(\sigma\) is said to be **regular** if \(\varphi'(\sigma) \neq \infty\).

**Theorem (F. Bracci, M.D. Contreras, S. Díaz-Madrigal, P.G., 2015)**

Let \(G\) be a H. v. f. and \((\varphi_{s,t})\) its (reverse) evolution family. Then:

\(\sigma\) is a BRFP for all \(\varphi_{s,t}\)’s \(\iff\) \(\lambda(t) := \angle \lim_{\zeta \to \sigma} \frac{G(\zeta, t)}{\zeta - \sigma} \) is \(L_{1}^{\text{loc}}([0, +\infty))\).

In this case, 
\[ \varphi'_{s,t}(\sigma) = \exp \left( \int_{s}^{t} \lambda(u) \, du \right). \]
CB-processes - 18.c -

Boundary f. pt. and
Differentiability problem

a CB-process \((Z_t)\) is \textit{conservative}, i.e. \(Z_t < +\infty\) a.s., \iff \(\zeta = 0\) is a boundary fixed point of \(\varphi_{s,t}\)’s, i.e. \(\angle \lim_{\zeta \to 0} \varphi_{s,t}(\zeta) = 0\).

\textbf{Expectation:} \(\mathbb{E}\left[Z_t \mid Z_s = x\right] = x \varphi'_{s,t}(0)\) [for the conservative case]

The angular derivative
\[
\varphi'(\sigma) := \angle \lim_{\zeta \to \sigma} \frac{\varphi(\zeta) - \sigma}{\zeta - \sigma} = \liminf_{\zeta \to \sigma} \frac{\text{Re} \varphi(\zeta)}{\text{Re} \zeta}
\]

\textbf{The second moment:} \(\mathbb{E}\left[Z_t^2 \mid Z_s = x\right] =
\[
= \begin{cases} 
+\infty, & \text{if } \varphi''_{s,t}(0) = -\infty, \\
(x \varphi'_{s,t}(0))^2 - x \varphi''_{s,t}(0), & \text{otherwise.}
\end{cases}
\]

\textbf{Theorem (T. Hasebe, J.L. Pérez, P.G., 2022)}

If \(t \mapsto \mathbb{E}\left[Z_t^k \mid Z_0 = 1\right], k = 1, 2,\) are AC\(_{1\text{oc}},\) then \((\varphi_{s,t})\) is AC.
Theorem (T. Hasebe, J.L. Pérez, P.G., 2022)

There is a one-to-one correspondence among:

- families \((k_{s,t})\) of transition kernels of CB-processes with \( t \mapsto \mathbb{E}\left[Z_t^k \mid Z_0 = 1\right], k = 1, 2, \) in \(AC_{1loc}([0, +\infty))\);
- AC reverse evolution families \((\varphi_{s,t}) \subset \{\varphi \in \mathscr{B} \mathfrak{G} : \varphi''(0) \neq -\infty\};\)
- the class of Herglotz vector fields given by Silverstein-type representation

\[
G(\zeta, t) = a_t \zeta - b_t \zeta^2 + \int_0^{+\infty} \left(1 - e^{-x\zeta} - x\zeta 1_{(0,1)}(x)\right) \pi_t(dx),
\]

where

(a) \(a_t \in \mathbb{R}, b_t \geq 0, (q_t \equiv 0),\)
    and \(\pi_t\) are a non-negative Borel measures on \((0, +\infty)\);

(b) \(t \mapsto \pi_t(B)\) is measurable for any Borel set \(B \subset (0, +\infty)\);

(c) \(t \mapsto a_t, t \mapsto b_t, t \mapsto \int_0^{+\infty} x^2 \pi_t(dx)\) are in \(L^1_{loc}\).
Probabilistic interpretation of $\tau = 0$

**Remark:** $\tau = 0$ is the DW-point of $\varphi : H_r \to H_r$ iff $\varphi(0) = 0$ and $\varphi'(0) \leq 1$.

**Recall:** $\mathbb{E}[Z_t \mid Z_s = x] = x \varphi'_{s,t}(0)$.

**Conclusion:** $\varphi_{s,t}$'s DW-point is at $\tau = 0$ iff $t \mapsto \mathbb{E}[Z_t \mid Z_0 = 1]$ is non-increasing.

**Extinction time:** $T^s_0 := \inf \{t \geq s : Z_t = 0\}$.

**Theorem (T. Hasebe, J.L. Pérez, P.G., 2022)** $[T^s_0 < +\infty]$

Suppose the Laplace exponents form an AC reverse EF $(\varphi_{s,t})$ with the DW-point $\tau = 0$ and with the H. v. f. $G$.

If $\int_0^{+\infty} G''(\infty, t) \, dt = -\infty$, then $q_{\infty}(s) := \lim_{t \to +\infty} \varphi_{s,t}(\infty) = 0$ and hence $
\mathbb{P}\left[T^s_0 < +\infty \mid Z_s = x\right] = e^{-q_{\infty}(s)} = 1 \quad \forall \ x \in (0, +\infty), \ s \geq 0.$
Theorem [monotonicity] (T. Hasebe, J.L. Pérez, P.G., 2022)

Let \((Z_t)\) be a CB-process with associated topological REF \((\varphi_{s,t})\). TFAE:

(i) the DW-point of all \(\varphi_{s,t}\)’s is at \(\infty\);

(ii) \(\Pr[Z_u \leq Z_t \mid Z_s = x] = 1\) whenever \(0 \leq s \leq u \leq t\) and \(x > 0\);

(iii) for any \((s, t) \in \Delta\) and for some (and hence all) \(x > 0\),

we have \(\Pr[Z_t \geq x \mid Z_s = x] = 1\).

Explosion time: \(T^s_\infty := \inf\{t \geq s : Z_t = +\infty\}\).

Theorem (T. Hasebe, J.L. Pérez, P.G., 2022) \([T^s_\infty < +\infty]\)

Suppose the Laplace exponents form an AC reverse EF \((\varphi_{s,t})\)

with the DW-point \(\tau = \infty\) and with the H. v. f. \(G\).

If

\[ \int_0^{+\infty} G(0, t) \exp\left(\int_0^t G'(\infty, s) \, ds\right) \, dt = +\infty, \tag{*} \]

then \(q_0(s) := \lim_{t \to +\infty} \varphi_{s,t}(0) = +\infty\), and hence

\[ \Pr[T^s_\infty < +\infty \mid Z_s = x] = 1 - e^{-q_0(s)} = 1 \]

for any \(s \geq 0\) and \(x \in (0, +\infty)\).
Consider again a branching process \((X_t)\) on the \textbf{discrete state} space \(\mathbb{N}_0^* = \{0, 1, 2, \ldots\} \cup \{+\infty\}\), with the transition probabilities \(p_{s,t}(k) := \mathbb{P} [X_t = k | X_s = 1]\).

**Definition (spatial embedding)** M. Jiřina, 1958

We say that \((X_t)\) embeds in (or extends to) a CB-process \((Z_t)\) on \([0, +\infty]\) with the transition kernels \((k_{s,t})\)

\[
\text{if } k_{s,t}(1; \{k\}) = p_{s,t}(k) \quad \text{for any } k \in \mathbb{N}_0^* \text{ and all } (s, t) \in \Delta.
\]

**Theorem (T. Hasebe, J.L. Pérez, P.G., 2022)**

\((X_t)\) embeds into some CB-process \((Z_t)\) iff the probability generating functions \((F_{s,t})\) of \((X_t)\) have common DW-point at \(\tau = 0\).

The Laplace exponents of \((Z_t)\) are given by

\[
\exp \left( -\varphi_{s,t}(\zeta) \right) = F_{s,t}(e^{-\zeta}).
\]