Methods of holomorphic dynamics in the study of branching processes

Pavel Gumenyuk
(Politecnico di Milano, Italy)

based on a joint work (still in progress) with
T. Hasebe (Hokkaido University, Japan) and
J. L. Pérez (Centro de Investigación en Matemáticas, Mexico)

Probabilistic Operator Algebra Seminar

University of California, Berkeley — June 5, 2023

Outline of the talk

- ☞ Simplest example: Galton – Watson processes
- ☞ Continuous-time inhomogeneous processes
  and evolution families of holomorphic self-maps
- ☞ Loewner's Parameteric Representation method
  in Conformal Mapping
- ☞ Modern Loewner Theory
  due to F. Bracci, M.D. Contreras, and S. Díaz-Madrigal
- ☞ REMARK: evolution families in Quantum Probabilities
- ☞ CB-processes and evolution families of Bernstein functions
- ☞ Differentiability problem
- ☞ Probabilistic interpretation of the Denjoy – Wolff point
- ☞ Spatial embedding

Galton – Watson process

is a Markov chain (X_n), where X_n is the *number of identical "particles"* in the n-th generation, $X_0 = 1$.
Each particle splits giving rise to $k \in \mathbb{N}_0$ offsprings with probability $p(k)$ *independently from others* (and prehistory).

Probability generating function $F(z) := \mathbb{E}\left[z^{X_1}\right] = \sum_{k=0}^{+\infty} p(k) z^k$

is a holomorphic self-map of $\mathbb{D} := \{z : |z| < 1\}$ with a *boundary fixed point* at 1 (except for the degenerate case $p(0) = 1$, $p(k) = 0$, $k \in \mathbb{N}$).

Relation to Dynamics:

If $p(k)$'s do not change in time, then the probability distribution of X_n is given by the Taylor coefficients of the n-th iterate of the function F,

$$F^{\circ n} := \underbrace{F \circ \ldots \circ F}_{n \text{ times}} : \mathbb{D} \to \mathbb{D}.$$

To a Galton – Watson process with continuous time one associates the family

$$(F_t)_{t\in\mathbb{R}_{\geqslant 0}} \subset \mathrm{Hol}(\mathbb{D},\mathbb{D}), \quad F_t(z) := \mathbb{E}[z^{X_t}]$$

(i) $F_0 = \mathrm{id}_{\mathbb{D}}$;

(ii) $F_t \circ F_s = F_{t+s}$ for any $t, s \in \mathbb{R}_{\geqslant 0}$;

Under a mild continuity assumption on the transition probabilities $p_{s,t}(k)$:

(iii) as $t \to 0^+$, $F_t \to \mathrm{id}_{\mathbb{D}}$ pointwise and hence locally uniformly in $\mathbb{D}$.

A family $(F_t)_{t\in\mathbb{R}_{\geqslant 0}} \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$ satisfying (i) – (iii) is usually called a *one-parameter semigroup* in $\mathbb{D}$.

Time-**inhomogeneous** case

Prob'ty generating f'ns: $(F_{s,t})_{t\geqslant s\geqslant 0} \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$, $F_{s,t}(z) := \mathbb{E}\left[z^{X_t} \,\middle|\, X_s = 1\right]$,

(i) $F_{s,s} = \mathrm{id}_{\mathbb{D}}$ for any $s \geqslant 0$;

(ii) $F_{s,u} \circ F_{u,t} = F_{s,t}$ whenever $0 \leqslant s \leqslant u \leqslant t$;

Again, under a mild continuity assumption on $p_{s,t}(k)$'s:

(iii) $\{(s,t) \in \mathbb{R}_{\geqslant 0} : s \leqslant t\} =: \Delta \ni (s,t) \mapsto F_{s,t} \in \mathrm{Hol}(\mathbb{D},\mathbb{D})$ is continuous.

A family $(F_{s,t})_{(s,t)\in\Delta} \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$ is said to be
a *topological reverse evolution family* if the above 3 conditions hold:

(i) $F_{s,s} = \mathrm{id}_{\mathbb{D}}$; (ii) $F_{s,u} \circ F_{u,t} = F_{s,t}$ whenever $0 \leqslant s \leqslant u \leqslant t$;

(iii) $(s,t) \mapsto F_{s,t}$ is continuous.

Special feature of the homogeneous case (E. Berkson, H. Porta, 1978)

If (F_t) is a one-parameter semigroup, then

$$dF_t(z)/dt = G(F_t(z)),\ t \geqslant 0, \quad F_0(z) = z, \qquad (*)$$

for a suitable $G \in \mathrm{Hol}(\mathbb{D},\mathbb{C})$ called the *infinitesimal generator* of (F_t).

The infinitesimal generators form a convex cone $\mathrm{Gen}(\mathbb{D}) \subset \mathrm{Hol}(\mathbb{D},\mathbb{C})$.

Inhomogeneous extension of $(*)$ [from the Dynamics viewpoint]

$$dF_{s,t}(z)/dt = G\big(F_{s,t}(z), t\big),\ t \geqslant s \geqslant 0, \quad F_{s,s}(z) = z;$$

$G(\cdot,t) \in \mathrm{Gen}(\mathbb{D})$ for a.e. $t \geqslant 0$. [(Generalized) *Loewner – Kufarev ODE*]

The Loewner – Kufarev ODE generates

absolutely continuous evolution families:

$(F_{s,t}) \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$

(i) $F_{s,s} = \mathrm{id}_{\mathbb{D}}$; (ii) $F_{u,t} \circ F_{s,u} = F_{s,t}$ whenever $0 \leqslant s \leqslant u \leqslant t$;

(iii) stronger than continuity of $(s,t) \mapsto F_{s,t}$.

☞ One-parameter semigroups: E. Schröder, 1871; G. Koenigs, 1884; E. Berkson, H. Porta, 1978:

$$G \in \mathrm{Gen}(\mathbb{D}) \quad \Longleftrightarrow \quad G(z) = (\tau - z)(1 - \overline{\tau} z)p(z),\ \tau \in \overline{\mathbb{D}},\ \mathrm{Re}\, p \geqslant 0.$$

☞ Evolution families: Ch. Loewner, Math. Ann. (1923)

$$\mathcal{S} := \left\{ f : \mathbb{D} \to \mathbb{C} \text{ injective holomorphic with } f(z) = z + a_2 z^2 + \ldots \right\}$$

A dense subclass $\mathcal{S}_{s1}$ is formed by *slit mappings*, i.e. by those $f \in \mathcal{S}$ for which $\Gamma := \overline{\mathbb{C}} \setminus f(\mathbb{D})$ is a Jordan arc with one end-point at ∞.

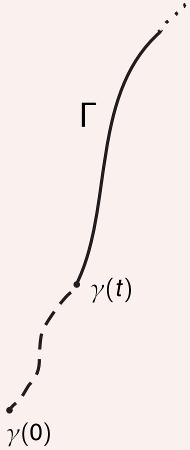
Consider a slit mapping $f \in \mathcal{S}_{\mathtt{sl}}$, $\Gamma := \overline{\mathbb{C}} \setminus f(\mathbb{D})$.

- ☞ Choose a parametrization $\gamma : [0, +\infty] \to \Gamma$, $\gamma(+\infty) = \infty$.
- ☞ Consider the domains $\Omega_t := \mathbb{C} \setminus \gamma([t, +\infty])$, $t \geqslant 0$.
- ☞ By Riem.'s Mapping Th'm $\forall t \geqslant 0$ $\exists!$ *conformal mapping*

$$f_t : \mathbb{D} \xrightarrow{\text{onto}} \Omega_t, \quad f_t(0) = 0,\ f_t'(0) > 0.$$

- ✔ Note that $\boxed{f_0 = f}$
- ✔ and that $\Omega_s = f_s(\mathbb{D}) \subset f_t(\mathbb{D}) = \Omega_t$ whenever $0 \leqslant s \leqslant t$.
- ☞ Reparameterizing Γ: $\forall\ t \geqslant 0$ $\boxed{f_t'(0) = e^t}$.

Figure 1



THEN: $\varphi_{s,t} := f_t^{-1} \circ f_s \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$, $0 \leqslant s \leqslant t$,

are C^1 in (s, t) and form an evolution family $(\varphi_{s,t})$.

Theorem (Ch. Loewner, 1923) f, (f_t), $(\varphi_{s,t})$ as above

$\exists!$ continuous function $\xi : [0, +\infty) \to \partial\mathbb{D}$ s.t. for any $z \in \mathbb{D}$ and $s \geqslant 0$, $w = w_{z,s}(t) := \varphi_{s,t}(z)$ solves the IVP for the *Loewner ODE*

$$\frac{dw}{dt} = -w\,\frac{1+\overline{\xi(t)}w}{1-\overline{\xi(t)}w}, \quad t \geqslant s; \qquad w(s) = z.$$

Moreover, $f_s = \lim_{t\to+\infty} e^t \varphi_{s,t}$ for any $s \geqslant 0$. **NB:** $f = f_0$

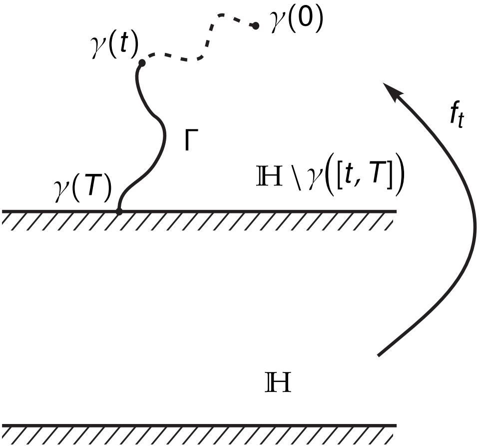
NB: a sort of converse is also true ☞ parametric representation of $\mathcal{S}'$, $\mathcal{S}_{\mathrm{sl}} \subset \mathcal{S}' \subset \mathcal{S}$.

Extension to the whole class $\mathcal{S}$: the *classical Loewner–Kufarev ODE*

P. P. Kufarev, 1943; Ch. Pommerenke, 1965; V. Ja. Gutljanskii, 1970:

$$\frac{dw}{dt} = -w(t)\,p(w(t), t), \quad t \geqslant s; \qquad w(s) = z;$$

$\mathrm{Re}\, p \geqslant 0$, $p(0, t) = 1$, measurable in t, not C^1 — Carathéodory's ODE!



Hydrodynamic normalization:

$$f_t(\zeta) = \zeta - \frac{c(t)}{\zeta} + o(1/\zeta) \quad \text{(HD)}$$

as $\mathbb{H} \ni \zeta \to \infty$, $c(t) \geqslant 0$;

The **evolution family**

$$\varphi_{s,t} := H^{-1} \circ (f_t^{-1} \circ f_s) \circ H,$$

$$\mathbb{D} \ni z \mapsto \zeta = H(z) := i\tfrac{1+z}{1-z},$$

satisfies the *chordal L.–K. ODE*

$$\mathrm{d}w/\mathrm{d}t = (1-w)^2 p(w,t).$$

Berkson – Porta: $G(z) = (\tau - z)(1-\overline{\tau}z)\,p(z), \quad \operatorname{Re} p \geqslant 0,\ \tau \in \overline{\mathbb{D}}.$

The r. h. s.' s $G_{\mathtt{cla}}(w,t) = -w\,p(w,t)$ and $G_{\mathrm{cho}} = (1-w)^2 p(w,t)$ are t-dependent infinitesimal generators, with $\tau := 0$ and $\tau := 1$.

What is the meaning of $\boldsymbol{\tau}$ in B.–P.'s $G(z) = (\tau - z)(1 - \overline{\tau}z)\,p(z)$?

Theorem (Denjoy and Wolff)

$\forall\, \varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\mathrm{id}_{\mathbb{D}}\}$, $\exists!\, \tau \in \overline{\mathbb{D}}$, called *the Denjoy – Wolff point* of φ, s.t.:

- if $\tau \in \mathbb{D}$, then $\varphi(\tau) = \tau$, and $\varphi^{\circ n} \xrightarrow{n\to+\infty} \tau$ l.u. in $\mathbb{D}$ if $\varphi \notin \mathrm{Aut}(\mathbb{D})$;
- if $\tau \in \partial\mathbb{D}$, then $\angle \lim_{z\to\tau} \varphi(z) = \tau$ and $\varphi^{\circ n} \xrightarrow{n\to+\infty} \tau$ l.u. in $\mathbb{D}$.

Convention: for $\varphi = \mathrm{id}_{\mathbb{D}}$, every $\tau \in \overline{\mathbb{D}}$ is its DW-point.

☞ The point $\boldsymbol{\tau}$ in B.–P.'s formula is the DW-point of all ϕ_t's in the one-parameter semigroup $(\phi_t) \sim G$.

☞ $\tau = 0$ is the DW-point of evolution families $(\varphi_{s,t})$ in Loewner's classical construction.

☞ $\tau = 1$ is the DW-point in the chordal version of Loewner's const'n.

In 2012, F. Bracci, M.D. Contreras, and S. Díaz-Madrigal proposed

generalized L.–K. ODE: $\dfrac{dw}{dt} = G\big(w(t), t\big), \quad t \geqslant s; \quad w(s) = z \in \mathbb{D},$

with $G(z,t) := \big(\tau(t) - z\big)\big(1 - \overline{\tau(t)}z\big)\, p(z,t)$, $\mathrm{Re}\, p \geqslant 0$, $\tau : [0, +\infty) \to \overline{\mathbb{D}}$.

More precisely, they assumed that G is a Herglotz vector field:

a function $G : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is called a *Herglotz vector field*, if:

HVF1: for a.e. $t \geqslant 0$, $G(\cdot, t) \in \mathrm{Gen}(\mathbb{D})$;

HVF2: for each $z \in \mathbb{D}$, $G(z, \cdot)$ is measurable on $[0, +\infty)$;

HVF3: for each compact $K \subset \mathbb{D}$,

$\exists$ a loc. integrable $M : [0, +\infty) \to [0, +\infty)$ s.t.

$$\max_K \big|G(\cdot, t)\big| \leqslant M(t) \quad \text{for a.e. } t \geqslant 0.$$

F. Bracci, M.D. Contreras, and S. Díaz-Madrigal proved that
the generalized Loewner – Kufarev ODE establishes a
one-to-one correspondence
between Herglotz vector fields and *abs. continuous evolution families* :

$(\varphi_{s,t})_{t\geqslant s\geqslant 0} \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$ is an *absolutely continuous evolution family* if:

EF1: $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$ for all $s \geqslant 0$; EF2: $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$, $0 \leqslant s \leqslant u \leqslant t$;

EF3: for every $z \in \mathbb{D}$, $\exists$ a locally integrable $k_z : [0,+\infty) \to [0,+\infty)$ s.t.

$$\left|\varphi_{s,t}(z) - \varphi_{s,u}(z)\right| \leqslant \int_u^t k_z(v)\,\mathrm{d}v \quad \text{whenever } 0 \leqslant s \leqslant u \leqslant t.$$

- If EF2 is replaced by $\varphi_{s,t} = \varphi_{s,u} \circ \varphi_{u,t}$, then we get the definition of an (absolutely continuous) *reverse* evolution family.
- If EF3 is replaced by (joint) continuity in (s,t), then we talk about *topological* (reverse) evolution families.

EFs ⟷ reverse EFs

$(\varphi_{s,t})$ is an (AC or topological) *reverse* evolution family **iff**

$\forall\ T>0,\quad (\psi^T_{s,t}):\quad \psi^T_{s,t}=\varphi_{u_T(t),u_T(s)},$ where $u_T(t):=\max\big\{T-t,\,0\big\},$

is a evolution family (AC or topological, resp.)

Theorem (Bracci, Contreras, Díaz-Madrigal)

- Let G be a Herglotz vector field. Then $\forall\, z\in\mathbb{D}\ \forall\, t>0$, the IVP

$$\mathrm{d}w/\mathrm{d}s \;=\; -\,G\big(w(s),s\big)\quad \text{a.e. } s\in[0,t];\qquad w\big|_{s=t}=z. \qquad (*)$$

has a unique solution $w=w_{z,t}:[0,t]\to\mathbb{D}$,

and $(\varphi_{s,t})$, $\boxed{\varphi_{s,t}(z):=w_{z,t}(s),}$ is an AC reverse evolution family.

- Conversely, any AC reverse evolution family $(\varphi_{s,t})$ is generated in the above sense by a corresponding Herglotz vector field G

[unique up to a null-set on the t-axis].

A remark - 12 - on parabolic evolution families

A self-map $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ is called *parabolic* if its DW-point $\tau \in \partial\mathbb{D}$

and
$$\varphi'(\tau) := \angle \lim_{z \to \tau} \frac{\varphi(z) - \tau}{z - \tau} = 1.$$

For a self-map $\Phi \in \mathrm{Hol}(\mathbb{H}, \mathbb{H})$, $\mathbb{H} := \{\zeta : \mathtt{Im}\,\zeta > 0\}$, this translates to

$$\angle \lim_{\zeta \to \infty} \Phi(\zeta) \big/ \zeta = 1 \quad \Longleftrightarrow \quad \frac{1}{\Phi(\zeta) - x} = \int_{\mathbb{R}} \frac{k(x; \mathrm{d}y)}{\zeta - y}, \; x \in \mathbb{R}, \quad (*)$$

for some (uniquely determined) Borel probability measures $k(x; \cdot)$ on $\mathbb{R}$.

R.O. Bauer, 2004:

Every reverse EF of parabolic self-maps of $\Phi_{s,t} : \mathbb{H} \to \mathbb{H}$
$\longrightarrow$ a family $(k_{s,t})$ of transition kernels of a Markov process.

Relation $\Phi_{s,u} \circ \Phi_{u,t} = \Phi_{s,t}$ is $\Longleftrightarrow$ to Chapman – Kolmogorov.

U. Franz, T. Hasebe, S. Schleißinger, 2020 [100+ page paper]:
a complete characterization and a 1-to-1 correspondence with SAIPs.

Continuous-state branching processes

Continuous-state branching processes (*CB-processes* for short)

are Markov stochastic processes analogous to Galton – Watson processes but with the state space $[0,+\infty]$. Their transition kernels

$$k_{s,t} : [0,+\infty] \times \mathcal{B}([0,+\infty]) \to [0,1], \quad 0 \leqslant s \leqslant t, \qquad \text{satisfy}$$

the *branching property*: $k_{s,t}(x;\cdot) * k_{s,t}(y;\cdot) = k(x+y;\cdot), \quad x, y \geqslant 0.$

Branching property $\Leftrightarrow$ the Laplace transform of $k_{s,t}(x,\cdot)$ is of the form

$$\mathcal{L}[k_{s,t}(x;\cdot)](\lambda) := \int_0^{+\infty} e^{-\lambda\xi} k_{s,t}(x; d\xi) = \exp\Big(-\varphi_{s,t}(\lambda)\, x\Big), \quad x, \lambda \in (0,+\infty),$$

where $\varphi_{s,t}$, referred to as the Laplace exponent, is a *Bernstein function*, i.e. non-negative C^∞-function in $(0,+\infty)$ with $(-1)^{n-1}\varphi_{s,t}^{(n)} \geqslant 0$, $n \in \mathbb{N}$.

☞ Every Bernstein function $\not\equiv 0$ is a restriction of a *holomorphic* self-map $\varphi : \mathbb{H}_r \to \mathbb{H}_r := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$;

☞ $\mathfrak{BF} := \big\{\text{Bernstein functions } \varphi \not\equiv 0\big\}$ is closed w.r.t. $\cdot \circ \cdot$ and also topologically closed in $\mathrm{Hol}(\mathbb{H}_r, \mathbb{H}_r)$.

Time-homogeneous case: $k_{s,t} = k_{0,t-s}$, $\varphi_{s,t} = \varphi_{0,t-s}$

The Laplace exponents $\phi_t := \varphi_{0,t-s}$ form a one-parameter semigroup in $\mathfrak{B}\mathfrak{F}$.

- M. Jiřina, 1958
- M.L. Silverstein, 1968: $\mathrm{Gen}[\mathfrak{B}\mathfrak{F}]$

$$G(\zeta) = q + a\zeta - b\zeta^2 + \int_0^{+\infty} \big(1 - e^{-x\zeta} - x\zeta\,\mathbf{1}_{(0,1)}(x)\big)\,\pi(\mathrm{d}x), \ \zeta \in \mathbb{H}_{\mathrm{r}},$$

where $a \in \mathbb{R}$, $q, b \geqslant 0$ and π is a Borel non-negative measure on $(0, +\infty)$

satisfying
$$\int_0^{+\infty} \min\{x^2, 1\}\,\pi(\mathrm{d}x) < +\infty \qquad (*)$$

The corresp'nce between $\phi \in \mathrm{Gen}[\mathfrak{B}\mathfrak{F}]$ and quadruples (q, a, b, π) is 1-to-1.

Inhomogeneous case ("varying environments")

- V. Bansaye, F. Simatos, 2015
- R. Fang, Z. Li, 2022: constructing CB-processes via an integral eq'n for the Laplace exponents $\varphi_{s,t}$. But no $\mathbb{C}$omplex Analysis so far!

Joint results with Takahiro Hasebe (Hokkaido Univ., Japan) and José Luis Pérez (CIMAT, México): arXiv:2206.04753, arXiv:2211.12442.

$$\mathcal{L}[k_{s,t}(x;\cdot)](\lambda) = \exp\Big(-\varphi_{s,t}(\lambda)\,x\Big), \quad x,\lambda \in (0,+\infty). \qquad (*)$$

The Chapman – Kolmogorov equation:

$$k_{s,t}(x;\cdot) = \int_{[0,+\infty]} k_{s,u}(x;dy)\, k_{u,t}(y;\cdot), \quad 0 \leqslant s \leqslant u \leqslant t,$$

$\Longleftrightarrow$ the composition rule $\varphi_{s,t} = \varphi_{s,u} \circ \varphi_{u,t}$.

Theorem (T. Hasebe, J.L. Pérez, P.G., 2022)

The Laplace transform $(*)$ establishes a 1-to-1 correspondence between topological reverse EFs $(\varphi_{s,t}) \subset \mathfrak{BF}$ and families $(k_{s,t})$ of transition kernels of CB-processes with

$$\underbrace{\Delta}_{\{(s,t)\,:\,0\leqslant s\leqslant t\}} \times [0,+\infty) \ni (s,t,x) \mapsto k_{s,t}(x;\cdot) \in \mathrm{P}([0,+\infty])$$ is weakly continuous.

- ☞ Homogeneous case: continuity $\Rightarrow$ differentiability in t.
- ☞ Inhomogeneous case: “AC” is *stronger* than “topological”.

Assume that $(\varphi_{s,t})$ is an AC reverse evolution family in $\mathbb{H}_{\mathrm{r}}$.

Problem

Characterize Herglotz vector fields $G : \mathbb{H}_{\mathrm{r}} \times [0, +\infty) \to \mathbb{C}$ whose REFs $(\varphi_{s,t}) \subset \mathfrak{BF}$.

General Theorem (T. Hasebe, J.L. Pérez, P.G., 2022)

Let $\mathfrak{S} \subset \mathrm{Hol}(D, D)$, $D \in \{\mathbb{D}, \mathbb{H}, \mathbb{H}_{\mathrm{r}}\}$.

Denote by $\mathrm{Gen}[\mathfrak{S}]$ the set of all inf. gen’tors G such that $(\phi_t^G) \subset \mathfrak{S}$.

Suppose:
- (i) $\mathfrak{S}$ is closed w.r.t. $\cdot \circ \cdot$ and $\mathrm{id}_D \in \mathcal{S}$;
- (ii) $\mathfrak{S}$ is (topogically) closed in $\mathrm{Hol}(D, D)$.

Then: $\mathrm{Gen}[\mathfrak{S}]$ is closed cone in $\mathrm{Hol}(D, \mathbb{C})$. Moreover,

$$(\varphi_{s,t}^G) \subset \mathfrak{S} \iff G(\cdot, t) \in \mathrm{Gen}[\mathfrak{S}] \text{ for a.e. } t \geqslant 0.$$

Recall Silverstein's representation formula for $G \in \mathrm{Gen}[\mathfrak{BF}]$:

$$G(\zeta) = q + a\zeta - b\zeta^2 + \int_0^{+\infty}\Big(1 - e^{-x\zeta} - x\zeta\,\mathbf{1}_{(0,1)}(x)\Big)\,\pi(\mathrm{d}x),\ \zeta \in \mathbb{H}_{\mathrm{r}},$$

where $a \in \mathbb{R}$, $q, b \geqslant 0$, and $\int_0^{+\infty} \min\{x^2, 1\}\,\pi(\mathrm{d}x) < +\infty$.

T. Hasebe, J.L. Pérez, P.G.: (less technical) $\mathbb{C}$omplex-analytic proof.

A family $(q_t, a_t, b_t, \pi_t)_{t\geqslant 0}$ is said to be a *Lévy family* if:

(a) $a_t \in \mathbb{R}$, $q_t, b_t \geqslant 0$, and π_t are a non-negative Borel measures on $(0, +\infty)$;

(b) $t \mapsto \pi_t(B)$ is measurable for any Borel set $B \subset (0, +\infty)$;

(c) $t \mapsto q_t$, $t \mapsto a_t$, $t \mapsto b_t$, $t \mapsto \int_0^{+\infty} \min\{x^2, 1\}\,\pi_t(\mathrm{d}x)$ are in L^1_{loc}.

Theorem (T. Hasebe, J.L. Pérez, P.G., 2022)

$G : \mathbb{H}_{\mathrm{r}} \times [0, +\infty) \to \mathbb{C}$ is a Herglotz vector field whose REF $(\varphi_{s,t}) \subset \mathfrak{BF}$ **iff** G admits the representation

$$G(\zeta, t) = q_t + a_t\zeta - b_t\zeta^2 + \int_0^{+\infty}\Big(1 - e^{-x\zeta} - x\zeta\,\mathbf{1}_{(0,1)}(x)\Big)\,\pi_t(\mathrm{d}x)$$

for all $\zeta \in \mathbb{H}_{\mathrm{r}}$ and a.e. $t \geqslant 0$, where $(q_t, a_t, b_t, \pi_t)_{t\geqslant 0}$ is a Lévy family.

☞ a CB-process (Z_t) is *conservative*, i.e. $Z_t < +\infty$ a.s., **iff**

$\zeta = 0$ is a boundary fixed point of $\varphi_{s,t}$'s, i.e. $\angle \lim_{\zeta \to 0} \varphi_{s,t}(\zeta) = 0$.

In the homogeneous case: conservative $\iff$ $\displaystyle\int_0^1 \frac{dx}{G(x)} = \infty$.

Explanation: G is related to the *Koenings map* of (ϕ_t),

$$h : \mathbb{H}_r \xrightarrow{\text{onto}} \Omega \subset \mathbb{C} \text{ conformal}, \quad h \circ \phi_t \circ h^{-1} = \begin{cases} \mathrm{id}_\Omega + t, & \text{if } \tau \in \partial\mathbb{H}_r, \\ e^{\lambda t}\,\mathrm{id}_\Omega, & \text{if } \tau \in \mathbb{H}_r, \end{cases}$$

where $\lambda := G'(\tau)$.

M.D. Contreras, S. Días-Madrigal, Ch. Pommerenke, 2004:

σ is a boundary f. pt. $\iff$ $\angle \lim_\sigma h = \infty$.

No characterization of boundary f. pt.'s in the inhomogeneous case!

☞ a CB-process (Z_t) is *conservative*, i.e. $Z_t < +\infty$ a.s., **iff**

$\zeta = 0$ is a boundary fixed point of $\varphi_{s,t}$'s, i.e. $\angle\lim_{\zeta\to 0} \varphi_{s,t}(\zeta) = 0$.

☞ **Expectation:** $\mathbb{E}\left[Z_t \,\middle|\, Z_s = x\right] = x\,\varphi'_{s,t}(0)$ [for the conservative case]

The angular derivative $\varphi'(\sigma) := \angle\lim_{\zeta\to\sigma} \dfrac{\varphi(\zeta)-\sigma}{\zeta-\sigma} = \liminf_{\zeta\to\sigma} \dfrac{\operatorname{Re}\varphi(\zeta)}{\operatorname{Re}\zeta}$

A boundary fixed point σ is said to be *regular* if $\varphi'(\sigma) \neq \infty$.

M.D. Contreras, S. Días-Madrigal, Ch. Pommerenke, 2006:

σ is a boundary regular f. pt. (BRFP) of (ϕ_t) $\iff$ $\lambda := \angle\lim_{\zeta\to\sigma} \dfrac{G(\zeta)}{\zeta-\sigma} \neq \infty$.

☞ a CB-process (Z_t) is *conservative*, i.e. $Z_t < +\infty$ a.s., **iff**

$\zeta = 0$ is a boundary fixed point of $\varphi_{s,t}$'s, i.e. $\angle \lim_{\zeta\to 0} \varphi_{s,t}(\zeta) = 0$.

☞ **Expectation:** $\mathbb{E}\left[Z_t \,\middle|\, Z_s = x\right] = x\,\varphi'_{s,t}(0)$ [for the conservative case]

The angular derivative $$\varphi'(\sigma) := \angle \lim_{\zeta\to\sigma} \frac{\varphi(\zeta)-\sigma}{\zeta-\sigma} = \liminf_{\zeta\to\sigma} \frac{\operatorname{Re}\varphi(\zeta)}{\operatorname{Re}\zeta}$$

A boundary fixed point σ is said to be *regular* if $\varphi'(\sigma) \neq \infty$.

Theorem (F. Bracci, M.D. Contreras, S. Días-Madrigal, P. G., 2015)

Let G be a H. v. f. and $(\varphi_{s,t})$ its (reverse) evolution family. Then:

σ is a BRFP for all $\varphi_{s,t}$'s $\Leftrightarrow$ $\lambda(t) := \angle \lim_{\zeta\to\sigma} \dfrac{G(\zeta,t)}{\zeta-\sigma}$ is $L^1_{\mathrm{loc}}([0,+\infty))$.

In this case, $$\varphi'_{s,t}(\sigma) = \exp\left(\int_s^t \lambda(u)\,du\right).$$

☞ a CB-process (Z_t) is *conservative*, i.e. $Z_t < +\infty$ a.s., **iff**

$\zeta = 0$ is a boundary fixed point of $\varphi_{s,t}$'s, i.e. $\angle\lim_{\zeta\to 0}\varphi_{s,t}(\zeta) = 0$.

☞ **Expectation:** $\mathbb{E}\left[Z_t \,\middle|\, Z_s = x\right] = x\,\varphi'_{s,t}(0)$ [for the conservative case]

The angular derivative $\varphi'(\sigma) := \angle\lim_{\zeta\to\sigma} \dfrac{\varphi(\zeta)-\sigma}{\zeta-\sigma} = \liminf_{\zeta\to\sigma} \dfrac{\operatorname{Re}\varphi(\zeta)}{\operatorname{Re}\zeta}$

☞ **The second moment:** $\mathbb{E}\left[Z_t^2 \,\middle|\, Z_s = x\right] =$

$$= \begin{cases} +\infty, & \text{if } \varphi''_{s,t}(0) = -\infty, \\ \left(x\,\varphi'_{s,t}(0)\right)^2 - x\,\varphi''_{s,t}(0), & \text{otherwise.} \end{cases}$$

Theorem (T. Hasebe, J.L. Pérez, P.G., 2022)

If $t \mapsto \mathbb{E}\left[Z_t^k \,\middle|\, Z_0 = 1\right]$, $k = 1, 2$, are $\mathrm{AC}_{\mathrm{loc}}$, then $(\varphi_{s,t})$ is AC.

Theorem (T. Hasebe, J.L. Pérez, P.G., 2022)

There is a one-to-one correspondence among:

- families $(k_{s,t})$ of transition kernels of CB-processes with $t \mapsto \mathbb{E}\left[Z_t^k \,\middle|\, Z_0 = 1\right]$, $k = 1, 2$, in $\mathrm{AC}_{\mathrm{loc}}([0, +\infty))$;
- AC reverse evolution families $(\varphi_{s,t}) \subset \left\{\varphi \in \mathfrak{BF} : \varphi''(0) \neq -\infty\right\}$;
- the class of Herglotz vector fields given by Silverstein-type representation

$$G(\zeta, t) = a_t\zeta - b_t\zeta^2 + \int_0^{+\infty} \left(1 - e^{-x\zeta} - x\zeta\,\mathbf{1}_{(0,1)}(x)\right)\pi_t(\mathrm{d}x),$$

where

(a) $a_t \in \mathbb{R}$, $b_t \geqslant 0$, $(q_t \equiv 0)$, and π_t are a non-negative Borel measures on $(0, +\infty)$;

(b) $t \mapsto \pi_t(B)$ is measurable for any Borel set $B \subset (0, +\infty)$;

(c) $t \mapsto a_t$, $t \mapsto b_t$, $t \mapsto \int_0^{+\infty} x^2\,\pi_t(\mathrm{d}x)$ are in L^1_{loc}.

Probabilistic interpretation of $\tau = 0$

Remark: $\tau = 0$ is the DW-point of $\varphi : \mathbb{H}_r \to \mathbb{H}_r$ **iff**
$\varphi(0) = 0$ and $\varphi'(0) \leqslant 1$.

Recall: $\mathbb{E}\left[Z_t \,\middle|\, Z_s = x\right] = x\,\varphi'_{s,t}(0)$.

Conclusion: $\varphi_{s,t}$'s DW-point is at $\tau = 0$ **iff**
$t \mapsto \mathbb{E}\left[Z_t \,\middle|\, Z_0 = 1\right]$ is *non-increasing*.

Extinction time: $T_0^s := \inf\{t \geqslant s : Z_t = 0\}$.

Theorem (T. Hasebe, J.L. Pérez, P.G., 2022) $[T_0^s < +\infty]$

Suppose the Laplace exponents form an AC reverse EF $(\varphi_{s,t})$ with the DW-point $\tau = 0$ and with the H. v. f. G.
If $\int_0^{+\infty} G''(\infty, t)\,dt = -\infty$, then $q_\infty(s) := \lim_{t\to+\infty} \varphi_{s,t}(\infty) = 0$ and

hence $\mathbb{P}\left[T_0^s < +\infty \,\middle|\, Z_s = x\right] = e^{-q_\infty(s)} = 1 \quad \forall\, x \in (0, +\infty),\ s \geqslant 0.$

Theorem [monotonicity] (T. Hasebe, J.L. Pérez, P.G., 2022)

Let (Z_t) be a CB-process with associated topological REF $(\varphi_{s,t})$. TFAE:

(i) the DW-point of all $\varphi_{s,t}$'s is at ∞;

(ii) $\mathbb{P}\left[Z_u \leqslant Z_t \,\middle|\, Z_s = x\right] = 1$ whenever $0 \leqslant s \leqslant u \leqslant t$ and $x > 0$;

(iii) for any $(s,t) \in \Delta$ and for some (and hence all) $x > 0$,

we have $\mathbb{P}\left[Z_t \geqslant x \,\middle|\, Z_s = x\right] = 1$.

Explosion time: $T^s_\infty := \inf\{t \geqslant s : \; Z_t = +\infty\}$.

Theorem (T. Hasebe, J.L. Pérez, P.G., 2022) $[T^s_\infty < +\infty]$

Suppose the Laplace exponents form an AC reverse EF $(\varphi_{s,t})$

with the DW-point $\tau = \infty$ and with the H. v. f. G.

If
$$\int_0^{+\infty} G(0,t)\, \exp\Big(\int_0^t G'(\infty, s)\, ds\Big)\, dt \;=\; +\infty, \qquad (*)$$

then $q_0(s) := \lim_{t \to +\infty} \varphi_{s,t}(0) = +\infty$, and hence

$\mathbb{P}\left[T^s_\infty < +\infty \,\middle|\, Z_s = x\right] \;=\; 1 - e^{-q_0(s)} \;=\; 1$ for any $s \geqslant 0$ and $x \in (0, +\infty)$.

Consider again a branching process (X_t)
on the **discrete state** space $\mathbb{N}_0^* = \{0, 1, 2, \ldots\} \cup \{+\infty\}$,
with the transition probabilities $p_{s,t}(k) := \mathbb{P}\Big[X_t = k \,\Big|\, X_s = 1\Big]$.

Definition (spatial embedding) M. Jiřina, 1958

We say that (X_t) *embeds in* (or *extends to*)
a CB-process (Z_t) on $[0, +\infty]$ with the transition kernels $(k_{s,t})$

if $\quad k_{s,t}(1; \{k\}) = p_{s,t}(k) \quad$ for any $k \in \mathbb{N}_0^*$ and all $(s, t) \in \Delta$.

Theorem (T. Hasebe, J.L. Pérez, P.G., 2022)

(X_t) embeds into some CB-process (Z_t) **iff**
the probability generating functions $(F_{s,t})$ of (X_t)
have common DW-point at $\tau = 0$.

The Laplace exponents of (Z_t) are given by $\exp\Big(-\varphi_{s,t}(\zeta)\Big) = F_{s,t}(e^{-\zeta})$.