Free probability and polynomial roots under repeated differentiation

Andrew Campbell

Based on joint work with Sean O'Rourke (CU Boulder) and David Renfrew (Binghamton University): arXiv:2307.11935

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How does differentiation affect polynomial roots?

Question

Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of degree n polynomials whose empirical root measures (ERM) $\mu_{p_n} = \frac{1}{n}\sum_{z:p_n(z)=0} \delta_z$ converge to a probability measure μ. For $t \in [0, 1)$, what is the limiting ERM of $p_n^{(\lfloor tn \rfloor)}$?

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This conjecture is not true in general. However, it is true for polynomials with real roots and (finite) free probability leads to a remarkably short proof. More precise versions of the conjecture are still open for random polynomials with complex roots.

Real root case

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In a seemingly unrelated paper Shlyakhtentko and Tao (2022) established the same dynamics (up to a rescaling) for the **fractional free convolution powers** of a measure μ.

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Corners of matrices/operators

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$$\mathcal{M}_p := \{[pap] : a \in \mathcal{M}\}$$

with

$$\tau_p([pap]) = \lambda^{-1}\tau(pap)$$

for any $a \in \mathcal{M}$.

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for any $a \in \mathcal{M}$. We then consider the map $\pi_\lambda : \mathcal{M} \to \mathcal{M}_p$ by $\pi_\lambda(a) := [pap]$.

Corners of matrices/operators

Nica and Speicher (1996) proved that if a is a self-adjoint element in $\mathcal{M}$ with law μ, that is freely independent of p, with $\lambda = 1/k$, then $k\pi(a)$ has the law $\mu^{\boxplus k}$ for $k \in \mathbb{N}$.

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From the PDE established by Steinerberger for polynomials and Shlyakhtentko and Tao for fractional free convolutions one would expect (at least formally) the limiting ERM of the tn-th derivative μ_t (if such a limit exists) to equal $\mu^{\boxplus \frac{1}{1-t}}$ up to a rescaling of the support.

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Theorem (Hoskins and Kabluchko 2021, Arizmendi, Garza-Vargas, Perales 2023)

Let μ *be a compactly supported probability measure on the real line, and let* $\{p_n\}$ *be a sequence of real rooted polynomials with limiting ERM* μ. *For any fixed* $t \in (0,1)$, *the ERM of the* $(\lceil tn \rceil)$*-th derivative of* $p_n((1-t)x)$ *converges weakly to* $\mu^{\boxplus \frac{1}{1-t}}$ *as* $n \to \infty$.

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The proof by Arizmendi, Garza-Vargas, and Perales, using the recently developed *finite* free convolutions of Marcus, Spielman, and Srivastava, expresses differentiation explicitly as the finite free multiplicative convolution with the polynomial $q(x) = x^a(1-x)^b$.

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Theorem (Kabluchko 2022+)

Let μ be a compactly supported probability measure on the real line, and let $\{p_n\}$ be a sequence of real rooted polynomials with limiting ERM μ. Define the polynomial $q_n(z;s) = \exp\left(-\frac{s}{2}\partial_z^2\right) p_n(z)$. Then, the limiting ERM of $q_n(z;t^2/n)$ is $\mu \boxplus SC_t$.

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- So instead of the spectral measure, we are forced to work with the Brown measure of any operator involved.

The Brown measure of an element a in $(\mathcal{M}, \tau)$ is given, in the distributional sense,

$$\mu_a = \frac{1}{2\pi} \Delta \log \lim_{\varepsilon \searrow 0} \left(\exp(\log \tau((a-z)^*(a-z)+\varepsilon)) \right).$$

Complex roots and non-self-adjoint elements cont.

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- O'Rourke and Steinerberger (2021) formally established a PDE for the root flow under differentiation.
- Hoskins and Kabluchko (2021) verified that the Feng–Yao limit satisfies the O'Rourke–Steinerberger PDE.

R-diagonal operators

An element $a \in \mathcal{M}$ is said to be R-diagonal if there exists $*$-free elements u and h in $\mathcal{M}$, such that u is Haar unitary (i.e. unitary with $\tau(u^n) = 0$ for any $n \in \mathbb{N}$), h is positive, and a has the same $*$-distribution as uh.

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One example is $a = s_1 + is_2$, where s_1 and s_2 self-adjoint free semicircular elements. In this case a is known as a circular element with Brown measure $d\mu_a = \frac{1}{\pi}\mathbf{1}_{|z|\leq 1}$.

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2. The set of R-diagonal operators is closed under powers, free sums, and free products.
3. The Brown measure of a is determined by the spectral measure of $|a| := \sqrt{a^*a}$.
4. For any freely independent R-diagonal operators a and b, $\tilde{\mu}_{|a+b|} = \tilde{\mu}_{|a|} \boxplus \tilde{\mu}_{|b|}$.

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$$\mu_a \oplus \mu_b := \mu_{a+b} = \mathcal{H}(\mathcal{H}^{-1}(\mu_a) \boxplus \mathcal{H}^{-1}(\mu_b)),$$

where a and b are $*$-freely independent R-diagonal operators.

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Proposition (C., O'Rourke, Renfrew)

For all $j \geq 1$, there exists a probability measure $\mu^{\oplus j}$ such that $\mu^{\oplus j}$ agrees with the j-th power of $\oplus$ for integer j and $\{\mu^{\oplus j}\}_{j \geq 1}$ forms a convolution semigroup:

$$\left(\mu^{\oplus j}\right)^{\oplus l} = \mu^{\oplus jl}$$

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As was the case with $\boxplus$, $\oplus$ is extended to fractional powers through a corner algebra $(\mathcal{M}_p, \tau_p)$.

Kac polynomials and Haar unitaries

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Kac polynomials and Haar unitaries

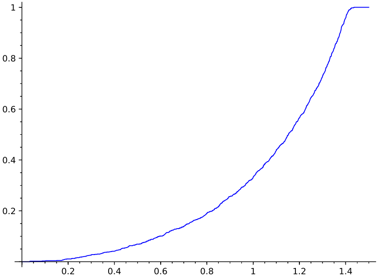

Figure: The left is a plot of the radial cumulative distribution function for $u + v$. The right it the radial cumulative distribution function for the 500-th derivative of a degree 1000 Kac polynomial (up to a push-forward by $x \mapsto \sqrt{x}$ and a rescaling).

Polynomials with independent coefficients

$$p_n(z) = \sum_{k=0}^{n} \xi_k p_{k,n} z^k$$

$$\mathbb{P}(\xi_0 = 0) = 0 \qquad \text{and} \qquad \mathbb{E}\log(1 + |\xi_0|) < \infty.$$

The coefficients $p_{k,n}$ are assumed to satisfy the following assumption.

Assumption

There exists a function $p : [0, \infty) \to [0, \infty)$ so that

1. $p(t) > 0$ for $t \in [0, 1)$ and $p(t) = 0$ for $t > 1$;
2. p is continuous on $[0, 1)$ and left continuous at 1; and
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3. $\lim_{n\to\infty} \sup_{0\le k\le n} \left||p_{k,n}|^{1/n} - p(\frac{k}{n})\right| = 0$.

Many radially symmetric probability measures can be recovered as the limiting ERM of some random polynomials with independent coefficients.

R-diagonal operators and random polynomials

Before we state the connection between polynomials with complex roots and R-diagonal operators we denote by ψ_2 the bijection on radially symmetric probability measures where $\psi_2\mu(\mathbb{D}_r) = \mu(\mathbb{D}_{\sqrt{r}})$.

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Theorem (C., O'Rourke, Renfrew)

Let $\{p_n\}$ be random polynomials with independent coefficients satisfying the assumptions, where μ is the limiting ERM of p_n. Additionally assume there exists an R-diagonal operator a with Brown measure $\psi_2^{-1}\mu$. For $t \in (0,1)$, let μ_t be the limiting ERM of the $\lceil tn \rceil$-th derivative of $p_n((1-t)^2 x)$. Then, $\mu_t = \psi_2(\psi_2^{-1}\mu)^{\oplus \frac{1}{1-t}}$.

R-diagonal operators and random polynomials

$$
\begin{array}{ccc}
\text{Brown Measure} & \xrightarrow{\psi_2} & \text{Polynomial Root Measure} \\
\Big\downarrow (\cdot)^{\boxplus \frac{1}{1-t}} & & \Big\downarrow \frac{d^{tn}}{dz^{tn}} \\
\text{Convolution Brown Measure} & \xleftarrow{\psi_2^{-1}} & \text{Derivative Root Measure}
\end{array}
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both have limiting ERM uniform on the unit circle. However, the limiting ERM of $p_n^{(\lfloor tn \rfloor)}$ is not δ_0.

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3. In the self-adjoint (real rooted) case this exactly determines the limiting spectral measure (differentiated root measure).
4. In the non-self-adjoint (complex root) case the instability of eigenvalues (roots) under small perturbations leads to counter examples.
5. However, for random objects one expects to avoid these counterexamples with high probability.

Why look for a free probability connection?

Conjecture (Kabluchko, Hoskins–Kabluchko)

The limiting ERM of $p_n^{(\lfloor tn \rfloor)}$ exists, and depends only on μ and t, given that p_n is random with independent identically distributed roots.

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Current approaches to this problem require sophisticated anti-concentration estimates and the state of the art can handle $tn \approx \log n / \log \log n$ (Michelen and T. Vu 2022+). Perhaps free probability could lead to progress on this conjecture, similar to Śniady's work on regularizing Brown measures and the circular law.

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Let μ be a measure arising as the limiting ERM of some random polynomials with independent coefficients, $p_n(z)$, such that $\mu(\mathbb{C} \setminus \mathbb{D}_r) \sim r^{-\frac{\alpha}{2-\alpha}}$ for some $\alpha \in (0, 2]$. Additionally let μ_t denote the limiting ERM (established by Feng and Yao) of $p_n^{(\lfloor tn \rfloor)}((1-t)^{2-\frac{2}{\alpha}} z)$. Then, μ_t converges weakly to μ_α as $t \to 1^-$, where μ_α is a probability measure depending only on α.

Consequences

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μ_2 is the probability measure on the unit disk with density $\frac{1}{2\pi|z|}$, i.e. the image of the circular law under ψ_2.

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The radial CDF $\Phi_\alpha(r) = \mu_\alpha(\mathbb{D}_r)$ has inverse (or radial quantile function)

$$\Phi_\alpha^{\langle -1 \rangle}(x) = \frac{x}{(1-x)^{\frac{2}{\alpha}-1}},$$

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$x \in [0,1)$. Kösters and Tikhomirov defined the notion of measure being $\oplus$-stable and observed a one-to-one correspondence between $\oplus$-stable distributions and symmetric $\boxplus$-stable distributions. In fact, $\psi_2^{-1}\mu_\alpha$ are the $\oplus$-stable distributions identified by Kösters and Tikhomirov.

Stable laws as operators and polynomials

1. $\alpha = 2$: The R-diagonal operator with Brown measure $\psi_2^{-1}\mu_2$ is the circular operator, and the polynomials associated to μ_2 is the (rescaled) random Taylor polynomials $p_n(z) = \sum_{k=0}^{n} \frac{n^k}{k!}\xi_k z^k$.

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2. $\alpha = 1$: The R-diagonal operator with Brown measure $\psi_2^{-1}\mu_1$ is xy^{-1}, where x and y are freely independent circular operators. The polynomials associated to μ_1 are $p_n(z) = \sum_{k=0}^{n} \binom{n}{k}\xi_k z^k$.

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3. $\alpha = \frac{2}{1+l}$, $l \in \mathbb{N}$: R-diagonal with Brown measure $\psi_2^{-1}\mu_\alpha$ is $x_0 x_1^{-1} \cdots x_l^{-1}$, and the polynomials are

$$p_n(z) = \sum_{k=0}^{n} \left(\frac{k!}{n^k}\right)^{l-1} \binom{n}{k}^{l} \xi_k z^k$$

The fractional convolution revisited

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Theorem (Haagerup and Larson)

Let a be an R-diagonal operator, then Brown measure of a is radially symmetric and its radial CDF is given by:

$$F_a(r) := \mu_a(\mathbb{D}_r) = 1 + \mathcal{S}_{a^*a}^{\langle -1 \rangle}(r^{-2})$$

for r in some suitable range, where $\mathcal{S}_{a^*a}$ is the S-transform of a^*a.

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Lemma (C., O'Rourke, Renfrew)

Let $p \in \mathcal{M}$ be a projection with $\tau(p) = \lambda \in (0,1]$, and $a \in \mathcal{M}$ be R-diagonal, such that p is $*$-free from a. Then

$$\mathcal{S}_{\lambda^{-2}\pi_\lambda(a)\pi_\lambda(a)^*}(z) = \frac{\lambda(1+\lambda z)}{1+z}\mathcal{S}_{aa^*}(\lambda z).$$

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The limiting radial quantile functions of random polynomials with independent coefficients established by Feng and Yao, and Hoskins and Kabluchko satisfy:

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These are equal, after applying ψ_2, up to a factor of $(1-t)^2$.

S-transform and polynomial coefficients

For an R-diagonal operator a the radial quantile function can be related to the S-transform of a^*a by:

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To build the coefficients which have limiting radial quantile function $\Phi^{\langle -1 \rangle}$ one should take the choice of polynomials

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Repeated differentiation is interpreted in terms of fractional $\oplus$ powers by observing (through these relations) the affects of both processes on the radial quantile functions of the measures.

Thank you!