Free Cumulants and Large Deviations

for Classical and Quantum Symmetric Exclusion Processess

Probabilistic Operator Algebra Seminar
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Exclusion Process

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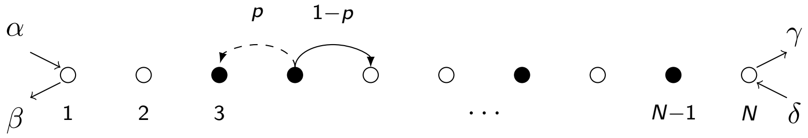

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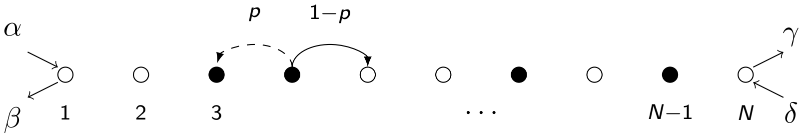

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In the large time limit a current is established and the configuration of particles converges to a *stationary measure* μ which is a probability measure on the set of configurations

$$\Omega = \{0, 1\}^N$$

Density profile

For large times and large N the *density profile* $n(x)$ gives the local density of particles according to their position $x \in [0,1]$.

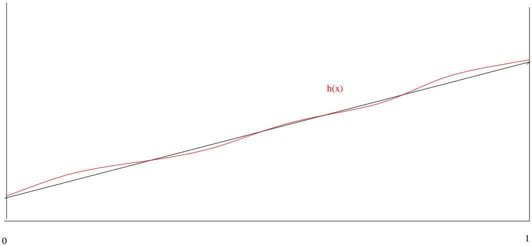
$$n(x) \sim \frac{\text{number of particles in } [xN, (x+\delta x)N]}{N\delta x}$$

It converges to a linear function on $[0,1]$ (which depends on $\alpha, \beta, \gamma, \delta$).

$$n(x) = n_0 + (n_1 - n_0)x \qquad 0 \le t \le 1$$

Without loss of generality one can assume that $n_0 = 0, n_1 = 1$.

Large deviations for the density profile



The probability that the density profile is close to some function $h(x)$ behaves as

$$e^{-NI_{ssep}(h)}$$

where I_{ssep} is the large deviation functional, computed by Derrida, Lebowitz, and Speer (2001).

$I_{ssep}(h)$ is the Legendre transform of F_{ssep} defined by the extremization problem

$$F_{\mathrm{ssep}}[h] = \max_{g(\cdot)} F[h;g]$$

$$F[h;g] := \int_0^1 dx \left[\log\left(1+g(x)e(x)\right) - \log(g'(x))\right]$$

with $e(x) = e^{h(x)} - 1$ and $g(x)$ solution of the non-linear differential equation,

$$\big(1+g(x)\,e(x))\big)\,g''(x) = g'(x)^2\,e(x)\ ,$$

with boundary conditions $g(0)=0$ and $g(1)=1$.

This formula is established using the *matrix ansatz*.

The aim of this talk is to reveal free cumulants hidden behind the formulas for the large deviation functional.

This is based on joint work with M. Bauer, D. Bernard, L. Hruzsa.

For this we use connections with a quantum version of the SSEP.

Quantum Symmetric Simple Exclusion Process

Fermionic particles on $\{1, 2, \ldots, N\}$ are subject to a Hamiltonian

$$H_t = \sum_{j=1}^{N-1} c_{j+1}^{\dagger} c_j W_t^j + c_j^{\dagger} c_{j+1} \bar{W}_t^j$$

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$c_i^{\dagger}, c_i$=fermionic creation and annihilation operators, satisfying the CAR, e.g.

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V is the quantum version of $\Omega = \{0, 1\}^N$.
A state of the form $e_{i_1} \otimes \ldots \otimes e_{i_N}$ corresponds to a classical configuration (e_0 =empty, e_1=occupied).

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It satisfies the evolution equation:

$$d\rho_t = -i[dH_t, \rho_t] - \frac{1}{2}[dH_t, [dH_t, \rho_t]] + \mathcal{L}_{bdry}(\rho_t)dt$$

$\mathcal{L}_{bdry}$ is a boundary term describing what happens at the boundary sites $1, N$.

ρ_t is a random matrix, if the initial configuration is diagonal on the classical states the expected value $\bar{\rho}_t$ satisfies the same evolution as the classical SSEP.

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In particular, for a fixed time t, the joint distribution of occupation numbers
τ_i:= 0 if site i is empty, 1 if occupied
can be expressed in terms of the QSSEP.

$$\mathbb{E}_{\text{ssep}}\left[e^{\sum_j h_j \tau_j}\right] = \mathbb{E}_\infty\left[\mathrm{Tr}\big(\rho\, e^{\sum_i h_i \hat{n}_i}\big)\right] = \mathrm{Tr}\left(\bar{\rho}_\infty\, e^{\sum_i h_i \hat{n}_i}\right)$$

with $\hat{n}_i := c_i^\dagger c_i$ the quantum number operators, $\bar{\rho}_\infty := \mathbb{E}_\infty[\rho]$ the mean Q-SSEP state, averaged w.r.t. the Q-SSEP steady measure denoted $\mathbb{E}_\infty$.

Large deviations for QSSEP and free cumulants

The two-point functions $G_{ij} = Tr(\rho c_i c_j^\dagger)$ form a random matrix

$$\mathbf{G} = (G_{ij})_{1 \le i,j \le N}$$

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The fluctuations of $\mathbf{G}$ are measured by their cumulants

$$E[G_{i_1 j_1} G_{i_2 j_2} \dots G_{i_p j_p}]^c = K_p(G_{i_1 j_1}, G_{i_2 j_2}, \dots, G_{i_p j_p})$$

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These are the quantities of interest.

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Only the ones for which $j_1, \ldots, j_p$ is a cyclic permutation of $i_1, \ldots, i_p$ have a nonzero limit.
If $i_1/N, i_2/N, \ldots, i_p/N \to u_1, u_2, \ldots, u_p \in [0,1]$ as $N \to \infty$, then

$$E[G_{i_1 i_p} G_{i_p i_{p-1}} \ldots G_{i_2 i_1}]^c = \frac{1}{N^{p-1}} g_p(u_1, \ldots, u_p) + O(\frac{1}{N^p})$$

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for some functions g_p.

The g_p are piecewise polynomial functions, polynomial in each sector corresponding to an ordering of the u_i.

Loop polynomials (Bernard and Jin, 2021)

Define $Q_\sigma(x_1, \ldots, x_p)$ for $0 \leq x_1 \leq x_2 \leq \ldots \leq x_p \leq 1$, indexed by circular permutations σ of $1, \ldots, p$ by

$$E[G_{i_1 i_{\sigma^{p-1}(1)}} G_{i_{\sigma^{p-1}(1)} i_{\sigma^{p-2}(1)}} \ldots G_{i_{\sigma(1)} i_1}]^c = \frac{1}{N^{p-1}} Q_\sigma(x_1, \ldots, x_p) + O(\frac{1}{N^p}).$$

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The Q_σ are the loop polynomials. They give the values of the functions g_p in each sector.

The loop polynomials as free cumulants

On $[0,1] \subset \mathbf{R}$ with Lebesgue measure let for $x \in [0,1]$

$$\Pi_x = 1_{[0,x]}$$

The Π_x for a commutative family of random variables.

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Theorem (B. 2022)

$$\boxed{Q_\sigma(x_1,\ldots,x_n) = \kappa_n(\Pi_{x_1}, \Pi_{x_{\sigma(1)}}, \Pi_{x_{\sigma^2(1)}}, \Pi_{x_{\sigma^{n-1}(1)}})}$$

Here the κ_n are the *free cumulants*.

The identification of the large deviation functional for the SSEP will be based on the computation of the asymptotics of the Laplace transform of the joint distribution of the occupation numbers τ_i, using their relation with the QSSEP.

$$\mathbb{E}_{\mathrm{ssep}}\left[e^{\sum_j h_j \tau_j}\right] = \mathbb{E}_\infty\left[\mathrm{Tr}\big(\rho\, e^{\sum_i h_i \hat{n}_i}\big)\right] = \mathrm{Tr}\left(\bar{\rho}_\infty\, e^{\sum_i h_i \hat{n}_i}\right)$$

This relation allows to relate non-coincident cumulants of the SSEP and the QSSEP.

Define the scaled cumulants of the SSEP as

$$g_n^{ssep}(x_1, \ldots, x_n) = \lim_{N \to \infty} N^{n-1} K(\tau_{i_1}, \ldots, \tau_{i_n})$$

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For $0 < x_k < 1$ all different, we have

$$g_n^{\mathrm{ssep}}(x_1, \cdots, x_n) = (-1)^{n-1} \sum_{\sigma \in S_n/\mathbb{Z}_n} g_n(x_{\sigma_1}, \cdots, x_{\sigma_n}).$$

The sum is over all permutations σ modulo cyclic permutations.
There are $(n-1)!$ terms in the sum.

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This relation follows from Wick's theorem. Together with the link between the Q-SSEP cumulants and free probability this will give the new construction of the classical SSEP large deviation function.

The occupation numbers are Bernoulli variables: they only take values 0 or 1.
In general the joint distribution of N Bernoulli variables depends on 2^N numbers.
In particular their cumulants depend on 2^N numbers, the non-coincident cumulants.

$$K(b_{i_1}, b_{i_2} \ldots, b_{i_k}); \quad i_1 < i_2 < \ldots < i_k$$

We first give a general formula for cumulants of Bernoulli variables in terms of non-coincident cumulants.

A general formula for cumulants of Bernoulli variables

Let b_i be a family of (commuting) Bernoulli variables,

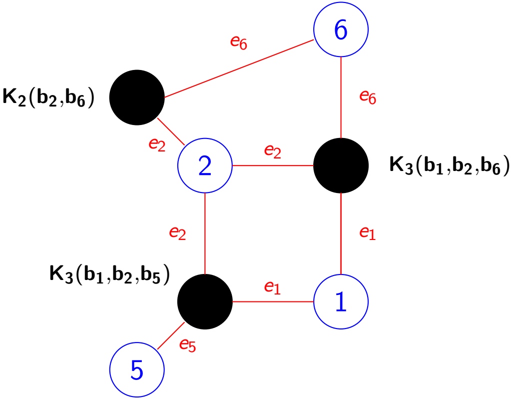
$$W = \log E\left[e^{\sum_{i=1}^N h_i b_i}\right] = \sum_H \frac{\mu(H^\bullet)}{|\mathrm{Aut}\, H|} \sum_{\mathcal{L} \in Lab(H)} w(\mathcal{L}), \tag{1}$$

where the sum is over connected bipartite graphs, $\mathcal{L}$ runs over all labellings of the white vertices of H by distinct indices and

$$w(\mathcal{L}) = \prod_\bullet K(b_\bullet) \prod_{\text{edges of } H} e_i \tag{2}$$

The graph $H^\bullet$ is the induced graph on black vertices, μ the value of a certain Möbius function associated to this graph and $|Aut(H)|$ the number of automorphisms of the unlabelled graph.

An example:



The graph $H^\bullet$ is a complete graph with three vertices so that $\mu(H^\bullet) = 2$ and there are no nontrivial automorphisms moreover the weight of the labelling is

$$w(\mathcal{L}) = e_1^2 e_2^3 e_5 e_6^2 K_3(b_1, b_2, b_6) K_2(b_2, b_6) K_3(b_1, b_2, b_5)$$

The preceding formula can be obtained using the theory of cumulants with products as entries (Leonov-Shiryaev formula) as well as some graph theory involving the chromatic polynomials.

A scaling limit

The previous formula simplifies if one makes the assumption that the cumulants scale, for $N \to \infty$, as

$$K_n(b_{i_1}, \ldots, b_{i_n}) \sim N^{1-n} \psi(i_1/N, \ldots, i_n/N)$$

for some continuous function ψ on $\Sigma = [0, 1]$.
In this case, in the sum only the contribution of graphs G which are trees survives as $N \to \infty$.

Inroduce the generating function of the ψ_n as

$$F_0(q) = \sum_n \frac{1}{n!} \int_{\Sigma^n} q(s_1)q(s_2)\dots q(s_n)\psi_n(s_1,\dots,s_n)ds_1\dots ds_n$$

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In the scaling limit, the free energy is obtained by solving the following variational problem

$$\lim_{N\to\infty} \frac{1}{N} W = \max_{g,q} \left[\int [\log(1+e(s)g(s)) - q(s)g(s)]ds + F_0(q) \right]$$

We can apply the preceding analysis to the case of the SSEP/QSSEP relation and make the connection with the free cumulants.
For this is it natural to introduce formulas related to the R-tranform:

Let $b(x) := -\int_x^1 dy\, a(y)$ define

$$F_0[a] = \int_0^1 dx\, \log(z - b(x)) - z + 1, \quad \text{with} \int_0^1 \frac{dy}{z - b(y)} = 1$$

A new formula for the SSEP large deviation functional

Bauer, Bernard, B., Hruza, 2023

$$
\begin{aligned}
I_{\rm ssep}[n] \quad = \quad & \max_{g(\cdot),q(\cdot)} \Big(\int_0^1 dx \Big[n(x) \log \Big(\frac{n(x)}{g(x)} \Big) \\
& + (1-n(x)) \log \Big(\frac{1-n(x)}{1-g(x)} \Big) + q(x) g(x) \Big] - F_0[q] \Big)
\end{aligned}
$$

One can check that this coincides with the previous formulation of Derrida, Lebowitz and Speer.

THANK YOU