Hermite trace polynomials

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Gaussian Hilbert space I.

- $\mathcal{H}_{\mathbb{R}} =$ real Hilbert space, $\mathcal{H}$ its complexification.
- Linear embedding $X : \mathcal{H}_{\mathbb{R}} \to (\mathcal{A}, \mathbb{E})$ into an algebra of random variables s.t. each $X(f)$ is Gaussian, induced map $\mathcal{H} \to L^2(\mathcal{A}, \mathbb{E})$ isometric.
- $\mathcal{P}_n =$ polynomials in $\{X(h) : h \in \mathcal{H}_{\mathbb{R}}\}$ of degree $\leq n$
- $\mathcal{P}(\mathcal{H}_{\mathbb{R}}) = \mathcal{P} =$ all such polynomials, $\simeq \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{R}}^{\odot_s n}$.
- WLOG $\mathcal{P} = \mathcal{A}$, $L^2(\mathcal{P}, \mathbb{E}) = \mathcal{F}_s(\mathcal{H})$, the symmetric Fock space, $X(h) =$ field operators on it.

Gaussian Hilbert space II.

- Denote $\mathrm{T}\,(h_1 \otimes \ldots \otimes h_n) = X(h_1) \ldots X(h_n)$, defined on $\bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{R}}^{\odot_s n}$.
- The projection $W : \overline{\mathcal{P}}_n \to \overline{\mathcal{P}}_n \cap \mathcal{P}_{n-1}^{\perp}$ is the Wick product. Defined on $\bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{R}}^{\otimes_s n}$.
- $W(h_1 \otimes \ldots \otimes h_n) =$ polynomial in $\{X(h_i) : 1 \leq i \leq n\}$, the Hermite polynomial.
- Explicitly,

$$W(h_1 \otimes \ldots \otimes h_n) = X(h_1) W(h_2, \ldots, h_n) - \sum_{i=2}^{n} \mathbb{E}\left[X(h_1) X(h_i)\right] \, W(h_1, \ldots, \hat{h}_i, \ldots, h_n).$$

- Properties later.

Guţă and Maassen (2002) construction.

- $\{V_n : n \in \mathbb{N}\}$ additional Hilbert spaces, unitary action of $S(n)$ on V_n.
- $V_n \otimes_s \mathcal{H}^{\otimes n} =$ fixed point subspace of the action

$$v \otimes F \mapsto (\sigma \cdot v) \otimes U_\sigma F,$$

where

$$U_\sigma(h_1 \otimes \ldots \otimes h_n) = h_{\sigma^{-1}(1)} \otimes \ldots \otimes h_{\sigma^{-1}(n)}.$$

- Symmetrized Fock space $\mathcal{F}_s = \bigoplus_{n=0}^{\infty} V_n \otimes_s \mathcal{H}^{\otimes n}$.
- No canonical creation operator $a^+(h)$, need a sequence of maps $j_n : V_n \to V_{n+1}$ which intertwine the actions.

Symmetric group.

Notation.

- $S_0(n) = S(\{0, 1, \ldots, n\})$.
- $S(n)$ acts on $\mathbb{C}[S_0(n)]$ by conjugation.
- For $\alpha \in S_0(n)$, $|\alpha| = (n+1) - \mathrm{cyc}(\alpha) = n - \mathrm{cyc}_0(\alpha)$.

Notation.

- $\mathrm{Par}(n) =$ (number) partitions of n elements
  $=$ Young diagrams with n boxes.
- $\mathrm{Par}(n; \leq N) =$ partitions with at most N parts
  $=$ Young diagrams with at most N rows.

Character χ_q.

- Define $\chi_q : S_0(n) \to \mathbb{C}$, $\chi_q(\alpha) = q^{|\alpha|}$, extend to the group algebra $\mathbb{C}[S_0(n)]$.
- χ_q are positive semi-definite for all n if

$$q \in \mathcal{Z} = \{0\} \cup \left\{ \pm \frac{1}{N} : N \in \mathbb{N} \right\}.$$

  (Gnedin, Gorin, Kerov 2013, Köstler, Nica 2021).
- $\chi_{1/N}$ is the normalized character of the standard representation

$$\pi_{n,q} : \mathbb{C}[S_0(n)] \to \mathrm{End}\left((\mathbb{C}^N)^{\otimes(n+1)} \right).$$

Kernel of χ_q.

- Wedderburn isomorphism

$$\mathcal{W} : \sum_{\lambda \in \mathrm{Par}(n+1)} M_{d_\lambda}(\mathbb{C}) \to \mathbb{C}[S_0(n)]$$

- Denote $\mathbb{C}[S_0(n)]_{\leq N} = \mathcal{W}\left(\sum_{\lambda \in \mathrm{Par}(n+1; \leq N)} M_{d_\lambda}(\mathbb{C}) \right)$ and similarly for $\mathbb{C}[S_0(n)]_{>N}$.
- The kernel

$$\mathcal{N}_{q,n} = \{\eta \in \mathbb{C}[S_0(n)] : \chi_q[\eta\eta^*] = 0\}$$

is $\mathcal{N}_{1/N,n} = \mathbb{C}[S_0(n)]_{>N}$, so that χ_q is faithful on $\mathbb{C}[S_0(n)]_{\leq N}$.

Fock space I.

- Guţă-Maassen Fock space

$$\overline{\mathcal{TP}}(\mathcal{H}) = \mathbb{C}(0) \oplus \bigoplus_{n=1}^{\infty} \left(\mathbb{C}[S_0(n)] \otimes_s \mathcal{H}^{\otimes n} \right),$$

where $S(n)$ acts on $\mathbb{C}[S_0(n)]$ by conjugation.

- For $q \in \mathcal{Z}$, inner product

$$\langle \alpha \otimes F, \beta \otimes G \rangle_q = \delta_{n=k} \sum_{\sigma \in S(n)} \chi_q(\alpha \sigma \beta^{-1} \sigma^{-1}) \langle F, U_\sigma G \rangle_{\mathcal{H}^{\otimes n}} .$$

Invariant under the $S(n)$ action.

Fock space II.

- The quotient by the kernel is $\overline{\mathcal{TP}}_q(\mathcal{H})$.

$$\overline{\mathcal{TP}}_{1/N}(\mathcal{H}) = \bigoplus_{n=0}^{N-1} \left(\mathbb{C}[S_0(n)] \otimes_s \mathcal{H}^{\otimes n}\right) \oplus \bigoplus_{n=N}^{\infty} \left(\mathbb{C}[S_0(n)]_{\leq N} \otimes_s \mathcal{H}^{\otimes n}\right).$$

- The completion is the Hilbert space $\mathcal{F}_q(\mathcal{H})$.

Star-algebra structure on the trace polynomials.

Definition. On $\mathcal{TP}(\mathcal{H}_\mathbb{R}) = \mathbb{C}(0) \oplus \bigoplus_{n=1}^{\infty} \left(\mathbb{C}[S_0(n)] \otimes_s \mathcal{H}_\mathbb{R}^{\odot n}\right)$, define the product

$$\mathrm{T}\left(\alpha \otimes_s F\right) \mathrm{T}\left(\beta \otimes_s G\right) = \mathrm{T}\left((\alpha \cup \beta) \otimes_s (F \otimes G)\right)$$

and the star

$$\mathrm{T}\left(\eta \otimes_s F\right)^* = \mathrm{T}\left(\eta^* \otimes_s F\right).$$

Example.

$(024)(13) \cup (021)(354) = (065)(798)\ (024)(13) = (02465)(13)(798)$

Proposition.

- Multiplication is well defined.
- $\left(\mathrm{T}\left(\alpha \otimes_s F\right) \mathrm{T}\left(\beta \otimes_s G\right)\right)^* = \mathrm{T}\left(\beta \otimes_s F\right)^* \mathrm{T}\left(\alpha \otimes_s G\right)^*$.

Contractions I.

Fix $\boxed{q \neq 0}$.

Definition. For a transposition $\pi = (ij) \in S(n)$, define the π-contraction by the linear extension of

$$\mathrm{T}\left(C_\pi(\alpha \otimes_s (h_1 \otimes \ldots \otimes h_n))\right) = q^{\mathrm{cyc}_0((\pi\alpha)|_{\{i,j\}^c}) - \mathrm{cyc}_0(\pi\alpha) + 1} \langle h_i, h_j \rangle$$
$$\mathrm{T}\left(P_{[0,n-2]}^{[0,n]\setminus\{i,j\}}(\pi\alpha)|_{\{i,j\}^c} \otimes_s (h_1 \otimes \ldots \otimes \widehat{h}_i \otimes \ldots \otimes \widehat{h}_j \otimes \ldots \otimes h_n)\right).$$

Remark. Recall:

- If i, j are in the same cycle of α, $(ij)\alpha$ splits this cycle into two.
- If i, j are in the different cycles of α, $(ij)\alpha$ merges these cycles.

Contractions II.

Example.

$$\begin{aligned} &\mathrm{T}\left(C_{(23)}((012)(345)\otimes(h_1\otimes\ldots\otimes h_5))\right) \\ &\quad\to\langle h_2,h_3\rangle\,\mathrm{T}\,((013452)\otimes(h_1\otimes h_4\otimes h_5)) \\ &\quad\to q\,\langle h_2,h_3\rangle\,\mathrm{T}\,((0145)\otimes(h_1\otimes h_4\otimes h_5)) \\ &\quad\to q\,\langle h_2,h_3\rangle\,\mathrm{T}\,((0123)\otimes(h_1\otimes h_4\otimes h_5))\,. \end{aligned}$$

Definition.

- Extend to C_π for $\pi\in\mathcal{P}_{1,2}(n)$ an (incomplete) matching / involution.
- Define the Laplacian $\mathcal{L}=\sum_\tau C_\tau$.

Wick products.

Definition. On $\mathcal{TP}(\mathcal{H}_{\mathbb{R}})$,

$$\mathrm{W}\left(\eta \otimes_s F\right) = \mathrm{T}\left(e^{-\mathcal{L}}(\eta \otimes_s F)\right) = \sum_{\pi \in \mathcal{P}_{1,2}} (-1)^{n-|\pi|} \mathrm{T}\left(C_\pi(\eta \otimes_s F)\right).$$

Remark. For $q = 1$, get ordinary Wick products.

Corollary.

$$\mathrm{T}\left(\eta \otimes_s F\right) = \mathrm{W}\left(e^{\mathcal{L}}(\eta \otimes_s F)\right) = \sum_{\pi \in \mathcal{P}_{1,2}} \mathrm{W}\left(C_\pi(\eta \otimes_s F)\right).$$

State.

Definition. Define a functional on $\mathcal{TP}(\mathcal{H}_{\mathbb{R}})$ by

$$\varphi_q[\mathrm{W}\,(\eta \otimes_s F)] = 0; \quad \varphi_q[1] = 1.$$

Theorem.

- φ_q is positive semi-definite exactly for $q \in \mathcal{Z} = \{0\} \cup \left\{\pm\frac{1}{N}, N \in \mathbb{N}\right\}$.
- $$\varphi_q\left[\mathrm{W}\,(\beta \otimes G)^* \,\mathrm{W}\,(\alpha \otimes F)\right] = \left\langle (\alpha \otimes F), (\beta \otimes G) \right\rangle_q .$$

  Thus $\mathcal{F}_q(\mathcal{H}) = L^2(\mathcal{TP}(\mathcal{H}_{\mathbb{R}}), \varphi_q)$ for $q \in \mathcal{Z}$.

Product formula and extension.

Proposition. Let $F \in \mathcal{H}_{\mathbb{R}}^{\odot n}$, $G \in \mathcal{H}_{\mathbb{R}}^{\odot k}$.

- Then

$$\mathrm{W}\left(\alpha \otimes_s F\right) \mathrm{W}\left(\beta \otimes_s G\right) = \sum_{\pi \in \mathcal{P}_{1,2}(n,k)} \mathrm{W}\left(C_\pi((\alpha \cup \beta) \otimes_s (F \otimes G))\right)$$

- and $\left\| \mathrm{W}\left(\alpha \otimes F\right) \mathrm{W}\left(\beta \otimes G\right) \right\|_{\varphi} \leq (n+k)!(2n)^k \left\| F \right\| \ \left\| G \right\|$.
- So can extend the star-algebra structure to

$$\overline{\mathcal{TP}}(\mathcal{H}_{\mathbb{R}}) = \left\{ \mathrm{W}\left(\eta \otimes_s F\right) : n \geq 0, \eta \in \mathbb{C}[S_0(n)], F \in \mathcal{H}_{\mathbb{R}}^{\otimes n} \right\}.$$

Note: $\mathrm{T}\left(\eta \otimes_s F\right)$ may not be defined for $F \in \mathcal{H}_{\mathbb{R}}^{\otimes n}$.

Conditional expectations.

Proposition. $\mathcal{H}'_{\mathbb{R}} \subset \mathcal{H}_{\mathbb{R}}$ a closed subspace, $P_{\mathcal{H}'} : \mathcal{H} \to \mathcal{H}'$ the orthogonal projection.

- The map $\overline{\mathcal{TP}}_q(\mathcal{H}_{\mathbb{R}}) \to \overline{\mathcal{TP}}_q(\mathcal{H}'_{\mathbb{R}})$ obtained by the linear extension of

$$\varphi\left[\mathrm{W}\left(\alpha \otimes F\right) \mid \mathcal{H}'\right] = \mathrm{W}\left(\alpha \otimes \left(P_{\mathcal{H}'}^{\otimes n} F\right)\right)$$

  is an algebraic conditional expectation.

- In the single-variable case, for $\alpha \in S_0(n)$, we have

$$\varphi\left[\mathrm{T}\left(\alpha \otimes h^{\otimes n}\right) \mid \mathcal{H}'\right] = \sum_{\pi \in \mathcal{P}_{1,2}(n)} \left\| P_{(\mathcal{H}')^\perp} h \right\|^{2|\mathrm{Pair}(\pi)|} \mathrm{T}\left(C_\pi(\alpha) \otimes (P_{\mathcal{H}'} h)^{\otimes |Sing\, \pi|}\right).$$

Fock representation.

- In the GNS representation of $(\mathcal{TP}(\mathcal{H}_{\mathbb{R}}), \varphi_q)$ on $\mathcal{F}_q(\mathcal{H})$, several choices for the creation operator: how to embed $S(n)$ into $S_0(n)$?
- The most interesting one: the field operator $X(h) = \mathrm{T}\left((01) \otimes h\right)$, corresponds to the creation operator

$$a^+_{(01)}(h)(\alpha \otimes F) = (0\, n+1)\alpha \otimes (h \otimes F).$$

Theorem. The distribution of $\mathrm{T}\left((01) \otimes h\right)$ is the unnormalized average empirical distribution of a GUE matrix with mean 0 and variance $\|h\|$.

Lack of cyclicity.

Proposition.

- Elements $\mathrm{T}\left((01) \otimes_s h\right)$ do not generate $\mathcal{T}\mathcal{P}(\mathcal{H}_{\mathbb{R}})$ as an algebra (vacuum not cyclic).
- Do generate if also allow conditional expectations $\varphi\left[\cdot|\mathcal{H}'\right]$.
- Or: if allow conditional expectations onto the center $\mathrm{Span}\left(\{\eta \otimes_s F : \eta \in S(n), n \in \mathbb{N}\}\right)$.

Bożejko and Guţă 2002.

- for $q = \pm\frac{1}{N}$, Fock space with the inner product

$$\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_k \rangle_q = \delta_{n=k} \sum_{\sigma \in S(n)} \chi_q[\sigma] \prod_{i=1}^{n} \left\langle f_i, g_{\sigma(i)} \right\rangle .$$

- For $\omega(h) = a^+(h) + a^-(h)$,

$$\langle \Omega, \omega(h_1) \ldots \omega(h_{2n}) \Omega \rangle = \sum_{\pi \in \mathcal{P}_2(2n)} q^{n - c(\pi)} C_\pi (h_1 \otimes \ldots \otimes h_{2n}).$$

- Compare with

$$\varphi_q \left[\mathrm{T} \left(\alpha \otimes (h_1 \otimes \ldots \otimes h_{2n}) \right) \right] = \sum_{\pi \in \mathcal{P}_2(2n)} q^{|\pi\alpha| - n} C_\pi (h_1 \otimes \ldots \otimes h_{2n}).$$

Commutation relation.

- Creation and annihilation operators satisfy a commutation relation
$$a^-(f)a^+(g) = \langle f, g\rangle + q\, d\Gamma(|g\rangle\langle f|), \qquad (*)$$
where $d\Gamma(A)$ is the standard second quantization operator.

Proposition. Denote

$$d\tilde{\Gamma}(A)(\alpha \otimes (h_1 \otimes \ldots \otimes h_n)) = \sum_{i=1}^{n} (0i)\alpha \otimes (h_1 \otimes \ldots \otimes Ah_i \otimes \ldots \otimes h_n).$$

- Well defined on $\mathcal{TP}(\mathcal{H})$.
- $a^-_{(01)}(f)$, $a^+_{(01)}(g)$, $d\tilde{\Gamma}(|g\rangle\langle f|)$ satisfy the relation ($*$).

Hermitian Gaussian matrices.

- Fix $N \in \mathbb{N}$. Let $\mathcal{K}_{\mathbb{R}} = M_N(\mathbb{C})^{sa} \otimes \mathcal{H}_{\mathbb{R}}$ be a real Gaussian Hilbert space, $\mathcal{K} = M_N(\mathbb{C}) \otimes \mathcal{H}_{\mathbb{R}}$ its complexification, with inner product $\frac{1}{N} \operatorname{Tr}[AB^*] \langle f, g \rangle$. Complex Gaussian Hilbert space.
- Denote $x_{ij}(h) = X(E_{ij} \otimes h)$. Note $\overline{x_{ij}(h)} = x_{ji}(h)$.
- $\mathcal{P}(\mathcal{K}) =$ polynomials in $\{x_{ij}(h) : 1 \leq i, j \leq N, h \in \mathcal{H}_{\mathbb{R}}\}$.
- Define the $N \times N$ Hermitian Gaussian process $\{X(h) : h \in \mathcal{H}_{\mathbb{R}}\}$ by $X(h)_{ij} = x_{ij}(h)$.

Matricial Wick products.

Lemma. In $(M_N(\mathbb{C}) \otimes \mathcal{P}(\mathcal{K}), \mathrm{Tr} \otimes \mathbb{E})$, let W be the Wick projection

$$M_N(\mathbb{C}) \otimes \overline{\mathcal{P}}_n \to (M_N(\mathbb{C}) \otimes \overline{\mathcal{P}}_n) \cap (M_N(\mathbb{C}) \otimes \mathcal{P}_{n-1})^{\perp}.$$

Then $\mathrm{W}\,(A)_{\ell r} = \mathrm{W}\,(A_{\ell r})$.

Compare with (Biane 1997).

Permutation notation.

Notation. For $\alpha \in S_0(n)$,

$$\mathrm{Tr}_\alpha[X(h_1), \ldots, X(h_n)]$$
$$= (X\text{'s in the cycle starting with } 0) \prod_{\text{other cycles}} \mathrm{Tr}[X\text{'s in the cycle}].$$

Example. For $\alpha = (024)(137)(56)$, define the trace polynomial

$$\mathrm{Tr}_\alpha[X(h_1), \ldots, X(h_7)]$$
$$= X(h_2)X(h_4)\,\mathrm{Tr}[X(h_1)X(h_3)X(h_7)]\,\mathrm{Tr}[X(h_5)X(h_6)].$$

Trace polynomials.

Procesi, Formanek, Leron etc. $\sim$ 1976+: trace identities.

Kemp, Cébron, Driver, Hall etc. $\sim$ 2013+: random matrices.

Klep, Špenko, Volčič etc. $\sim$ 2014+: free analysis.

Jekel, etc. $\sim$ 2019+: operator algebras.

Huber, etc. $\sim$ 2021+: quantum information.

Evaluation map.

Definition. Let $q = \frac{1}{N}$.

Define the map $\mathcal{E}$ on $\bigoplus_{n=0}^{\infty} \mathbb{C}[S_0(n)] \otimes \mathcal{H}_{\mathbb{R}}^{\odot n}$ by

$$\mathcal{E}[\mathrm{T}\,(\alpha \otimes (h_1 \otimes \ldots \otimes h_n))] = \mathrm{Tr}_\alpha(X(h_1), \ldots, X(h_n)).$$

Theorem.

- $\mathcal{E}$ extends to a star-homomorphism from $\overline{\mathcal{TP}}(\mathcal{H}_{\mathbb{R}})$.
- $\mathcal{E}$ extends to an isometry from $\mathcal{F}_{1/N}(\mathcal{H})$.
- $\mathcal{E}$ intertwines conditional expectations.
- $\mathcal{E}$ intertwines Wick products.

Idea of proof. For $\alpha \in S_0(2n)$ and $F = h_1 \otimes \ldots \otimes h_{2n}$,

$$\varphi_q\left[\mathrm{T}\,(\alpha \otimes F)\right] = \sum_{\pi \in \mathcal{P}_2(2n)} q^{|\pi\alpha| - n} C_\pi(F) = \mathbb{E}\left[\mathrm{Tr}_\alpha(X(h_1), \ldots, X(h_{2n}))\right].$$

Hermite trace polynomials.

Definition. Hermite trace polynomials are

$$\mathrm{W}\left(\alpha \otimes (\xi_{i(1)} \otimes \ldots \otimes \xi_{i(n)})\right)$$

or perhaps

$$\mathrm{Tr}_\alpha(X(\xi_{i(1)}), \ldots, X(\xi_{i(n)})).$$

Proposition.

- Contraction formulas.
- Product formulas.
- Conditional expectation: martingale property.
- Orthogonality?

Univariate pure trace polynomials I.

Let $\mathcal{H} = \mathbb{C}$ and $\eta \in \mathbb{C}[S(n)]$ (rather than $S_0(n)$).

- For an $N \times N$ matrix X,

$$\mathrm{Tr}_\alpha(X) = p_\alpha(x_1, \ldots, x_N),$$

  the power sum symmetric polynomial in the eigenvalues of X.
- Depends only on the (number) partition λ.
- $\{\mathrm{W}(\alpha)\}$ orthogonal for different n but not for different λ.
- $\chi^\lambda =$ character of the irreducible representation indexed by the partition λ.

Theorem. $\left\{\mathrm{W}\left(\chi^\lambda\right) : \lambda \in \mathrm{Par}(n; \leq N)\right\}$ form an orthogonal basis with respect to $\varphi_{1/N}$.

Hermite polynomials of matrix argument I.

Schur polynomial

$$s_\lambda = \frac{1}{n!}\sum_{\nu\vdash n}\frac{n!}{z_\nu}\chi^\lambda(\nu)p_\nu = \frac{1}{n!}\sum_{\alpha\in S(n)}\chi^\lambda(\alpha)p_\alpha.$$

Denote

$$D^* = \sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{i\neq j}\frac{1}{x_i - x_j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)$$

and

$$E^* = \sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i}.$$

Hermite polynomials of matrix argument II.

For $\lambda \in \mathrm{Par}(n)$, the Hermite polynomial of matrix argument (for $\beta = 2$) is the symmetric polynomial in $\{x_1, \ldots, x_N\}$ with leading term $\frac{|\lambda|!}{c_\lambda} s_\lambda$ which is an eigenfunction of the operator $D^* - E^*$ with eigenvalue $-n$ (James 1975, Baker, Forrester 1997).

Theorem. In terms of $\{x_1, \ldots, x_N\}$, $\mathcal{E}[\mathrm{W}\left(\chi^\lambda\right)] = \mathrm{Tr}_{\chi^\lambda}(X)$ is (a multiple of) the Hermite polynomials of matrix argument.

Idea of proof. For $\eta \in \mathbb{C}[S_0(n)]$, define the Euler operator E on $\mathcal{TP}(\mathcal{H}_\mathbb{R})$ by

$$E\mathrm{T}\left(\eta \otimes F\right) = n\mathrm{T}\left(\eta \otimes F\right)$$

Then $\mathrm{W}\left(\eta \otimes F\right)$ is the unique eigenfunction of the operator $E - 2\mathcal{L}$ with eigenvalue n and leading term $\mathrm{T}\left(\eta \otimes F\right)$.

Chaos decomposition I: univariate trace polynomials.

Proposition. Let $\mathcal{H} = \mathbb{C}$, so that $\mathcal{TP}(\mathbb{C}) = Z(\mathbb{C}[S_0(n)] : \mathbb{C}[S(n)])$ is the centralizer. Then

$$\left\{ \mathrm{W}\left(\chi^{\lambda':\lambda}\right) : \lambda' = \lambda + \square, \lambda' \in \mathrm{Par}(n+1; \leq N) \right\}$$

form an orthogonal basis for $L^2(\mathcal{TP}(\mathbb{C}), \varphi_{1/N})$.

Definition. Let $\chi^{\lambda':\lambda}$ is the character of the compression of the λ'-irreducible representation of $S_0(n)$ to the (unique) component giving a λ-irreducible representation of $S(n)$. Then $\mathrm{W}\left(\chi^{\lambda':\lambda}\right)$ or $\mathrm{Tr}_{\chi^{\lambda':\lambda}}(X) =$ univariate Hermite trace polynomials.

Chaos decomposition III: general.

Theorem. Let $\{\xi_i : i \in \Xi\} =$ ONB for $\mathcal{H}_{\mathbb{R}}$. Denote

$$\Delta(\Xi^n) = \{\mathbf{u} \in \Xi^n : u(1) \leq u(2) \leq \ldots \leq u(n)\},$$

$$\ker(\mathbf{u}) = \pi = (I_1, \ldots, I_k) \in Int(n) : u(i) = u(j) \Leftrightarrow i \stackrel{\pi}{\sim} j.$$

$$Z(\mathbb{C}[S_0(n)] : \pi) = Z(\mathbb{C}[S_0(n)] : \mathbb{C}[S(I_1)] \times \ldots \times \mathbb{C}[S(I_k)]).$$

Then $A \in L^2(\mathcal{TP}(\mathcal{H}_{\mathbb{R}}), \varphi_{1/N})$ has a unique orthogonal decomposition in terms of Hermite trace polynomials

$$A = \sum_{n=0}^{\infty} \sum_{\mathbf{u} \in \Delta(\Xi^n)} \mathrm{W}\left(\eta_{\mathbf{u}} \otimes_s \xi_{\mathbf{u}}\right),$$

where $\eta_{\mathbf{u}} \in Z(\mathbb{C}[S_0(n)] : \ker(\mathbf{u})) \cap \mathbb{C}[S_0(n)]_{\leq N}$.

Questions.

- Real Hermite trace polynomials (Redelmeier, Mingo et al.)?
- Laguerre and Jacobi trace polynomials (Graczyk, Vostrikova 2007, Bryc 2008)?
- General orthogonal trace polynomials?
- Relation to the construction by [Köstler, Nica 2021]?
- Generating functions?

Thank you!