Infinitesimal Operators

Jamie Mingo (Queen’s University at Kingston)

(*joint work with Pei-Lun Tseng, NYU Abu Dhabi*)

POAS, Sept. 25 2023, Berkeley

limit distributions of random matrices

- $\{A_N\}_N$ is an ensemble of random matrices, $\mathrm{Tr} =$ un-normalized trace and $\mathrm{tr} = N^{-1}\mathrm{Tr} =$ normalized trace
- for a polynomial p, $\mu_N(p) = \mathrm{E}(\mathrm{tr}(p(A_N)))$
- if μ_N converges pointwise on polynomials to μ, we say the ensemble $\{A_N\}_N$ has the *limit distribution* μ
- let $\mu_N' = N(\mu_N - \mu)$, μ_N' is linear on polynomials and $\mu_N'(1) = 0$
- we say the pair (μ_N, μ_N') is an *infinitesimal law* on $\mathcal{A} = \mathbf{C}[t]$ (polynomials in t) and the triple $(\mathcal{A}, \mu_N, \mu_N')$ is an (*infinitesimal probability space*)
- if μ_N' converges pointwise to μ' we have that (μ, μ') is an infinitesimal law on $\mathcal{A}$ and we say that $\{A_N\}_N$ has the *infinitesimal limit distribution* (μ, μ')

examples (Johansson (1998), Dumitriu & Edelman (2006))

- $G = (g_{ij})_{i,j=1}^{N}$ with $\{g_{ij}\}_{i,j}$ real independent $\mathcal{N}(0,1)$
- $A_N = \frac{1}{\sqrt{2N}}(G + G^t) = N \times N$ *Gaussian Orthogonal Ensemble*
- it has the limit infinitesimal distribution (μ, μ') where μ is Wigner's semi-circle law and $\mu' = \frac{1}{2}(\nu_1 - \nu_2)$ is a signed measure with $\nu_1 = \frac{1}{2}(\delta_{-2} + \delta_2)$ (Dirac masses at ± 2) and $d\nu_2(t) = \frac{1}{\pi}\frac{1}{\sqrt{4-t^2}}$ (arcsine law on $[-2, 2]$)

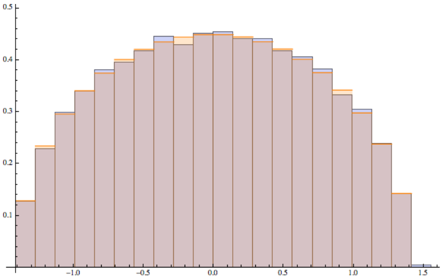
- $G = (g_{ij})_{\substack{i=1,\ldots N \\ j=1,\ldots,M}}$ $(N \times M)$, $\{g_{ij}\}_{i,j}$ real independent $\mathcal{N}(0,1)$
- $W_N = \frac{1}{N}GG^t$, $N \times N$ *real Wishart matrix*
- if $\frac{M}{N} \longrightarrow c$ then $\{W_N\}_N$ has the limit distribution $\mu =$ the Marchenko-Pastur Law with parameter c
- if $N(\frac{M}{N} - c) \longrightarrow c'$ then $\{W_N\}_N$ has the infinitesimal limit distribution $\mu' =$ with parameter c'

Infinitesimal Marchenko-Pastur (details)

- μ_c (where $M/N \to c > 0$)
- $a = (1 - \sqrt{c})^2$, and $b = (1 + \sqrt{2})^2$
- $d\mu_c(t) = (1-c)\delta_0 + \dfrac{\sqrt{(b-t)(t-a)}}{2\pi t}\, dt$
- $\mu_c' = \frac{1}{2}(\nu_1 - \nu_2) - c'(\rho_1 - \rho_2)$
- $\nu_1 = \frac{1}{2}(\delta_a + \delta_b)$, $d\nu_2(t) = \frac{dt}{\pi\sqrt{(b-t)(t-a)}}$ on $[a, b]$
- $\rho_1 = \delta_0$ and $\rho_2 = \frac{t+1-c}{2\pi t\sqrt{(b-t)(t-a)}}$ on $[a, b]$
- ρ_1 is absent when $c > 1$

Finite Rank Ex. (Shlyakhtenko (2018), Collins, Hasebe, Sakuma (2018))

- A a fixed finite matrix padded with 0 to $N \times N$.
- $\mu = \delta_0$ (because we normalize the trace)
- $\mu'(p) = \mathrm{Tr}(p(A) - p(0))$
- if A is $k \times k$ and normal, then $\mu' = \delta_{\lambda_1} + \delta_{\lambda_2} + \cdots + \delta_{\lambda_k} - k\delta_0$ where the eigenvalues of A are $\lambda_1, \ldots, \lambda_k$



a figure from Shlyakhtenko (2018) showing a perturbed GUE with a spike of mass 1/100 at 0.4, orange shows predicted eigenvalue density using infinitesimal freeness

Infinitesimal Freeness: Belinschi & Shylakhtenko (2012), Fevrier, Nica (2010), Tseng (2019)

- $\varphi' : \mathcal{A} \overset{\text{linear}}{\longrightarrow} \mathbf{C}$, $\varphi'(1) = 0$, $(\mathcal{A}, \varphi, \varphi')$ *infinitesimal probability space*
- $\tilde{\mathcal{A}} = \left\{ \begin{bmatrix} a & a' \\ 0 & a \end{bmatrix} \mid a, a' \in \mathcal{A} \right\}, \quad \widetilde{\mathbf{C}} = \left\{ \begin{bmatrix} \alpha & \alpha' \\ 0 & \alpha \end{bmatrix} \mid \alpha, \alpha' \in \mathbf{C} \right\},$

  $\tilde{\varphi} = \begin{bmatrix} \varphi & \varphi' \\ 0 & \varphi \end{bmatrix}, \quad \tilde{\varphi} : \tilde{\mathcal{A}} \to \widetilde{\mathbf{C}}$
- $(\tilde{\mathcal{A}}, \tilde{\varphi})$ is an algebraic probability space
- $\mathcal{A}_1, \dots, \mathcal{A}_s \subseteq \mathcal{A}$ are *infinitesimally free* if $\tilde{\mathcal{A}}_1, \dots, \tilde{\mathcal{A}}_s$ are $\tilde{\varphi}$-free
- $\mathcal{A}_1, \dots, \mathcal{A}_s \subseteq \mathcal{A}$ infinitesimally free $\overset{\text{THM}}{\Leftrightarrow}$ $\mathcal{A}_1, \dots, \mathcal{A}_s$ φ-free **and** whenever:
  - $a_1, \dots, a_n \in \mathcal{A}$ with $\varphi(a_i) = 0$, $1 \leqslant i \leqslant n$, and
  - $a_i \in \mathcal{A}_{j_i}$ with $j_1 \neq j_2 \neq \cdots \neq j_n$;

  we have $\varphi'(a_1 \cdots a_n) = \varphi(a_1 \cdots \varphi'(a_{\frac{n+1}{2}}) \cdots a_n)$ for n **odd** and 0 otherwise

asymptotic freeness

- $\{A_N\}_N$ and $\{B_N\}_N$ two random matrix ensembles with joint distribution $\mu_N : \mathbf{C}\langle s, t\rangle \to \mathbf{C}$ are *asymptotically free* if μ_N converges pointwise to a distribution for which $\mathbf{C}\langle s\rangle$ and $\mathbf{C}\langle t\rangle$ are free
- $\{A_N\}_N$ and $\{B_N\}_N$ two random matrix ensembles with joint distribution $\mu_N : \mathbf{C}\langle s, t\rangle \to \mathbf{C}$ are *asymptotically infinitesimally free* if (μ_N, μ_N') converges pointwise to a distribution for which $\mathbf{C}\langle s\rangle$ and $\mathbf{C}\langle t\rangle$ are infinitesimally free
- independent GOE random matrices are asymptotically free but *not* asymptotically infinitesimally free (M. 2019)
- a unitarily invariant ensemble and a fixed finite rank matrix are asymptotically infinitesimally free (Shlyakhtenko (2018), Collins, Hasebe, Sakuma, (2018))

infinitesimal operators

- $(\mathcal{A}, \varphi, \varphi')$ is an infinitesimal probability space
- $a \in \mathcal{A}$ is *infinitesimal* if $\varphi(a^k) = 0$ for $k = 1, 2, 3, \ldots$
- the infinitesimal cumulants of infinitesimal operators are particularly simple
- in a C^*-probability space a self-adjoint infinitesimal will have spectral measure δ_0 (Dirac mass at 0) with respect to φ
- if a and b are infinitesimally free and a is infinitesimal then ab is infinitesimal
- if a and b are infinitesimally free and both a and b are infinitesimal then $a + b$ is infinitesimal

free cumulants (Speicher)

$\kappa_1(x_1) = \varphi(x_1),$ $\qquad G(z) = \varphi((z - X)^{-1})$

$\kappa_2(x_1, x_2) = \varphi(x_1 x_2) - \varphi(x_1)\varphi(x_2)$ $\qquad R(z) = G^{\langle -1 \rangle}(x) - z^{-1}$

$\kappa_{|\,\sqcup}(x_1, x_2, x_3) = \kappa_1(x_1)\,\kappa_2(x_2, x_3)$ $\qquad R(z) = \kappa_1 + \kappa_2 z + \kappa_3 z^2 + \cdots$

$$\varphi(x_1\, x_2\, x_3) = \kappa_{\sqcup\!\sqcup} + \kappa_{|\,\sqcup} + \kappa_{\sqcup\!\!|\!\!\sqcup} + \kappa_{\sqcup\,|} + \kappa_{|\,|\,|}$$

$$\varphi(x_1 x_2 x_3 x_4) = \kappa_{\sqcup\!\sqcup\!\sqcup} + \kappa_{|\,\sqcup\!\sqcup} + \kappa_{\sqcup\!|\!\sqcup\!\sqcup} + \kappa_{\sqcup\!\sqcup\!|\!\sqcup} + \kappa_{\sqcup\!\sqcup\,|}$$
$$+ \kappa_{\sqcup\,\sqcup} + \kappa_{\sqcup\!\sqcup\!\!\sqcup}$$
$$+ \kappa_{|\,|\,\sqcup} + \kappa_{|\,\sqcup\!|\!\sqcup} + \kappa_{\sqcup\!|\,|\!\sqcup} + \kappa_{|\,\sqcup\,|} + \kappa_{\sqcup\!|\!\sqcup\,|} + \kappa_{\sqcup\,|\,|}$$
$$+ \kappa_{|\,|\,|\,|}$$

$$\boxed{\varphi(x_1 \cdots x_n) = \sum_{\pi \in NC(n)} \kappa_\pi(x_1, \ldots, x_n)}$$

where $NC(n)$ is the set of non-crossing partitions of $[n] = \{1, 2, \ldots, n\}$

infinitesimal cumulants, Fevrier & Nica (2010)

- $$\varphi(x_1 \cdots x_n) = \sum_{\pi \in NC(n)} \kappa_\pi(x_1, \ldots, x_n)$$
- $$\boxed{\varphi'(x_1 \cdots x_n) = \sum_{\pi \in NC(n)} \partial\kappa_\pi(x_1, \ldots, x_n)}$$
- where $\partial\kappa_\pi(x_1 \cdots x_n)$
  $= \sum_{V \in \pi} \kappa'_{|V|}(x_1, \ldots, x_n \mid V) \prod_{W \neq V} \kappa_{|W|}(x_1, \ldots, x_n \mid W)$
- where $\partial\kappa_\pi$ is defined by the Leibniz rule, for example when $\pi = \{(1,3),(2),(4)\}$

$$\begin{aligned}\partial\kappa_\pi(x_1, x_2, x_3, x_4) &= \partial(\kappa_2(x_1, x_3)\kappa_1(x_2)\kappa_1(x_4)) \\ &= \kappa'_2(x_1, x_3)\kappa_1(x_2)\kappa_1(x_4) + \kappa_2(x_1, x_3)\kappa'_1(x_2)\kappa_1(x_4) \\ &\quad + \kappa_2(x_1, x_3)\kappa_1(x_2)\kappa'_1(x_4)\end{aligned}$$

higher orders

Fix $m \geqslant 1$. For $0 \leqslant k \leqslant m-1$, let $\{\kappa_n^{(k)}\}_{n\geqslant 1}$ be an infinitesimal cumulant sequence of order $m-1$. We are following the notation of calculus and writing $\kappa_n^{(1)}$ for κ_n' and $\kappa_n^{(2)}$ for κ_n''. In this notation $\kappa_n^{(0)} = \kappa_n$. We define a derivation, ∂, by setting $\partial\kappa_n^{(k)} = \kappa_n^{(k+1)}$ for all k and n and extend to sums by linearity and to products by the Leibnitz rule. Then by the Leibnitz rule

$$(*) \qquad \frac{\partial^i(\kappa_{l_1}\kappa_{l_2}\cdots\kappa_{l_k})}{i!} = \sum_{\substack{j_1,\ldots,j_k\geqslant 0 \\ j_1+j_2+\cdots+j_k=i}} \frac{\kappa_{l_1}^{(j_1)}}{j_1!}\frac{\kappa_{l_2}^{(j_2)}}{j_2!}\cdots\frac{\kappa_{l_k}^{(j_k)}}{j_k!}.$$

Given a partition $\pi \in \mathcal{P}(n)$ with blocks $\{V_1,\ldots,V_k\}$ of size $l_1,\ldots,k_k$ respectively, we set, as usual, $\kappa_\pi = \kappa_{l_1}\cdots\kappa_{l_k}$. By the Leibnitz rule we have

$$\partial\kappa_\pi = \kappa_{l_1}'\kappa_{l_2}\cdots\kappa_{l_k} + \kappa_{l_1}\kappa_{l_2}'\cdots\kappa_{l_k} + \cdots + \kappa_{l_1}\kappa_{l_2}\cdots\kappa_{l_k}'.$$

Let us define a $m \times m$ upper triangular matrix K_n by

$$K_n = \begin{bmatrix} \frac{\kappa_n^{(0)}}{0!} & \frac{\kappa_n^{(1)}}{1!} & \cdots & \frac{\kappa_n^{(m-1)}}{(m-1)!} \\ 0 & \frac{\kappa_n^{(0)}}{0!} & \cdots & \frac{\kappa_n^{(m-2)}}{(m-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\kappa_n^{(0)}}{0!} \end{bmatrix} = \begin{bmatrix} \kappa_n & \frac{\partial^1 \kappa_n}{1!} & \cdots & \frac{\partial^{m-1} \kappa_n}{(m-1)!} \\ 0 & \kappa_n & \cdots & \frac{\partial^{m-2} \kappa_n}{(m-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \kappa_n \end{bmatrix}.$$

We extend this to partitions in the usual way and set $K_\pi = K_{l_1} \cdots K_{l_k}$. Note that the matrices $K_{l_1}, \ldots, K_{l_k}$ commute so the order of the blocks does not matter, and thus the extension is well defined. Then by $(*)$ we have

(LEMMA)
$$K_\pi = \begin{bmatrix} \kappa_\pi & \frac{\partial^1 \kappa_\pi}{1!} & \cdots & \frac{\partial^{m-1} \kappa_\pi}{(m-1)!} \\ 0 & \kappa_\pi & \cdots & \frac{\partial^{m-2} \kappa_\pi}{(m-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \kappa_\pi \end{bmatrix}.$$

For $0 \leqslant k \leqslant m-1$, let $\{m_n^{(k)}\}_{n\geqslant 1}$ be an infinitesimal moment sequence of order $m-1$, using the same convention as with cumulants. We let

$$M_n = \begin{bmatrix} \frac{m_n^{(0)}}{0!} & \frac{m_n^{(1)}}{1!} & \cdots & \frac{m_n^{(m-1)}}{(m-1)!} \\ 0 & \frac{m_n^{(0)}}{0!} & \cdots & \frac{m_n^{(m-2)}}{(m-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{m_n^{(0)}}{0!} \end{bmatrix} = \begin{bmatrix} m_n & \frac{\partial^1 m_n}{1!} & \cdots & \frac{\partial^{m-1} m_n}{(m-1)!} \\ 0 & m_n & \cdots & \frac{\partial^{m-2} m_n}{(m-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}.$$

$$M_\pi = \begin{bmatrix} m_\pi & \frac{\partial^1 m_\pi}{1!} & \cdots & \frac{\partial^{m-1} m_\pi}{(m-1)!} \\ 0 & m_\pi & \cdots & \frac{\partial^{m-2} m_\pi}{(m-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_\pi \end{bmatrix} \text{ and } M_n = \sum_{\pi \in NC(n)} K_\pi$$

For $z_0, \ldots, z_{m-1} \in \mathbf{C}$, let

$$Z = \begin{bmatrix} z_0 & z_1 & \cdots & z_{m-1} \\ 0 & z_0 & \cdots & z_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_0 \end{bmatrix},$$

$$G(Z) = \sum_{n=0}^{\infty} M_n Z^{-(n+1)} \text{ and } R(Z) = \sum_{n=1}^{\infty} K_n Z^{n-1}.$$

Then

$$G(Z)^{-1} + R(G(Z)) = Z = G(Z^{-1} + R(Z)).$$

- when $m = 2$ we have $Z = \begin{bmatrix} z_0 & z_1 \\ 0 & z_0 \end{bmatrix}$
- $G(Z) = \begin{bmatrix} G(z_0) & G'(z_0)z_1 + g(z_0) \\ 0 & G(z_0) \end{bmatrix}$ where $g(z_0) = \sum_{n=1}^{\infty} \frac{m_n'}{z_0^{n+1}}$
- $R(Z) = \begin{bmatrix} R(z_0) & R'(z_0)z_1 + r(z_0) \\ 0 & R(z_0) \end{bmatrix}$ where $r(z_0) = \sum_{n=1}^{\infty} \kappa_n' z_0^{n-1}$
- $G(Z)^{-1} + R(G(Z)) = Z \Rightarrow g(z) = -r(G(z))\, G'(z)$
- if $\varphi(x^n) = 0$ for $n \geqslant 1$ (i.e. x is infinitesimal) then $G(z) = z^{-1}$ and

$$z^{-1} g(z^{-1}) = z r(z)$$

- in particular $m_n' = \kappa_n'$ for $n \geqslant 1$

commutators and anti-commutators

- $(\mathcal{A}, \varphi)$ algebraic probability space, $x_1, x_2 \in \mathcal{A}$ freely independent
- $q = i(x_1x_2 - x_2x_1)$ is the *commutator* of x_1 and x_2
- $p = x_1x_2 + x_2x_1$ the *anti-commutator* of x_1 and x_2
- Nica & Speicher found the distribution of p and q in terms of x_1 and x_2, many subsequent investigations: Daniel Perales and Jacob Campbell
- Collins, Hasebe, Sakuma (2018) considered the spectrum of $P_N = B_NA_N + A_NB_N$ and $Q_N = i(B_NA_N - A_NB_N)$ with B_N a unitarily invariant ensemble and A_N trace class operators, $P_N \xrightarrow{\mathcal{D}} p$ and $Q_N \xrightarrow{\mathcal{D}} q$
- generalization: $(\mathcal{A}, \varphi, \varphi')$ infinitesimal probability space, $x_1, x_2 \in \mathcal{A}$ infinitesimally freely independent, x_2 infinitesimal
- find the distribution of p and q

main results (M. & Tseng)

- $(\mathcal{A}, \varphi, \varphi')$ infinitesimal probability space, $x_1, x_2 \in \mathcal{A}$ infinitesimally freely independent, x_2 **infinitesimal**
- $q = i(x_1x_2 - x_2x_1)$

THM. $\kappa_n'(q) = \kappa_n'(\omega x_2) + \kappa_n'(-\omega x_2)$ for n even and 0 otherwise, where $\omega = \sqrt{\kappa_2(x_1)}$

THM. $q \overset{\mathcal{D}}{\sim} \omega x_2 \boxplus_B -\omega x_2$ (this means that q has the same distribution as the free additive infinitesimal convolution of two operators each of which is a scaled version of x_2)

- $p = x_1x_2 + x_2x_1$

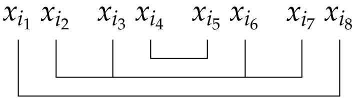
THM. $\kappa_n'(p) = \kappa_n'(\alpha x_2) + \kappa_n'(\beta x_2)$ where $\alpha = \varphi(x_1) + \sqrt{\varphi(x_1^2)}$ and $\beta = \varphi(x_1) - \sqrt{\varphi(x_1^2)}$ and $\pm\sqrt{\varphi(x_1^2)}$ are the two square roots of $\varphi(x_1^2)$

THM. $p \overset{\mathcal{D}}{\sim} \alpha x_2 \boxplus_B \beta x_2$ (this means that p has the same distribution as the free additive infinitesimal convolution of two operators each of which is a scaled version of x_2)

moment calculations

$$\varphi'(x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5}x_{i_6}x_{i_7}x_{i_8}) = \sum_{\pi \in NC(8)} \partial\kappa_\pi(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}, x_{i_8})$$

- for each l the pair $\{x_{2l-1}, x_{2l}\}$ contains one x_1 and one x_2
- by vanishing of mixed cumulants, no block can connect an x_1 to an x_2
- by infinitesimality of x_2, only one x_2-block



$$\pi = \{\{1,8\},\{2,3,6,7\},\{4,5\}\},$$

$$i_1 = i_4 = i_5 = i_8 = 1, \qquad i_2 = i_3 = i_6 = i_7 = 2$$

$$\partial\kappa_\pi(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}, = \kappa_2(x_1)^2\kappa_4'(x_2)$$

operator valued cumulants

- $a_1, \ldots, a_n, b_1, \ldots, b_n \in (\mathcal{A}, \varphi, \varphi')$
- $\{a_1, \ldots, a_n\}$ infinitesimally free from $\{b_1, \ldots, b_n\}$
- $a_1, \ldots, a_n$ infinitesimal operators
- $x = b_1 a_1 b_1^* + \cdots + b_n a_n b_n^*$
- $A = \begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{bmatrix}$, $B = (\beta_{ij})_{i,j,=1,\ldots,n}$ where $\beta_{ij} = \varphi(b_i b_j^*)$
- $X = \sqrt{B} A \sqrt{B}$

THM. $\varphi'(x^m) = \mathrm{Tr}(K'_m(AB, AB, \ldots, AB)) = \mathrm{Tr}(K'_m(X, X, \ldots, X)) = \mathrm{Tr} \otimes \varphi'(X^m)$

- K'_n is the n^{th} infinitesimal cumulant with values in $M_n(\mathbf{C})$

an infinitesimal manifesto

- Borot, Charbonnier, Garcia-Failde, Leid, and Shadrin (2022) found a connection between topological recursion and higher order freeness
- in this model infinitesimal freeness corresponds to genus $\frac{1}{2}$ (i.e. planar but not orientable)
- independent GUE random matrices are asymptotically free (Voiculescu, 1991) and have trivial infinitesimal laws , and thus are asymptotically infinitesimally free
- Fevrier (2012) defined higher order freeness (think 3×3 matrices), but independent GUEs are not asymptotically second order infinitesimally free (Harer, Zagier 1986)
- independent GOE random matrices are not asymptotically infinitesimally free, **but** there is a universal rule (M. 2019)

a connection to Boolean independence

- $(\mathcal{A}, \varphi, \varphi')$ an infinitesimal probability space, $j \in \mathcal{A}$ is infinitesimal and idempotent, and $\varphi'(j) \neq 0$
- $\mathcal{A}_1, \dots, \mathcal{A}_s \subseteq \mathcal{A}$, unital subalgebras with $\{\mathcal{A}_1, \dots, \mathcal{A}_s\}$ infinitesimally free from j
- define $\psi : \mathcal{A} \to \mathbf{C}$ by $\psi(a) = \varphi'(aj)/\varphi'(j)$.
- for $a \in \mathcal{A}_i$ we have $\varphi'(aj) = \partial\kappa_\pi(a, j)$ with $\pi = \{\{1\},\{2\}\}$, so $\varphi'(aj) = \varphi'(a)\varphi(j) + \varphi(a)\varphi'(j) = \varphi(a)\varphi'(j)$. Thus on $\mathcal{A}_i$ we have $\psi = \varphi$
- $\mathcal{J}(\mathcal{A}_k) = \text{alg}\{ j^{(\epsilon_1)} a_1 j^{(\epsilon_2)} \cdots j^{(\epsilon_{n-1})} a_{n-1} j^{(\epsilon_n)} \mid a_1, \dots a_{n-1} \in \mathcal{A}_k\}$ where $j^{(1)} = 1 - j$ and $j^{(-1)} = j$
- $\mathcal{J}_a(\mathcal{A}_k)$ is the sub-algebra generated by the same words as above but we require that the ϵ's alternate in sign
- If $\{\mathcal{A}_1, \dots, \mathcal{A}_s\}$ infinitesimally free from j, then the subalgebras $j\mathcal{J}(\mathcal{A}_1)j, \dots, j\mathcal{J}(\mathcal{A}_s)j$ are Boolean independent
- If $\{\mathcal{A}_1, \dots, \mathcal{A}_s\}$ infinitesimally free from j, and φ-free amongst themselves, then then the subalgebras $\mathcal{J}_a(\mathcal{A}_1), \dots,$ $\mathcal{J}_a(\mathcal{A}_s)$ are Boolean independent

key relation

- recall $j^{(1)} = 1 - j$ and $j^{(-1)} = j$
- given $\epsilon_2, \dots, \epsilon_k \in \{-1, 1\}$ we let $\sigma_\epsilon \in \mathcal{P}(n)$ be the interval partition (i.e. all blocks are intervals) where we start a new block at l whenever $\epsilon_l = -1$, here we assume that $k < n$.
- For example, suppose $n = 9$ and $k = 6$ and $\epsilon_2 = -1, \epsilon_3 = 1, \epsilon_4 = 1, \epsilon_5 = -1, \epsilon_6 = -1$; then $\sigma_\epsilon = \{(1), (2, 3, 4), (5), (6, 7, 8, 9)\}$.
- Given two strings $(\epsilon_2, \dots, \epsilon_n)$ and $(\eta_2, \eta_3, \dots, \eta_n)$ in $\{-1, 1\}^{n-1}$, we say that $\epsilon \leqslant \eta$ if $\epsilon_k \leqslant \eta_k$ for $k = 2, \dots, n$. Then $\epsilon \leqslant \eta \Leftrightarrow \sigma_\epsilon \leqslant \sigma_\eta$. In addition $\sigma_{-\epsilon} = \sigma_{B(\sigma_\epsilon)}$ where $B(\sigma)$ is the Boolean complement of σ, namely the smallest $\pi \in I(n)$ such that $\pi \vee \sigma = 1_n$.

▶ $\psi(ja_1 j^{(\epsilon_2)} \cdots j^{(\epsilon_n)} a_n j) = \psi(ja_1 j^{(\epsilon_2)} \cdots j^{(\epsilon_n)} a_n) = \beta_{\sigma_\epsilon}(a_1, \dots, a_n)$ where β_{σ_ϵ} is the Boolean cumulant corresponding to the interval partition σ_ϵ.

previous work I mentioned

- S. T. Belinschi and D. Shlyakhtenko. Free probability of type B: Analytic interpretation and applications. *Amer. J. Math.*, 134:193–234, 2012.
- B. Collins, T. Hasebe, and N. Sakuma. Free probability for purely discrete eigenvalues of random matrices. *J. Math. Soc. Japan*, 70(3):1111–1150, 2018.

- S. T. Belinschi and D. Shlyakhtenko. Free probability of type B: Analytic interpretation and applications. *Amer. J. Math.*, 134:193–234, 2012.

- M. Fevrier and A. Nica. Infinitesimal non-crossing cumulants and free probability of type B. *J. Functional Analysis*, 258(9):2983–3023, 2010.
- R. Lenczewski. Limit distributions of random matrices. *Adv. in Math.*, 263:253–320, 2014.

- C. Male. Traffic distributions and independence: permutation invariant random matrices and the three notions of independence. *Mem. Amer. Math. Soc.*, 1300, 2021.
- D. Perales. On the anti-commutator of two free random variables. *arXiv*, 2101:09444, (2021).
- D. Shlyakhtenko. Free probability of type B and asymptotics of finite rank perturbations of random matrices. *Indiana Univ. Math. J.*, 67:971–991, 2018.
- P.-L. Tseng. A unified approach to infinitesimal freeness with amalgamation. *Int. J. Math.*, (*to appear*), *arXiv*:1904.11646.