Structure of free group factors

Srivatsav Kunnawalkam Elayavalli aka Sri

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VIRGINIA CAVALIERS
CHAMPIONS

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Definition

Given a tracial von Neumann algebra (M, τ) and $x \in M_{sa}^d$, we define the law of x, denoted ℓ_x, to be the linear functional $\ell_x \colon \mathbb{C}\langle t_1, \cdots, t_d\rangle \to \mathbb{C}$ given by

$$\ell_x(f) = \tau(f(x)).$$

$\Sigma_{d,R}$: set of all linear maps $\mathbb{C}\langle t_1, \ldots, t_d\rangle \to \mathbb{C}$ arising as the law of some tuple in a tracial von Neumann algebra.

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Equip $\Sigma_{d,R}$ with the weak* topology (topology of pointwise convergence on $\mathbb{C}\langle t_1, \ldots, t_d \rangle$).

Voiculescu's microstate space

Let (M, τ) be a diffuse tracial von Neumann algebra, and $X, Y \subset M_{\text{sa}}$ finite such that $\|x\| \leq R$ for all $x \in X \cup Y$. For each weak* neighborhood $\mathcal{O}$ of $\ell_{X \sqcup Y}$ in $\Sigma_{d,R}$ and $n \in \mathbb{N}$, we define

$$\Gamma_R^{(n)}(X : Y; \mathcal{O}) = \{A \in \mathbb{M}_n(\mathbb{C})_{\text{sa}}^X : \exists B \in \mathbb{M}_n(\mathbb{C})_{\text{sa}}^Y \mid \ell_{A \sqcup B} \in \mathcal{O}, \|A_x\|, \|B_y\| \leq R\}.$$

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Orbital covering numbers

We say that Ξ orbitally $(\varepsilon, \|\cdot\|_2)$-covers Ω if for every $A \in \Omega$, there is a $B \in \Xi$ and an $n \times n$ unitary matrix V so that $\|A - VBV^*\|_2 < \varepsilon$.

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Let $X, Y \subset M_{\mathrm{sa}}$ not necessarily finite, satisfying $X'' \subset Y''$ and $\|x\| \leq R$ for all $x \in X \cup Y$. Let F, G be finite subsets of X, Y respectively. For a weak*-neighborhood $\mathcal{O}$ of $\ell_{X \sqcup Y}$, we define

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$$h_\varepsilon(F : G; \mathcal{O}) := \limsup_{n\to\infty} \frac{1}{n^2} \log K_\varepsilon^{\mathrm{orb}}(\Gamma_R^{(n)}(F : G; \mathcal{O})),$$

$$h_\varepsilon(F : G) := \inf_{\mathcal{O} \ni \ell_{F \sqcup G}} h_\varepsilon(\mathcal{O}),$$

$$h_\varepsilon(X : Y) := \sup_{F \subset_{\mathrm{finite}} X} \inf_{G \subset_{\mathrm{finite}} Y} h_\varepsilon(F : G)$$

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M is strongly 1-bounded if $-\infty < h(M) < \infty$.

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Fact 5:

$h(M) = \infty$ if M is diffuse, and $M = \mathrm{W}^*(x_1, \cdots, x_n)$ where $x_j \in M_{sa}$ for all $1 \leq j \leq n$ and $\delta_0(x_1, \cdots, x_n) > 1$. For example this applies if $M = L(\mathbb{F}_n)$, for $n > 1$.

Lemma 1:

$h(N : M) = h(N : M^{\mathcal{U}})$ if $N \subset M$ is diffuse, and $\mathcal{U}$ is an ultrafilter on a set I. In particular, $h(M^{\mathcal{U}}) \geq h(M)$.

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Lemma 2:

$h(N_1 \vee N_2 : M) \leq h(N_1 : M) + h(N_2 : M)$ if $N_1, N_2 \subset M$ and $N_1 \cap N_2$ is diffuse. In particular, $h(N_1 \vee N_2) \leq h(N_1) + h(N_2)$.

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Lemma 3:

Suppose that $(N_\alpha)_\alpha$ is an increasing chain of diffuse von Neumann subalgebras of a von Neumann algebra M. Then

$$h\left(\bigvee_\alpha N_\alpha : M\right) = \sup_\alpha h(N_\alpha : M).$$

Lemma 4:

Let I be a countable set, and $M = \bigoplus_{i \in I} M_i$ with M_i diffuse for all i. Suppose that τ is a faithful trace on M, and that λ_i is the trace of the identity on M_i. Endow M_i with the trace $\tau_i = \frac{\tau|_{M_i}}{\lambda_i}$. Then

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Lemma 6:

$h(pNp : pMp) = \frac{1}{\tau(p)^2} h(N : M)$, if $N \leq M$ is diffuse, p is a nonzero projection in N, and M is a factor.

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3. N has Property (T) (H-J-KE'21A).
4. N is a graph product of finite dimensional von Neumann algebras, with vanishing cohomology (AIM Hexagon).
5. First example of ultrapower of non Gamma factor M with $h(M) \leq 0$ (CIKE'23).

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5. Matrix ultraproducts (Jekel).
6. $L(\Gamma)$ where Γ is a hyperbolic tower in the sense of Sela (KE'22).

Definition:

Let (M, τ) be a tracial von Neumann algebra. We say that $P \leq M$ is Pinsker if $h(P : M) \leq 0$ and for any $P \leq Q \leq M$ with $P \neq Q$ we have $h(Q : M) > 0$.

If $Q \leq M$ is diffuse and $h(Q : M) \leq 0$, then there is a unique Pinsker algebra $P \leq M$ with $Q \subseteq P$. E.g.

$$P = \bigvee_{N \leq M, N \supseteq Q, h(N:M) \leq 0} N.$$

We say P the Pinsker algebra of $Q \subseteq M$.

Theorem: (Hayes, BC1, BC2)

Fix $r \in \mathbb{N} \cup \{\infty\}$. Then:

(i) $Q \leq L(\mathbb{F}_r)$ is amenable if and only if $h(Q : L(\mathbb{F}_r)) = 0$,

(ii) $P \leq L(\mathbb{F}_r)$ is Pinsker if and only if it is maximal amenable.

For an inclusion $N \leq M$ of tracial von Neumann algebras, we let

$$\mathcal{H}_{\text{anti-c}}(N \leq M) = \bigcap_{T \in \text{Hom}_{N-N}(L^2(M), L^2(N) \otimes L^2(N))} \ker(T).$$

Where $\text{Hom}_{N-N}(L^2(M), L^2(N) \otimes L^2(N)$ is the space of bounded, linear, N-N bimodular maps $T\colon L^2(M) \to L^2(N) \otimes L^2(N)$.

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This contains by Hayes the following generalizations of the normalizer of $N \leq M$

$$q^1\mathcal{N}_M(N) = \left\{ x \in M : \text{ there exists } x_1, \cdots, x_n \in M \text{ so that } xN \subseteq \sum_{j=1}^{n} Nx_j \right\}.$$

$$\mathcal{N}_M^{wq}(N) = \{u \in \mathcal{U}(M) : uNu^* \cap N \text{ is diffuse}\}.$$

Definition

Let (M,τ) be a tracial von Neumann algebra. For $Q, P \leq M$ diffuse, we define the intertwining space from Q to P inside M, denoted $I_M(Q,P)$, to be the set of $\xi \in L^2(M)$ so that

$$\overline{\text{Span}\{a\xi b : a \in Q, b \in P\}}^{\|\cdot\|_2}$$

has finite dimension as a right P-module. We define the weak intertwining space from Q to P inside M by

$$wI_M(Q,P) = \bigcup_{Q_0 \leq Q \text{ diffuse}} I_M(Q_0, P).$$

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One can show that

$$wI_M(Q, Q) \subseteq \mathcal{H}_{\text{anti-c}}(Q \leq M).$$

Theorem (Hayes 16)

$h(W^*(\mathcal{H}_{\text{anti-c}}(N \leq M)) : M) = h(N : M)$ if $N \leq M$ is diffuse.

Theorem 1 (HJKE'23)

Fix $t > 1$. Then $Q \leq L(\mathbb{F}_t)$ is amenable if and only if $h(Q : L(\mathbb{F}_t)) = 0$.

Theorem 2 (HJKE'23)

Let $M = L(\mathbb{F}_t)$ for some $t > 1$. For any maximal amenable $P \leq L(\mathbb{F}_t)$ we have that

$$_P(L^2(M) \ominus L^2(P))_P \leq [L^2(P) \otimes L^2(P)]^{\oplus\infty}.$$

Let $A \leq M$ is a maximal abelian $*$-subalgebra. Write $A = L^{\infty}(X, \mu)$. The representation

$$\pi\colon C(X) \otimes C(X) \to B(L^2(M) \ominus L^2(A))$$

given by

$$\pi(f \otimes g)\xi = f\xi g,$$

gives rise to a spectral measure E on $X \times X$ whose marginals are Radon-Nikodym equivalent to μ. We say that $\nu \in \mathrm{Prob}(X \times X)$ is a left/right measure of $A \leq M$ if it is Radon-Nikodym equivalent to E.

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Note that if ν is a left/right measure, and if $\phi \colon C(X) \otimes C(X) \to L^\infty(X \times X, \nu)$ is the map sending an element of $C(X) \otimes C(X) \cong C(X \times X)$ to its $L^\infty(\nu)$-equivalence class, then there is a unique normal $*$-isomorphism $\rho \colon L^\infty(X \times X, \nu) \to \overline{\pi(C(X) \otimes C(X))}^{SOT}$ so that $\pi = \rho \circ \phi$.

Theorem 3 (HJKE'23)

Let $M = L(\mathbb{F}_t)$ for $t > 1$. Suppose that $A \leq M$ is abelian and a maximal amenable subalgebra of M. Write $A = L^\infty(X, \mu)$ for some compact metrizable space X and some Borel probability measure on X. Then the left/right measure of $A \leq M$ is absolutely continuous with respect to $\mu \otimes \mu$.

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Given a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ we say that M is spectrally ω-solid if for any diffuse, amenable $Q \leq M^\omega$ we have that $W^*(\mathcal{H}_{\text{anti-c}}(Q \leq M^\omega)) \cap M$ is amenable. We say that M is spectrally ultrasolid if it is spectrally ω-solid for every free ultrafilter ω.

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Theorem 4 (HJKE'23)

We have that $L(\mathbb{F}_t)$ is spectrally ultrasolid. If $Q \leq L(\mathbb{F}_t)$, $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ is a free ultrafilter and $Q' \cap L(\mathbb{F}_t)^\omega$ is diffuse, then Q is amenable.

If M is a finite von Neumann algebra, and $P, Q \leq M$ we say that a corner of Q intertwines into P inside of M and write $Q \prec P$ if there are nonzero projections $f \in Q$, $e \in P$, a unital $*$-homomorphism $\Theta\colon fQf \to ePe$ and a nonzero partial isometry $v \in M$ so that:

- $xv = v\Theta(x)$ for all $x \in fQf$,
- $vv^* \in (fQf)' \cap fMf$,
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Theorem 5 (HJKE'23)

Fix $t > 1$, and let Q, P be maximal amenable subalgebras of $L(\mathbb{F}_t)$. Then exactly one of the following occurs:

1. either there are nonzero projections $e \in Q, f \in P$ and a unitary $u \in L(\mathbb{F}F_t)$ so that $u^*(ePe)u = fQf$, or
2. for any diffuse $Q_0 \leq Q$ we have that $Q_0 \not\prec P$.

In particular, if Q, P are hyperfinite subfactors of $L\mathbb{F}_t)$ that are maximal amenable subalgebras in $L(\mathbb{F}_t)$, then either they are unitarily conjugate or no corner of any diffuse subalgebra of one can be intertwined into the other inside of $L(\mathbb{F}_t)$.

Theorem 6 (HJKE'23)

Let $t > 1$ and let $N \leq L(\mathbb{F}_t)$ be a nonamenable subfactor. Then there is a free ultrafilter ω and an embedding $\iota\colon N \to \prod_{k\to\omega} M_k(\mathbb{C})$ with $\iota(N)' \cap \prod_{k\to\omega} M_k(\mathbb{C}) = \mathbb{C}1$.

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Stay tuned until next episode of free entropy theory and Peterson Thom conjecture!