A conjugate system for twisted Araki-Woods algebras of finite dimensional spaces

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Plan of the talk

- Recall the construction of Twisted Araki-Woods algebras $\mathcal{L}_T(H)$ and seperability of the vacuum state [da Silva, Lechner 22']
- Brief review of conjugate variables and free fisher information
- Explain how to concretely compute the conjugate system for $\mathcal{L}_T(H)$: Wick formula
- Some consequences: factoriality, free monotone transport.

Fock spaces and Commutation relations

$\mathcal{H}$ a complex Hilbert space.

- Consider the symmetric, antisymmetric, and full Fock space

$$\mathcal{F}_F(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{sym}}^n \mathcal{H}, \quad \mathcal{F}_{-F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \Lambda^n \mathcal{H}, \quad \mathcal{F}_0(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \otimes^n \mathcal{H}$$

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- The creation operator $a^*(f) : g_1 \otimes \cdots \otimes g_n \to f \otimes g_1 \otimes \cdots \otimes g_n$, and its adjoint $a(f)$ (annihilation operator) satisfies the Bosonic and Fermion commutation relations

$$a_F(f)a_F^*(g) - a_F^*(g)a_F(f) = \langle f, g \rangle, \quad a_{-F}(f)a_{-F}^*(g) + a_{-F}^*(g)a_{-F}(f) = \langle f, g \rangle, \quad \forall f, g \in \mathcal{H},$$

and for the full Fock space $a(f)a^*(g) = a(f)a^*(g) + 0a^*(g)a(f) = \langle f, g \rangle$.

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- Bożejko and Speicher considered the q-commutation relation $(-1 \leq q \leq 1)$

$$a(f)a^*(g) - qa^*(f)a(g) = \langle f, g \rangle,$$

  and showed that it can be realized as the creation and annihilation operators on the q-Fock space $\mathcal{F}_{qF}(\mathcal{H})$.

q and twisted Fock space

- F the tensor flip $F(f \otimes g) = g \otimes f$ and $T = qF$, denote

$$T_k = 1^{k-1} \otimes T \otimes 1^{n-k-1} \in B(\mathcal{H}^{\otimes n}).$$

- **Kernel**: $P_{T,n} = \sum_{\sigma \in S_n} \Psi_T(\sigma) \in B(\mathcal{H}^{\otimes n})$ with $\Psi_T(\sigma) : S_n \to B(\mathcal{H}^{\otimes n})$ be the quasi-multiplicative map (w.r.t the Cayley graph)

$$\Psi_T((12)(45)) = T_1 T_4, \quad \Psi_T((123)) = \Psi_T((12)(23)) = T_1 T_2.$$

- $T = qF$, $P_{T,n}(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}$.
- Alternatively, we can also define recursively

$$R_{T,n} := 1 + T_1 + T_1 T_2 + \cdots + T_1 \cdots T_{n-1},$$
$$P_{T,1} := R_{T,1}, \quad P_{T,n} := (1 \otimes P_{T,n-1}) R_{T,n}.$$

- Let $\mathcal{H}_{T,n}$ be the closure of $(\mathcal{H}^{\otimes n}, \langle \cdot, P_{T,n} \cdot \rangle)$, (possibly module off the kernel), then the q-Fock space (or in general T-twisted Fock space) is

$$\mathcal{F}_T(\mathcal{H}) := \bigoplus_{n=0}^{\infty} (\mathcal{H}^{\otimes n}, \langle \cdot, P_{T,n} \cdot \rangle)^{-} = \bigoplus_{n=0}^{\infty} \mathcal{H}_{T,n}.$$

Positivity of $P_{T,n}$

To define $\mathcal{F}_T(\mathcal{H})$, we need the **positivity** of $P_{T,n}$ for $n \geq 1$ which is nontrivial even for $T = qF$.

Theorem

*Bożejko,Speicher'94 If T satisfies **Yang-Baxter** eq.*

$$T_1 T_2 T_1 = T_2 T_1 T_2$$

and $\|T\| \leq 1$ ($\|T\| < 1$) then for all $n \geq 0$, $P_{T,n} \geq 0$ ($P_{T,n} > 0$).

Definition

$T = T^* \in B(\mathcal{H} \otimes \mathcal{H})$ is

- a **twist**: $P_{T,n} \geq 0, \quad \forall n \geq 1.$
- a **strict twist**: $P_{T,n} > 0, \quad \forall n \geq 1.$

Field operators

- T a twist. Left field operator on $\mathcal{F}_T(\mathcal{H})$:

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- Want $\Omega \in \mathcal{H}_{T,0} = \mathbb{C}\Omega$ to be standard (cyclic and separating), therefore we only take 'half' of the vectors:

$$\mathcal{L}_T(H) := \{X_T(f) : f \in H\}'', \quad H \subset \mathcal{H} \text{ a } \textbf{standard} \text{ real subspace.}$$

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- For standard H, it is more convenient to write

$$X_T(f) := a_T(S_H f) + a_T^*(f), \quad f \in H + iH$$

Definition

For a **standard** (real) subspace $H \subseteq \mathcal{H}$, and a twist $T \in B(\mathcal{H} \otimes \mathcal{H})_{s.a.}$, the T-twisted Araki-Woods algebra is $\mathcal{L}_T(H) := \{X_T(f) : f \in H\}''$.

Standard subspace

Definition

A real linear subspace $H \subseteq \mathcal{H}$ is a **standard** subspace if

- H is a closed real subspace of $(\mathcal{H}, \mathrm{Re}\langle\cdot,\cdot\rangle)$.
- H is cyclic: $H + iH$ is dense in $\mathcal{H}$.
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Example

1) $\mathbb{R}^n \subseteq \mathbb{C}^n$; 2) $(M, \mathcal{H})$ von Neumann algebra with an vector state $\varphi = \langle \xi, \cdot \xi \rangle$, then $H = \overline{M_{s.a.}\xi}^{\mathbb{R}}$ is cyclic (separating) iff ξ is cyclic (separating) for M. ($\mathcal{H}$ is called the standard representation if both holds.)

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Modular data for standard subspace $H \subset \mathcal{H}$: Involution $S_H : \mathcal{H} \to \mathcal{H}$ with $D(S_H) = H + iH$,

$$S_H(f + ig) = f - ig, \quad \forall f, g \in H.$$

Polar decomposition $S_H = J_H \Delta_H^{1/2}$, then

$$\Delta_H^{it} H = H, \quad \forall t \in \mathbb{R}.$$

J_H anti-unitary and $H' = J_H H$ is the symplectic complement of H w.r.t $\mathrm{Im}\langle \cdot, \cdot \rangle$.
[Shlyakhtenko'97] $H \subseteq \mathcal{H}$ standard subspace $\iff$ one-parameter orthogonal groups Δ_H^{-it} acting on H.

Examples of $\mathcal{L}_T(\mathcal{H})$

$H \subset \mathcal{H}$ standard, $\mathcal{L}_T(\mathcal{H}) := \{X_T(f) = a(f) + a^*(f) : f \in H\}$.

- $T = F$, $X_F(f)$'s satisfies CCR relation $[X(f), X(g)] = 2\mathrm{Im}\langle f, g\rangle$ with respect to the symplectic form $2\mathrm{Im}\langle\cdot,\cdot\rangle$ on H. ($W(tf) := \exp(itX_F(f))$ generates the Weyl's algebra.)

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- $T = -F$, $X_{-F}(f)$'s satisfies CAR relation $\{X(f), X(g)\} = 2\mathrm{Re}\langle f, g\rangle$ with respect to the symmetric bilinear form $2\mathrm{Re}\langle \cdot, \cdot\rangle$ on H. [Araki, Woods'69]

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- $T = 0$, $\mathcal{L}_0(H)$ is the free Araki-Woods algebra [Shlyakhtenko'97].
- $T = qF$, $\mathcal{L}_{qF}(H)$ is the q-Araki-Woods algebra [F.Hiai'01].

Example

If $\Delta_H = \mathrm{id}$, ($\mathcal{H} = \mathbb{C} \otimes H$), then

- $\mathcal{L}_{-F}(H) = M_{2^n}(\mathbb{C})$ when $\dim\mathcal{H} = 2n$;
- $\mathcal{L}_F(H)$ is diffuse abelian;
- $\mathcal{L}_0(H) \simeq L(F_{\dim H})$;
- $\mathcal{L}_{qF}(H)$ is the q-Gaussian algebra for $-1 < q < 1$.
- If $T(e_i \otimes e_j) = q_{ij} e_j \otimes e_i$, then $\mathcal{L}_T(H)$ is the mixed q-Gaussian algebra.

Separability of the vacuum Ω for $\mathcal{L}_T(\mathcal{H})$

We have choosen $H \subseteq \mathcal{H}$ to be a standard subspace, but when is $\mathcal{F}_T(\mathcal{H})$ a standard representation of $\mathcal{L}_T(H)$? Namely, when is Ω separating? (Ω obviously cyclic)

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Theorem (G. Lechner,R.C. da Silva'22+)

T a twist, $H \subset \mathcal{H}$ standard subspace. Assume T is compatible with H: $[T, \Delta_H^{it} \otimes \Delta_H^{it}] = 0$. Then Ω is separating for $\mathcal{L}_T(H)$ if and only if

- $T_1 T_2 T_1 = T_2 T_1 T_2$.
- *T is crossing symmetric: for all $\psi_1, \cdots, \psi_4 \in \mathcal{H}$, the function on $\mathbb{R}$*

$$T^{\psi_2,\psi_1}_{\psi_3,\psi_4}(t) := \langle \psi_2 \otimes \psi_1, (\Delta_H^{it} \otimes 1) T (1 \otimes \Delta_H^{-it})(\psi_3 \otimes \psi_4) \rangle$$

is holomorphic on the strip $\{z \in \mathbb{C} : 0 \leq Im(z) \leq 1/2\}$

$$T^{\psi_2,\psi_1}_{\psi_3,\psi_4}(t + \frac{i}{2}) := \langle \psi_1 \otimes J_H\psi_4, (1 \otimes \Delta_H^{it}) T (\Delta_H^{-it} \otimes 1)(J_H\psi_2 \otimes \psi_3) \rangle.$$

Denote the contraction $C(f_1 \otimes f_2) = \langle S_H f_1, f_2 \rangle$. $C_i = \mathrm{id}^{i-1} \otimes C \otimes \mathrm{id}^{n-i-1}$.

Proposition (Y)

T crossing symmetric iff $C_1 T_2 = C_2 T_1$.

crossing symmetry $\iff C_1 T_2 = C_2 T_1$

For the tracial case, crossing symmetry is simple: Let $\{e_i\}$ be an orthogonal basis of H, and $T_{ij}^{kl} = \langle e_k \otimes e_l, e_i \otimes e_j \rangle$, then

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Apply $C_1 T_2$ to $e_{ijk} := e_i \otimes e_j \otimes e_k$, we get
$C_1 T_2(e_{ijk}) = \sum_{x,y} C_1(T_{jk}^{xy} e_{ixy}) = \sum_{x,y} \delta_{ix} T_{jk}^{xy} e_y = \sum_y T_{jk}^{iy} e_y$. And similarly,
$C_2 T_1(e_{ijk}) = \sum_{x,y} C_2(T_{ij}^{xy} e_{xyk}) = \sum_{x,y} \delta_{yk} T_{ij}^{xy} e_x = \sum_x T_{ij}^{xk} e_x$.

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- Therefore, crossing symmetry $\iff C_1 T_2 = C_2 T_1$.
- In general, we need to be careful about S_H and Δ_H^{it} when taking contractions, and use uniqueness of analytic extension.

An example of (non q_{ij}) crossing symmetric twist

Let $\mathcal{H} = M_n(\mathbb{C})$, with inner product induced by Tr, and $H = M_n(\mathbb{C})_{s.a.}$. Denote $m : M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \to M_n(\mathbb{C})$ the multiplication operator:

$$m(a \otimes b) = ab.$$

Then $T = cm^* m$ is a crossing symmetric Yang-Baxter solution (due to Frobenious structure of finite dimensional C^*-algebras). And $\|T\| = cn$.

Conjugate variables and free Fisher information

Let $(X_1, \cdots, X_d)$ be a family of noncommutative random variables (not necessarily self-adjoint).

The free Fisher information of $(X_1, \cdots, X_d)$ is defined via conjugate variables $\Xi_i := \partial_i^*(1 \otimes 1)$ w.r.t. the free difference quotient $\partial_i : L^2(M, \varphi) \to L^2(M, \varphi) \otimes L^2(M, \varphi)$:

$$\partial_i(X_j) = \delta_{ij} 1 \otimes 1, \quad \partial_i(pq) = (\partial_i p) \cdot q + p \cdot (\partial_i q).$$

Free Fisher information:

$$\Phi(X_1, \cdots, X_d) := \sum_{i=1}^{d} \|\Xi_i\|_2^2 = \sum_{i=1}^{d} \|\partial_i^*(1 \otimes 1)\|_2^2.$$

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Example

$(S_1, \cdots, S_d)$ a family of freely independent semicircular variables, then $\xi_i = S_i$, hence $\Phi(X_1, \cdots, X_d) := \sum_{i=1}^{d} \|\Xi_i\|_2^2 = \sum_{i=1}^{d} \|S_i\|_2^2 = d$.

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Properties when conjugate variables exists

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- Factoriality: A finite von Neumann algebra $(W^*(X_1, \cdots, X_d), \tau)$ with finite non-microstate free fisher information is non-Γ II_1 factor by Dabrowski 2010. Nontracial cases: B. Nelson '17.

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- Free monotone transport: If the conjugate variables $(\Xi_1, \cdots, \Xi_d)$ exists and $\|\Xi_i - X_i\|_R$ is small, then by [Guionnet, Shlyakhtenko,'13] there exists free monotone transport from M to $L(F_d)$, hence

$$W^*(X_1, \cdots, X_d) \simeq L(F_d).$$

  Nontracial cases: B. Nelson '15.
- Example: q-Gaussian for small q [Dabrowski'14], for all $-1 < q < 1$ [Miyagawa, Speicher'22]. q-Araki-Woods algebras for $-1 < q < 1$ [A. Skalski, M. Kumar, M. Wasilewski'23].

From polynomials to tensors

Recall $\forall f \in H + iH, X_T(f) = a_T^*(f) + a_T(f)$. The formula for a_T:

$$a_T(f) = a(f)(1 + T_1 + \cdots + T_1 \cdots T_k)$$

where $a(f) : f_1 \otimes \cdots \otimes f_k = \langle f, f_1 \rangle f_2 \otimes f_k$ is the free annihilation.

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$$a(Sf)(f_1 \otimes \cdots \otimes f_k) = C_1(f \otimes f_1 \otimes \cdots \otimes f_k)$$

$$a_T(Sf)(f_1 \otimes \cdots \otimes f_k) = C_1(1 + T_2 + \cdots + T_2 \cdots T_{k+1})(f \otimes f_1 \otimes \cdots \otimes f_k)$$

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where $a(f) : f_1 \otimes \cdots \otimes f_k = \langle f, f_1 \rangle f_2 \otimes f_k$ is the free annihilation.
$C(f \otimes g) = \langle Sf, g \rangle$, and $C_i = \mathrm{id}_{\mathcal{H}}^{\otimes(i-1)} \otimes C \otimes \mathrm{id}$

$$a(Sf)(f_1 \otimes \cdots \otimes f_k) = C_1(f \otimes f_1 \otimes \cdots \otimes f_k)$$
$$a_T(Sf)(f_1 \otimes \cdots \otimes f_k) = C_1(1 + T_2 + \cdots + T_2 \cdots T_{k+1})(f \otimes f_1 \otimes \cdots \otimes f_k)$$

- $X_T(\xi_1)\Omega = \xi_1$. $X_T(\xi_1)X_T(\xi_2)\Omega = X_T(\xi_1)\xi_2 = (a_T^*(\xi_1) + a_T(S\xi_1))\xi_2 = \xi_1 \otimes \xi_2 + \langle S\xi_1, \xi_2 \rangle \Omega$.

From polynomials to tensors

Recall $\forall f \in H + iH, X_T(f) = a_T^*(f) + a_T(f)$. The formula for a_T:

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-

$$\begin{aligned} X_T(\xi_1)X_T(\xi_2)X_T(\xi_3)\Omega &= (a_T^*(\xi_1) + a_T(S\xi_1))(\xi_2 \otimes \xi_3 + \langle S\xi_2, \xi_3 \rangle \Omega) \\ =&\xi_1 \otimes \xi_2 \otimes \xi_3 + \langle S\xi_2, \xi_3 \rangle \xi_1 + a(S\xi_1)(1 + T_1)(\xi_2 \otimes \xi_3) \\ =&\xi_1 \otimes \xi_2 \otimes \xi_3 + \langle S\xi_2, \xi_3 \rangle \xi_1 + \langle S\xi_1, \xi_2 \rangle \xi_3 + a(S\xi_1) T(\xi_2 \otimes \xi_3) \end{aligned}$$

From polynomials to tensors

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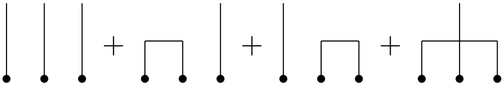
$$X_T(\xi_1)X_T(\xi_2)X_T(\xi_3)\Omega =$$ 

From polynomials to tensors

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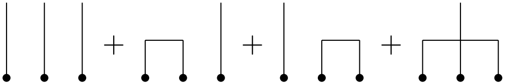
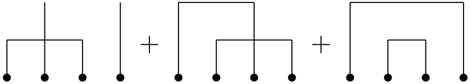
$$X_T(\xi_1)X_T(\xi_2)X_T(\xi_3)\Omega =$$ 

$$X_T(\xi_1)X_T(\xi_2)X_T(\xi_3)X_T(\xi_4)\Omega = \cdots +$$ 

$$= \cdots + a(S\xi_1)\,T(\xi_2 \otimes \xi_3) \otimes \xi_4 + a(S\xi_1)a(S\xi_2)\,T(\xi_3 \otimes \xi_4)\Omega + \langle S\xi_2, \xi_3\rangle\langle S\xi_1, \xi_4\rangle\Omega$$

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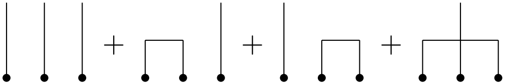
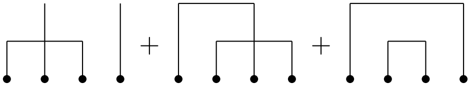
$$X_T(\xi_1)X_T(\xi_2)X_T(\xi_3)\Omega =$$ 

$$X_T(\xi_1)X_T(\xi_2)X_T(\xi_3)X_T(\xi_4)\Omega = \cdots +$$ 

$$= \cdots + a(S\xi_1)\, T(\xi_2 \otimes \xi_3) \otimes \xi_4 + a(S\xi_1) a(S\xi_2)\, T(\xi_3 \otimes \xi_4)\Omega + \langle S\xi_2, \xi_3 \rangle \langle S\xi_1, \xi_4 \rangle \Omega$$

In general, $X_T(\xi_1) \cdots X_T(\xi_k)$ is summing over partitions that only contain pairings and singletons: $\mathcal{P}_{1,2}(k)$.

Wick product: From tensors to polynomials

Definition (Wick product)

For $f_i \in \mathcal{H}$, $\Phi(f_1 \otimes \cdots \otimes f_n)$ is the unique operator in $\mathcal{L}_T(H)$ such that

$$\Phi(f_1 \otimes \cdots \otimes f_n)\Omega = f_1 \otimes \cdots \otimes f_n.$$

That is: $\Phi(f_1 \otimes \cdots \otimes f_n)$ the unique polynomial $p(X(f) : f \in \mathcal{H})$ in $X_T(f)$'s such that $p\Omega = f_1 \otimes \cdots \otimes f_n$.

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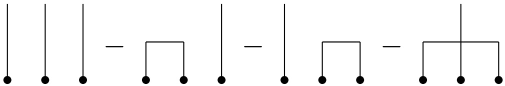
$$\begin{aligned} \Phi(\xi_1 \otimes \xi_2 \otimes \xi_3) = {} & X(\xi_1)X(\xi_2)X(\xi_3) - \langle S\xi_1, \xi_2 \rangle X(\xi_3) \\ & - \langle S\xi_2, \xi_3 \rangle X(\xi_1) - X(a(S\xi_1) T(\xi_2 \otimes \xi_3)) \end{aligned}$$

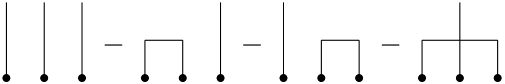
$$\Phi(\xi_1 \otimes \xi_2 \otimes \xi_3) = X(\xi_1)X(\xi_2)X(\xi_3) - \langle S\xi_1, \xi_2\rangle X(\xi_3) - \langle S\xi_2, \xi_3\rangle X(\xi_1) - X(a(S\xi_1)T(\xi_2 \otimes \xi_3))$$

$$\Phi(\xi_1 \otimes \xi_2 \otimes \xi_3) =$$

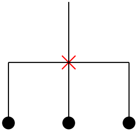


$$\Phi(\xi_1 \otimes \xi_2 \otimes \xi_3) = X(\xi_1)X(\xi_2)X(\xi_3) - \langle S\xi_1, \xi_2\rangle X(\xi_3) - \langle S\xi_2, \xi_3\rangle X(\xi_1) - X(a(S\xi_1)\,T(\xi_2 \otimes \xi_3))$$

$$\Phi(\xi_1 \otimes \xi_2 \otimes \xi_3) =$$



$$C_1\,T_2(\xi_1 \otimes \xi_2 \otimes \xi_3) = a(S\xi_1)\,T(\xi_2 \otimes \xi_3) =$$



$$:= W_\pi^T(\xi_1 \otimes \xi_2 \otimes \xi_3)$$

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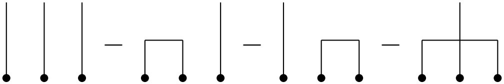

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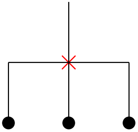

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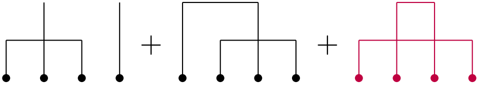
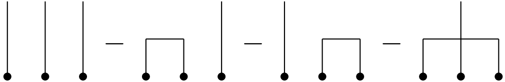
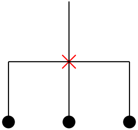
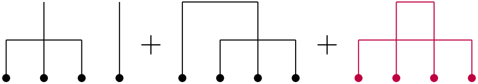
$$\Phi(\xi_1 \otimes \xi_2 \otimes \xi_3 \otimes \xi_4) = \cdots +$$



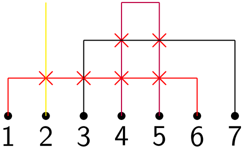
$$\Phi(\xi_1 \otimes \xi_2 \otimes \xi_3) = X(\xi_1)X(\xi_2)X(\xi_3) - \langle S\xi_1, \xi_2\rangle X(\xi_3) - \langle S\xi_2, \xi_3\rangle X(\xi_1) - X(a(S\xi_1)T(\xi_2 \otimes \xi_3))$$

$\Phi(\xi_1 \otimes \xi_2 \otimes \xi_3) =$ 

$C_1 T_2(\xi_1 \otimes \xi_2 \otimes \xi_3) = a(S\xi_1)T(\xi_2 \otimes \xi_3) =$  $:= W^T_\pi(\xi_1 \otimes \xi_2 \otimes \xi_3)$

$\Phi(\xi_1 \otimes \xi_2 \otimes \xi_3 \otimes \xi_4) = \cdots +$ 

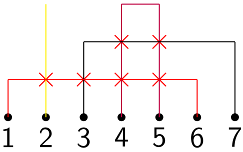
Number of crossings = Number of T_i's. Height of chord = Order of contractions. Number of pairings = Number of C_i's



$= W^T_{\pi'} = C_2 C_2 T_3 T_4 C_1 T_2 T_3 T_4 T_5$

More about W_π^T

$\pi' = \{\{1,6\},\{3,7\},\{2\},\{4\},\{5\}\} \in \mathcal{P}_{1,2}(7)$.



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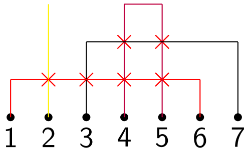

$= \; W_{\pi'}^T = C_2 C_2 T_3 T_4 C_1 T_2 T_3 T_4 T_5$

For q-Araki-Woods algebra, $T = qF$, this is precisely taking contraction of $\langle S\xi_1, \xi_6\rangle$ and $\langle S\xi_3, \xi_7\rangle$:

$$W_{\pi'}^{qF}(\xi_1 \otimes \cdots \otimes \xi_7) = q \qquad \xi_2$$

More about W^T_π

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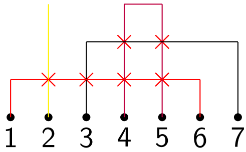

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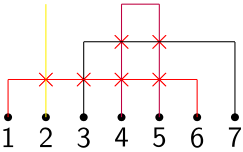

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$$W_{\pi'}^{qF}(\xi_1 \otimes \cdots \otimes \xi_7) = q^6 \langle S\xi_1, \xi_6\rangle \langle S\xi_3, \xi_7\rangle \langle S\xi_4, \xi_5\rangle \xi_2$$

More about W_π^T

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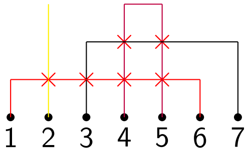

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For general T, $W_{\pi'}^T$ involves information about all ξ_i, and there are no explicit formula other than the definition $C_2 T_3 T_4 C_1 T_2 T_3 T_4 T_5$.

Wick product: From tensors to polynomials

Theorem (Wick formula, Y)

$$\Phi(\xi_1 \otimes \cdots \otimes \xi_n) = X\Big(\sum_{\pi \in \mathcal{P}_{1,2}(n)} (-1)^{|p(\pi)|} W_\pi^T(\xi_1 \otimes \cdots \otimes \xi_n) \Big),$$

where X *is the linear map*
$X(\eta_1 \otimes \cdots \otimes \eta_k) = X_T(\eta_1) X_T(\eta_2) \cdots X_T(\eta_k) \in \mathcal{L}_T(H)$ *for all* $\eta_i \in \mathcal{H}$ *and* $X(\Omega) = 1$. $\mathcal{P}_{1,2}(n)$ *partitions with only pairings and singletons.*

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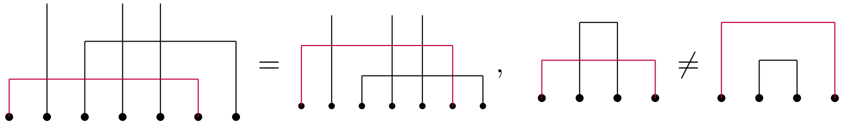
Lemma (W_π^T is Order invariant)

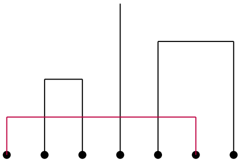
If T is braided and crossing symmetric: $T_1 T_2 T_1 = T_2 T_1 T_2$, $C_1 T_2 = C_2 T_1$, then W_π^T is invariant under change of order (preserving the nesting).



Computing $\partial_i \Phi(\xi_1 \otimes \cdots \otimes \xi_n)$

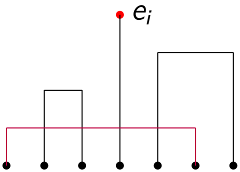
Fix orthonomal basis $(e_1, \cdots, e_d)$ of $\mathcal{H}$, $X_i = X_T(e_i)$.

$\partial_i \Phi(\xi_1 \otimes \cdots \otimes \xi_7) = \cdots + \partial_i$



$+ \cdots$

$= \cdots +$



$+ \cdots$

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Fix orthonomal basis $(e_1, \cdots, e_d)$ of $\mathcal{H}$, $X_i = X_T(e_i)$.

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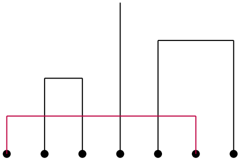
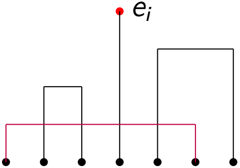


$+ \cdots$

$= \cdots +$



$+ \cdots$

Idea: ∂_i splits each $\pi \in \mathcal{P}_{1,2}(n)$ into the left part and the right part.

Computing $\partial_i \Phi(\xi_1 \otimes \cdots \otimes \xi_n)$

Decompose partitions π w.r.t a singleton k:

$$\left\{ \begin{array}{c} \pi \in \mathcal{P}_{1,2}(n) \\ k\text{: singleton of } \pi \end{array} \right\} \iff \left\{ \begin{array}{c} \pi^m \in \mathcal{P}_{1,2}(n) \\ k\text{: singleton of } \pi^m, k \in \cap p(\pi^m) \\ \pi_l \in \mathcal{P}_{1,2}(s_l(\pi^m)), \pi_r \in \mathcal{P}_{1,2}(s_r(\pi^m)) \end{array} \right\}$$

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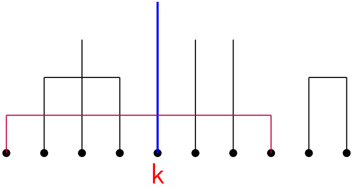
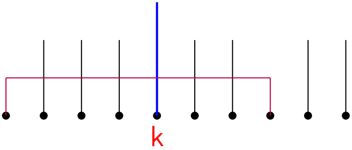
$\pi =$  $\Longrightarrow$ $\pi^m =$ 

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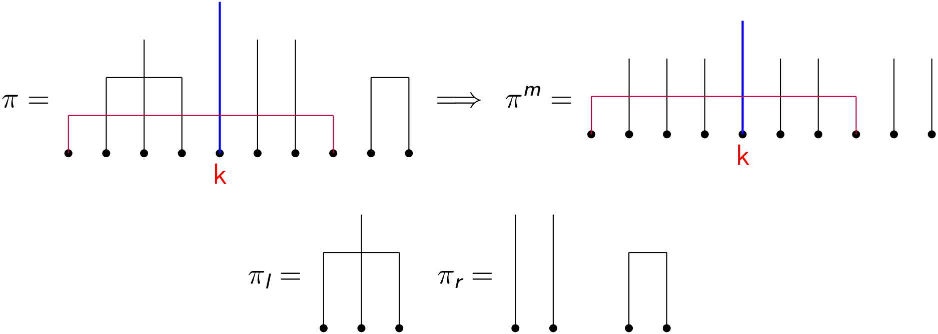

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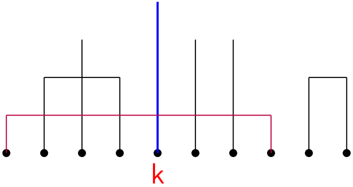
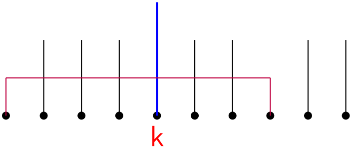
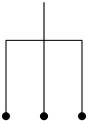
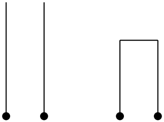
$\pi =$  $\Longrightarrow$ $\pi^m =$ 

$\pi_l =$  $\pi_r =$ 

$$W_\pi^T = (W_{\pi_l}^T \otimes \mathrm{id}_{\mathcal{H}} \otimes W_{\pi_r}^T) W_{\pi^m}^T$$

Computing $\partial_i \Phi(\xi_1 \otimes \cdots \otimes \xi_n)$

$$W^T_\pi = (W^T_{\pi_l} \otimes \mathrm{id}_{\mathcal{H}} \otimes W^T_{\pi_r}) W^T_{\pi^m}$$

Set $\nabla^k_i : f_1 \otimes \cdots \otimes f_n \mapsto \langle e_i, f_k \rangle (f_1 \otimes \cdots \otimes f_{k-1}) \otimes (f_{k+1} \otimes \cdots \otimes f_n)$

$$\partial_i \Phi = \partial_i \sum_{\pi \in P_{1,2}(n)} (-1)^{\cdots} X \circ W^T_\pi = \sum_{k \in s(\pi)} \sum_{\pi \in P_{1,2}(n)} (-1)^{\cdots} X \circ \nabla^k_i W^T_\pi$$

$$= \sum_{\pi^m} \sum_{k \in \cap p(\pi^m)} \sum_{\pi_l} \sum_{\pi_r} (-1)^{\cdots} X \circ \nabla^k_i (W^T_{\pi_l} \otimes \mathrm{id}_{\mathcal{H}} \otimes W^T_{\pi_r}) W^T_{\pi^m}$$

$$= \sum_{\pi^m} \sum_{k \in \cap p(\pi^m)} \sum_{\pi_l} \sum_{\pi_r} (-1)^{\cdots} X \circ (W^T_{\pi_l} \otimes W^T_{\pi_r}) \nabla^{k'}_i W^T_{\pi^m}$$

apply inverse formula for Φ for the sum π_l, π_r, (and identify polynomials as its L^2-image)

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$\partial_i \Phi(\xi_1 \otimes \cdots \otimes \xi_n)$ and Ξ_i

Corollary (Y)

If T is braided and crossing symmetric,

$$\partial_i = \sum_{\pi \in \mathcal{P}_{1,2}(n)} \sum_{k \in \cap p(\pi)} (-1)^{|p(\pi)|} \nabla_i^{k'} W_\pi^T,$$

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Need to compute $\Xi_i = \partial_i^*(\Omega \otimes \Omega)$. Note that

$$\begin{aligned} \langle \partial_i(\xi_1 \otimes \cdots \otimes \xi_n), \Omega \otimes \Omega \rangle &= \langle \sum (-1)^{\cdots} \nabla_i^{k'} W_\pi^T(\xi_1 \otimes \cdots \otimes \xi_n), \Omega \otimes \Omega \rangle \\ &= \sum (-1)^{\cdots} \langle W_\pi^T(\xi_1 \otimes \cdots \otimes \xi_n), e_i \rangle \end{aligned}$$

We only need to focus on π with the only singleton $k \in \cap p(\pi)$.

Computing Ξ_i

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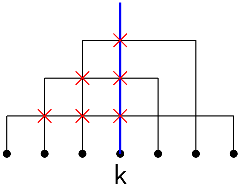

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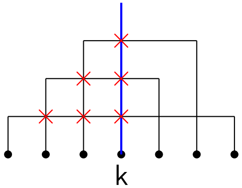

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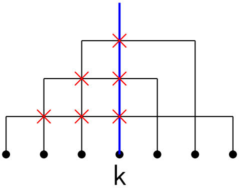

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In particular, for all such π, it has 'MAXIMAL' Crossing Number on the left:

$$\mathrm{wcr}(\pi) \geq \frac{m(m+1)}{2} \implies \|W_\pi^T\| \lesssim \|T\|^{m(m+1)/2}$$

Ξ_i

Continue the computation, we obtain:

Theorem (Y)

Let $\mathcal{H}$ be a finite dimensional complex Hilbert space with a standard subspace $H \subset \mathcal{H}$, and T be a compatible crossing symmetric and braided twist on $\mathcal{H}$ with $\|T\| < 1$. Let $(e_1, \cdots, e_d)$ be a orthonomal basis of $\mathcal{H}$. Then the conjugate system $(\Xi_1, \cdots, \Xi_d)$ for $(X_T(e_1), \cdots, X_T(e_d))$ exists and

$$\Xi_i = \sum_{n=0}^{\infty} (-1)^n P_{T,2n+1}^{-1} \sum_{\pi \in B(2n+1)} (W_\pi^T)^* e_i, \quad \forall 1 \leq i \leq d,$$

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Key observation: $\|(W_\pi^T)^*\| = \|W_\pi^T\| \lesssim \|T\|^{m(m+1)/2} \simeq e^{-cn^2}$ beats all other terms!!!

Conclusion

Corollary (Factoriality)

If $2 \leq \dim \mathcal{H} < \infty$, $\|T\| < 1$, *let* $G < \mathbb{R}^*_\times$ *be the closed subgroup generated by* $Sp(\Delta_H)$, *then then* $\mathcal{L}_T(H)$ *is a* factor *of type*

$$\begin{cases} III_1, & \text{if } G = \mathbb{R}^\times_* \\ III_\lambda, & \text{if } G = \lambda^{\mathbb{Z}}, 0 < \lambda < 1 \\ II_1, & \text{if } G = \{1\}. \end{cases}$$

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Corollary (Free monotone transport)

For $H \subset \mathcal{H}$ *with* $2 \leq \dim \mathcal{H} < \infty$, *there is a constant* $q_H > 0$ *depending on* H, *such that for all* $\|T\| < q_H$,

$$\mathcal{L}_T(H) \simeq \mathcal{L}_0(H).$$

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- What about $\mathcal{L}_T(H)$ when $\|T\| = 1$ (and when $T \geq 0$)? (There are examples: $\mathcal{L}_T(H)$ is still free group factor but $\|T\| = 1$.)

Thank you!