Biexact von Neumann algebras

joint with Jesse Peterson.

A II_1 factor M is prime if $M \not\cong N_1 \bar{\otimes} N_2$, N_i is a II_1 factors.

Popa '83: $L\mathbb{F}_S$ S uncountable.

Ge '98: $L\mathbb{F}_n$, $n \geq 2$, uses Voiculescu's free entropy theory.

Ge asked if $L\mathbb{F}_n$ is solid: $\forall A \subset L\mathbb{F}_n$ diffuse subalgebra, is $A' \cap L\mathbb{F}_n$ amenable?

Note: solidity passes to subalgebras
solidity + nonamen $\Rightarrow$ primeness.

Thm (Ozawa '03) $L\mathbb{F}_n$ $n \geq 2$ is solid, i.e.
$\forall A \subset L\mathbb{F}_n$ diffuse, $A' \cap L\mathbb{F}_n$ is amenable.

Proof involves two ingredients:

$\Gamma = \mathbb{F}_2$.

1) Akemann-Ostrand '75

$\nu: C^*_\lambda\Gamma \otimes C^*_\rho\Gamma \to a \otimes b \mapsto ab + \mathbb{K}(\ell^2\Gamma) \in \mathbb{B}(\ell^2\Gamma)/\mathbb{K}(\ell^2\Gamma)$

is min-continuous.

2) Γ is exact. (uses exact $\Rightarrow C^*_\lambda\Gamma$ is locally reflexive (gives good approx of $L\Gamma$ using $C^*_\lambda\Gamma$)).

Proof idea: take $A \subset L\Gamma$ diffuse.

pick $u_n \in \mathcal{U}(A)$, $u_n \to 0$ weakly.

$\theta := \lim \mathrm{Ad}(u_n): \mathbb{B}(\ell^2\Gamma) \longrightarrow \mathbb{B}(\ell^2\Gamma)$

$\phi: C^*_\lambda\Gamma \otimes_{\min} C^*_\rho\Gamma \xrightarrow{\nu} \mathbb{B}(\ell^2\Gamma)/\mathbb{K}(\ell^2\Gamma) \xrightarrow{\theta} \mathbb{B}(\ell^2\Gamma)$

Assume $P \cap C^*_\lambda\Gamma \subset P$ dense, $P = A' \cap L\Gamma$.

$\phi(a \otimes b) = ab \quad \forall a \in P \cap C^*_\lambda\Gamma, \ b \in C^*_\rho\Gamma$.

$\tilde{\phi}: \mathbb{B}(\ell^2\Gamma) \longrightarrow L\Gamma \xrightarrow{E_P} P$, $\tilde{\phi}|_{P \cap C^*_\lambda\Gamma} = \mathrm{id}$

$\Rightarrow P$ is amenable.

Turns out a wider class of groups satisfies

1) Akemann - Ostrand property.
2) exact.

is called biexact groups.

Ex: hyperbolic groups, $\mathbb{Z}^2 \rtimes SL_2\mathbb{Z}$,

$\Lambda \wr \Gamma$ with Λ amenable, Γ biexact (Ozawa).

- Ozawa's idea offers much more flexibility

combine with deformation/rigidity.

Popa - Vaes' 14: $\mathbb{F}_n \curvearrowright (X,\mu)$, $\mathbb{F}_m \curvearrowright (Y,\nu)$ free ergodic p.m.p. actions, then

$L^\infty(X) \rtimes \mathbb{F}_n \cong L^\infty(Y) \rtimes \mathbb{F}_m$ iff $m = n$.

- Another dynamical definition for biexact groups.

$\underline{S(\Gamma)} = \{ f \in \ell^\infty\Gamma \mid f - R_t f \in \underline{c_0\Gamma}, \forall t \in \Gamma \}$

$\subset \ell^\infty \Gamma$.

$(R_t f)(s) = f(st)$

$S(\Gamma) = C(\overline{\Gamma}^S)$, $\overline{\Gamma}^S$ is a compactification of Γ.

$\Gamma \curvearrowright^R \Gamma \rightsquigarrow \overline{\Gamma}^S \curvearrowleft^R \Gamma$

$\overline{\Gamma}^S$ is the universal compactification s.t. $\overline{\Gamma}^S \setminus \Gamma \curvearrowleft^R \Gamma$ is a trivial action.

- when Γ is hyperbolic, $\partial_{Grom}\Gamma$ is a factor of $\overline{\Gamma}^S \setminus \Gamma$.

- Def: Γ is biexact if and only if $\Gamma \curvearrowright^L \overline{\Gamma}^S$ is top. amen $\Leftrightarrow$ $S(\Gamma) \rtimes_r \Gamma$ is a nuclear C^*- algebra.

• although biexactness is characterized by $\Gamma \curvearrowright \bar{\Gamma}^s$ being top. amen. Sako '09 showed that if $\Gamma \sim_{ME} \Lambda$, then Γ is biexact iff Λ is biexact.

Q: if $L\Gamma \cong L\Lambda$, does Γ being biexact imply Λ being biexact.

Approach: to define biexactness for von Neuman algebras and show Γ is biexact iff $L\Gamma$ is biexact.

Recall Γ biexact if

i) $\underline{S(\Gamma)} = \{ f \in \ell^\infty\Gamma \mid f - R_t f \in c_0\Gamma, \forall t \in \Gamma \}$ $\subset \ell^\infty\Gamma$.

ii) $\Gamma \overset{L}{\curvearrowright} S(\Gamma)$ is top. amen. $\Leftrightarrow$
$S(\Gamma) \rtimes_r \Gamma$ is nuclear $\Leftrightarrow$
$C^*_r\Gamma \subset S(\Gamma) \rtimes_r \Gamma$ is a nuclear map

i.e. $C^*_r\Gamma \xrightarrow{\phi_i} \mathbb{M}_{n(i)}(\mathbb{C}) \xrightarrow{\psi_i} S(\Gamma) \rtimes_r \Gamma$ $\quad$ ϕ_i, ψ_i u.c.p. $\psi_i \circ \phi_i \to id_{C^*_r\Gamma}$ in pt-norm.

i) what is the small at infinity compt. for (M, τ)?

$\{ T \in \mathbb{B}(L^2M) \mid [T, x] \in \mathbb{K}(L^2M), \forall x \in M' \}$
$= M + \mathbb{K}(L^2M)$ $\quad$ Johnson - Parrot '75
i.e. this is not the right object.

$\underline{\mathbb{S}(M)} = \{ T \in \mathbb{B}(L^2M) \mid [T, x] \in \overline{\mathbb{K}(L^2M)}^{\|\cdot\|_{\infty,1}}, \forall x \in M' \}$.

$\|T\|_{\infty,1} = \sup\limits_{a, b \in (M)_1} \langle T\hat{a}, \hat{b} \rangle$

($\mathbb{S}(M)$ is first considered in D - KE - P '22).

• $X \subset M'$ a generating set. If $T \in \mathbb{B}(L^2M)$ satisfies $[T, x] \in \mathbb{K}(L^2M)$ $\forall x \in X$, then $[T, a] \in \overline{\mathbb{K}(L^2M)}^{\|\cdot\|_{\infty,1}}$ $\forall a \in M'$.

in particular: $S(\Gamma) \hookrightarrow \mathbb{S}(L\Gamma)$ because $[f, \rho_t] \in \mathbb{K}(\ell^2\Gamma)$ $\forall t \in \Gamma$.

ii) what is a top. amen action for (M,τ)?

$\Gamma \curvearrowright S(\Gamma)$ is top. amen.

$\Leftrightarrow$ $C^*_r\Gamma \subset S(\Gamma)\rtimes_r\Gamma$ is a nuclear map.

for M, we look at

$$\begin{tikzcd} M \arrow[d, "\phi_i"'] & \subset & S(M) = \{ T\in \mathbb{B}(L^2M) \mid [T,x]\in \overline{\mathbb{K}(L^2M)}^{\|\cdot\|_{\infty,1}} \ \forall x\in M' \}. \\ \mathbb{M}_{n(i)}(\mathbb{C}) \arrow[urr, "\psi_i"'] & & \end{tikzcd}$$

$\psi_i\circ\phi_i \to \mathrm{id}_M$ in what topology?

norm is too strong for vNa.
weak* topology, but $S(M)$ is not weak* - closed.

$S(M)^\# = \{ \varphi\in S(M)^* \mid M\times M \ni (a,b)\mapsto \varphi(aTb)$ is sep. normal $\forall\, T\in S(M) \}$.

a state $\varphi\in S(M)^*$, then $\varphi\in S(M)^\#$ iff $\varphi|_M$ is normal.

$\psi_i\circ\phi_i \to \mathrm{id}_M$ in pt-$\sigma(S(M), S(M)^\#)$-topology

i.e. $\varphi(\psi_i(\phi_i(x)) - x)\to 0$, $\forall x\in M$, $\varphi\in S(M)^\#$.

We say $M\subset S(M)$ is M-nuclear if

$$\begin{tikzcd} M \arrow[d, "\phi_i"'] & \subset & S(M) \\ \mathbb{M}_{n(i)}(\mathbb{C}) \arrow[urr, "\psi_i"'] & & \end{tikzcd}$$

s.t. $\psi_i\circ\phi_i \to \mathrm{id}_M$ in pt-$\sigma(S(M), S(M)^\#)$ topology.

And M is biexact if $M\subset S(M)$ is M-nuclear.

Γ group. biexact

$S(\Gamma) = \{ f\in \ell^\infty\Gamma \mid f - R_t f\in c_0\Gamma \ \forall t\in\Gamma \}$

$$\begin{tikzcd} C^*_r\Gamma \arrow[dr] & \subset & S(\Gamma)\rtimes_r\Gamma \\ & \mathbb{M}_n(\mathbb{C}) \arrow[ur] & \end{tikzcd}$$

almost commute.

M vNa biexact

$S(M) = \{ T\in \mathbb{B}(L^2M) \mid [T,x]\in \overline{\mathbb{K}(L^2M)}^{\|\cdot\|_{\infty,1}} \ \forall x\in M' \}$

$$\begin{tikzcd} M \arrow[dr] & \subset & S(M) \\ & \mathbb{M}_n \arrow[ur] & \end{tikzcd}$$

almost commutes.
in $\sigma(S(M), S(M)^\#)$

Thm: Γ biexact iff $L\Gamma$ is biexact.

($\Rightarrow$) $C^*_r\Gamma \subset S(\Gamma)\rtimes_r\Gamma$ is nuclear

$S(\Gamma) \overset{\Gamma}{\hookrightarrow} \mathbb{S}(L\Gamma)$

$S(\Gamma)\rtimes_r\Gamma \hookrightarrow \mathbb{S}(L\Gamma)$

$\rightsquigarrow$ $C^*_r\Gamma \subset \mathbb{S}(L\Gamma)$ is nuclear

local reflexivity $\rightsquigarrow$ $L\Gamma \subset \mathbb{S}(L\Gamma)$ is $L\Gamma$-nuclear.

($\Leftarrow$) $\Gamma \subset L\Gamma$

$$\begin{tikzcd} & \mathbb{S}(L\Gamma) \arrow[r, "E"] & S(\Gamma) \\ L\Gamma \arrow[d, "\phi_i"] & & \\ \mathbb{M}_{n(i)}(\mathbb{C}) \arrow[uur, "\psi_i"'] & & \end{tikzcd}$$

$E: \mathbb{B}(\ell^2\Gamma) \to \ell^\infty\Gamma$ cond. exp. to diagonal.

$h_i : \Gamma \to S(\Gamma)$

by $h_i(t) = E(\psi_i(\phi_i(\lambda_t))\lambda_t^*)$

$h_i \to 1$ in pt-norm. $\Rightarrow$ $\Gamma \curvearrowright S(\Gamma)$ is top amen. $\square$

h_i p.d. fin. supp

Ex: 1) q-Gaussian vNa $M_q(\mathcal{H})$

$\dim(\mathcal{H}) < \infty$.

2) Free Araki–Woods factors.

3) biexact is closed under free product.

passes to subalgebras w/ exp.

Properties: M biexact $\Rightarrow$ M is solid, full.

Application: if $L\Gamma \hookrightarrow L\mathbb{F}_n$, $n \geq 2$,

then Γ is biexact.

Recall AO property for Γ is

$$C^*_\lambda\Gamma \otimes C^*_\rho\Gamma \ni a\otimes b \mapsto ab + \mathbb{K}(\ell^2\Gamma) \in C^*(C^*_\lambda\Gamma, C^*_\rho\Gamma)/\mathbb{K}(\ell^2\Gamma)$$

is min-continuous.

W*AO:

$$L\Gamma \otimes R\Gamma \ni a\otimes b \mapsto ab + \mathbb{K}(L\Gamma) \in C^*(L\Gamma, R\Gamma)/\mathbb{K}(L\Gamma)$$

is min-continuous

$$\mathbb{K}(L\Gamma) = C^*\left(\left(\overline{\mathbb{K}(\ell^2\Gamma)}^{\|\cdot\|_{\infty,1}}\right)_+\right)$$

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W^*AO:

$$L\Gamma \otimes R\Gamma \ni a\otimes b \mapsto ab + \mathbb{K}(L\Gamma) \in C^*(L\Gamma, R\Gamma)/\mathbb{K}(L\Gamma)$$

is min-continuous

$$\mathbb{K}(L\Gamma) = C^*\left(\left(\overline{\mathbb{K}(\ell^2\Gamma)}^{\|\cdot\|_{\infty,1}}\right)_+\right)$$

Γ biexact $\Rightarrow$ $L\Gamma$ has W^*AO.

M biexact $\Rightarrow$ M has W^*AO

$W^*AO \Rightarrow$ solidity

Caspers'22: $M_q(\mathcal{H})$, $q \neq \pm 1, 0$, $\dim(\mathcal{H}) = \infty$.

then $M_q(\mathcal{H})$ does not have W^*AO.

$\Rightarrow$ $M_q(\mathcal{H})$ is not biexact

$\Rightarrow$ $M_q(\mathcal{H}) \not\leftrightarrow L\mathbb{F}_n$.

Thm: $M_q(\mathcal{H})$ is strongly solid.

but not biexact when $\dim(\mathcal{H}) = \infty$.

idea: weak amenable.

$M_q(\mathcal{H}\oplus 0) \subset M_q(\mathcal{H}\oplus\mathcal{H})$.

$\bigcup_n M_q(\mathcal{H}_n) \subset M_q(\mathcal{H})$ is weakly dense.

Q: is there a group Γ which is not biexact but $L\Gamma$ is solid.

$$A \otimes_{\min} A' \ni a\otimes b \mapsto ab + \mathbb{K}(\mathcal{H}) \in \mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H}).$$

$S(\Gamma) \rtimes_r \Gamma = S(A)$ $\quad A = C^*_r\Gamma$

$A \subset S(A)$ is nuclear