

The space of traces of the free group and free product C^* -algebras

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Outline

- 1 Introduction
 - Traces on C^* -algebras
 - The topological structure
 - Types of Simplices
- 2 Some Group Theory
 - Examples of Traces on Groups
 - Character Rigidity
- 3 Free Products of Matrix Algebras
 - Some Quantum Information Theory
- 4 Proof
- 5 Other Groups
- 6 Free Products of Matrix Algebras
 - Faces of finite-dimensional traces
- 7 Property (T)

Traces on C^* -algebras

Let A be a unital C^* -algebra.

A tracial state on A is a positive linear functional $\tau : A \rightarrow \mathbb{C}$ such that:

- 1 $\tau(1_A) = 1$
- 2 $\tau(ab) = \tau(ba)$

The set of traces on A is denoted by $\mathbb{T}(A)$. It is equipped with the weak- $*$ topology, which is the point-wise convergence topology. With this topology, $\mathbb{T}(A)$ is a compact and convex set. The space of traces was identified as an important invariant of C^* -algebras. For example, it plays a crucial role in Elliot's classification program.

Why are traces important?

As we know, a great way to study an algebra A is by studying its different representations.

Given such a representation, $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$, we can look at $\pi(A)''$, the von Neumann algebra of π , sometimes denoted by M_π .

When M_π carries a faithful normal trace, this gives a tracial state on A .

In fact, such tracial representations stand in one-to-one correspondence (up to quasi-equivalence) with traces on A (GNS).

The Simplex Structure

We said that the space of traces $\mathbb{T}(A)$ is compact and convex. Thus, by the Krein-Milman theorem, it is the convex hull of its extreme points, and focusing on these extreme points is in a sense enough to understand the entire space. In our case, extreme points are traces which can not be written as convex combinations of other traces.

$\mathbb{T}(A)$ is moreover a Choquet simplex.

Definition

A compact convex set K inside a topological vector space is a Choquet Simplex if every point in K is the barycenter of a *unique* probability measure on $\partial_e(K)$.

Types of Simplices

The theory of Choquet simplices is quite developed as a generalization of finite-dimensional simplices. While these simplices come in various structures, there are two "extreme" cases for their possible behaviour:

Let K be a Choquet simplex.

- 1 If $\overline{\partial_e K} = \partial_e K$ we say that K is a *Bauer* simplex.
- 2 If $\overline{\partial_e K} = K$ we say that K is a *Poulsen* simplex.

And while there are a lot of very different Bauer simplices (for example, take $\text{Prob}(X)$ for X a compact space), there is only one Poulsen simplex.

Theorem (Olsen, Lindenstrauss, Sternfeld 78')

Every two Poulsen simplices are affinely homeomorphic. Moreover, $\partial_e K \cong \ell^2(\mathbb{Z})$.

Our first result is therefore the following theorem:

Theorem (Orovitz, S', Vigdorovich 23')

Let $A = C^(F_n)$ be the maximal C^* -algebra of the free group on $2 \leq n \leq \infty$ elements. Then $T(A)$ is a Poulsen simplex.*

In particular, free groups of different ranks cannot be distinguished by their trace simplex.

Moreover, the space of traces is huge and the space of indecomposable traces is homeomorphic to an infinite dimensional Banach space.

Our perspective as group theorists

One can do the same thing without saying the word C^* -algebra. (Not that they should, of course).

People like Frobenius have studied traces on finite simple groups more than a 120 years ago.

Let $\rho : G \rightarrow GL_n(\mathbb{C})$ be a finite dimensional representation of a finite group G .

Group theorists say that $\chi_\rho(g) := \text{Tr}(\rho(g))$ is a *character* on G . Sometimes also interested in the normalized version, $\overline{\chi}_\rho := \frac{\chi_\rho}{n}$

This became a powerful tool in studying finite groups. It allows one to do harmonic analysis on finite groups, it was used in the classification of finite simple groups and helped solve many conjectures (see the Ore conjecture).

Now people who study infinite discrete groups got quite jealous of these techniques, and tried to come up with a definition. So consider now an infinite discrete group and say that a function $\varphi : G \rightarrow \mathbb{C}$ is a trace if:

- 1 φ is positive definite, i.e. ,the matrix $(\varphi(g_i g_j^{-1}))_{i,j}$ is positive for any choice of n elements g_1, \dots, g_n .
- 2 $\varphi(1_G) = 1$
- 3 $\varphi(ghg^{-1}) = \varphi(h)$ for every $g, h \in G$.

Now, these traces are in one-to-one correspondence with tracial states on the maximal C^* -algebra of G . But with this "intrinsic" definition, we can give examples which show why the group theory community got interested in them in the last few years. We denote by $T(G)$ the space of such functions on G , and by $\text{Ch}(G)$ the extremal ones.

Examples of Traces on Groups

- 1 $\text{Hom}(G, S^1) \subset \text{Ch}(G)$ (equality if G is abelian, and this is the Pontryagin dual of G).
- 2 Given $N \trianglelefteq G$ a normal subgroup, we have $1_N \in \text{T}(G)$.
- 3 Given a finite-dimensional representation $\pi : G \rightarrow GL_n(\mathbb{C})$, construct $\overline{\chi_\pi} \in \text{T}(G)$. If π was irreducible, then $\overline{\chi_\pi}$ is extremal.
- 4 Given a probability measure preserving action on (X, μ) one can construct a trace $\varphi(g) := \mu(\{x \in X \mid g.x = x\})$
- 5 $\pi : G \rightarrow (M, \tau)$ a uni. rep. into a tracial von Neumann algebra. If M is a factor, the resulting trace is extremal.

Character Rigidity

One of the most remarkable theorems in this area is so called "Character Rigidity".

Theorem (Bekka, 07', Peterson, Thom, Boutonnet, Houdayer, Bader...)

Let $G = SL_3(\mathbb{Z})$. Then the only extremal traces on G , respectively, extremal tracial states on $C^(G)$, are δ_e and ones coming from finite-dimensional irreducible representations.*

This implies two celebrated results. The first is the Margulis Normal Subgroup Theorem, which says that every normal subgroup of such G is either finite or of finite-index. The second is the Stuck-Zimmer Theorem, which says that such groups can not have very interesting actions.

Thus, we view our result on the free group as a strong negation of character rigidity in so called rank 1 (the free groups is a lattice in $SL_2(\mathbb{R})$, the main example of a rank 1 Lie group).

Free Products of Matrix Algebras

The second result in our paper deals with the following C^* -algebra, which does not come from a group, $M_n(\mathbb{C}) * M_n(\mathbb{C})$, the universal unital free product.

We answer positively a question of Musat and Rørdam for $n \geq 4$:

Theorem (Orovitz, S', Vigdorovich 23')

*Let $A = \mathbf{M}_n * \mathbf{M}_n$ for $n \geq 4$, then $T(A)$ is a Poulsen simplex.*

We remark that this is probably true for $n = 3$ and the difficulty is a technical one in our proof.

Time for some Quantum Information Theory

A map $T : \mathbf{M}_n(\mathbb{C}) \rightarrow \mathbf{M}_n(\mathbb{C})$ is a quantum channel if it is completely positive, unital and trace-preserving. These channels model communication of both classical and quantum information. We denote the collection of such channels by $\mathcal{FM}(n)$.

We say that T is *factorizable* if there exists a tracial von Neumann algebra (N, τ) and representations $\pi_1, \pi_2 : \mathbf{M}_n \rightarrow N$ such that $T = \pi_1^* \circ \pi_2$. In this case, N is called the Ancilla of the factorization.

In 2011 Haagerup and Musat showed that there exists non-factorizable channels for every $n \geq 3$, and deduced that the Asymptotic Quantum Birkhoff Conjecture is false.

Another interesting question is - Is there a channel which can be factorized only through an infinite dimensional ancilla? This was answered positively by Musat and Rørdam in 2019 for $n \geq 11$.

Connection to Connes' Embedding Problem - equivalent to $\mathcal{FM}(n)_{fin}$ being dense for every $n \geq 3$ (Haagerup and Musat 2015).

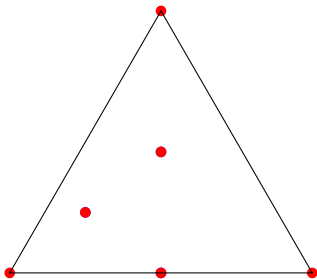
What is the connection to our result?

A choice of two tracial representations of \mathbf{M}_n is but a choice of a single representation of $\mathbf{M}_n * \mathbf{M}_n$. These representations are parametrized by the space of traces, and hence allowed Musat and Rørdam to construct a map $\Phi : T(\mathbf{M}_n * \mathbf{M}_n) \rightarrow \mathcal{FM}(n)$ which is affine, surjective and continuous. Moreover, $\Phi(T_{\text{fin}}(\mathbf{M}_n * \mathbf{M}_n)) = \mathcal{FM}_{\text{fin}}(n)$.

Using this map they draw connections to the Conne's Embedding Problem and posed the question of whether the trace space is a Poulsen simplex.

Proof for Free Groups

We want to show that every trace can be approximated by extremal ones. First, it is enough to approximate finite convex combinations of extremal ones. In fact, it is enough to approximate traces of the form $\varphi = \frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$ where φ_1, φ_2 are extremal.



Proof - continued

So, we have two *factor* representations π_1, π_2 which are associated with φ_1, φ_2 . Our first attempt in approximating φ is therefore:

$$\pi : F_2 \rightarrow N_1 \otimes N_2 \otimes M_2(\mathbb{C})$$

given by

$$x \mapsto \begin{bmatrix} \pi_1(x) \otimes 1_{N_2} & 0 \\ 0 & 1_{N_1} \otimes \pi_2(x) \end{bmatrix}$$

This was too good to be true. We approximated φ on the nose. However, the von Neumann algebra generated by this representation is not a factor, it has a two-dimensional center.

So we want to perturb this representations slightly such that it will generate the entire algebra $M = N_1 \otimes N_2 \otimes M_2(\mathbb{C})$ (which is a factor). To do that, we are going to introduce our tool:

Our Hammer

Denote by $W^*(S)$ the von Neumann algebra generated by S and 1 . We prove the following lemma:

Lemma

Let u, v be commuting unitaries in $\mathcal{B}(\mathcal{H})$. Then for every $\varepsilon > 0$ there exists a unitary u' such that $W^(u') = W^*(u, v)$ and $\|u - u'\|_{op} < \varepsilon$.*

This is a sort of a packing lemma in the sense that we want to add v to u . u is going to be a unitary we are given, but we're not completely happy about. We want it to do something more. We thus perturb it a bit, and get u' which is very close, but "upgraded" in the sense that it can also generate v . The proof uses borel functional calculus and some analysis of functions on a torus.

As for the second generator, we define:

$$v \mapsto \left(\begin{array}{ccc|ccc} V_1 \otimes 1 & & 0 & & & \\ & \ddots & & & & \\ 0 & & V_1 \otimes 1 & & & \\ \hline & & & 0 & & \\ & & & 1 \otimes V_2 & & 0 \\ & & 0 & & \ddots & \\ & & & & & 1 \otimes V_2 \end{array} \right) \begin{array}{l} \updownarrow \frac{1}{2}n + 1 \\ \\ \updownarrow \frac{1}{2}n - 1 \end{array}$$

What did we gain?

First of all, by enlarging the matrices, we have gained more operators which commute with the image of the generators. For example, every block matrix of size $\frac{1}{2}n \times \frac{1}{2}n$ commutes with the image of u since it is diagonal there.

Second, by de-symmetrizing the representation, we can now "insert" two elements. One that commutes with u , and one that commutes with v , but which do not commute with each other.

Of course, we did not change the trace by much, since the images do not agree with the "obvious" embedding only on 2 entries, so for large n this change is very small.

Preparing the nails

Consider the following k by k matrix:

$$C_k = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}$$

And define the following matrices:

$$X = \left(\begin{array}{c|c} I_{\frac{1}{2}n} & 0 \\ \hline 0 & C_{\frac{1}{2}n} \end{array} \right), \quad Y = \left(\begin{array}{c|c} C_{\frac{1}{2}n+1} & 0 \\ \hline 0 & I_{\frac{1}{2}n-1} \end{array} \right),$$

Observe that X, Y and E_{11} generate $M_2(\mathbb{C})$.

Using the hammer

Since the block structure fits, and U, V are diagonal in each block, we see that X commutes with U , and Y commutes with V .

This means that using the lemma, we can find U' and V' which are very close to U and V , but which now generate U, V, X, Y . We need just one more operator to generate $M_2(\mathbb{C})$, which is E_{11} , but this commutes with U' , so we use the lemma a third time.

We now see that the case of free groups with more generators is in a sense "easier" since we have more operators to perturb. For example, we can "add" E_{11} to the third generator without any worry.

In fact, we also see that the same result is true for all algebras of the form $C^*(F_2) * A$, where A is some separable unital C^* -algebra.

Possible upgrades - our lemma can be stated more generally for certain abelian von Neumann algebras. For example, take a perfect compact space X , we can look at $\pi : C(X) \rightarrow \mathcal{B}(\mathcal{H})$ and v an element which commutes with the image of π , and perturb the image of the representation to include v as well. This allows us to prove the same result for $G = A * B$ where A, B are infinite abelian groups. [Can we do more?](#)

Other Groups

What about virtually-free groups?

First, we realized that the following groups are not Poulsen:

Proposition

*Let A, B be finite groups, then $A * B$ is not Poulsen.*

Proof.

For simplicity, take $A = \langle a \rangle = \mathbb{Z}/n\mathbb{Z}$ and $B = \langle b \rangle = \mathbb{Z}/m\mathbb{Z}$, and denote $G = A * B$. We claim that the trivial trace, 1_G , is an isolated point in the set of extremal traces. This will show that the space of traces is not a Poulsen simplex, since its extreme points are homeomorphic to $\ell^2(\mathbb{Z})$.

Suppose $\pi : G \rightarrow (M, \tau)$ is a representation which is very close to 1 in trace. But $\pi(a), \pi(b)$ are finite order elements, and are thus diagonalizable with eigenvalues which are roots of unity of order n and m .

This means that most of these eigenvalues must be 1, and thus there are large subspace of fixed-vectors. Their intersection must be non-empty, and thus M is not a factor. □

As an application, the groups $SL_2(\mathbb{Z})$ or $PSL_2(\mathbb{Z})$ are *not* Poulsen.

But as we've seen before, the group $F_2 * \mathbb{Z}/n\mathbb{Z}$ is Poulsen, and it is also virtually-free, so there is no nice dichotomy.

Very interesting question: what about $G = \pi_1(S_g)$, a surface group? This question has connections to spectral properties of the surface, and there exists interesting action of $MCG(S_g)$ on $T(G)$.

Another interesting question: we have an action of $\text{Aut}(F_n)$ on $T(F_n)$. For $n \geq 4$ this is a property (T) group with an action on the Poulsen simplex. Understanding this action is interesting and related to the so-called "Wiegold conjecture" and the product-replacement algorithm (how fast can you shuffle a deck of cards).

Free Products of Matrix Algebras

The general strategy is the same, we have two factor representations π_1, π_2 of $A = \mathbf{M}_n * \mathbf{M}_n$ corresponding to φ_1, φ_2 and we want to approximate $\varphi = \frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$. Fact: a factor rep. of M_n into M can be seen as a representation into $M_n(N)$, where N is some factor. Thus our initial attempt:

$$\pi : \mathbf{M}_n * \mathbf{M}_n \rightarrow M_1 \otimes M_2 \otimes M_2(\mathbb{C})$$

Can be actually a rep. $\pi : \mathbf{M}_n * \mathbf{M}_n \rightarrow M_n(\mathbb{C}) \otimes N_1 \otimes N_2 \otimes M_2(\mathbb{C})$ where we have this common M_n factor.

Changing this representation is much harder than for the free group which had no relations. So we use the following trick (inspired by Musat and Rørdam): If N is a factor, and u_2, \dots, u_n are unitaries in N , we can define a rep. $\mathbf{M}_n * \mathbf{M}_n$ into $M_n(N)$ by $\pi(e_{ij}) = E_{ij} \otimes 1_N$, and $\pi(f_{1j}) = E_{1j} \otimes u_{1j}$.

Continuing proof for Free product of Matrices

Notice that this representation is surjective when u_2, \dots, u_n generate N . These types of representations are special, they have the property that $\pi(e_{ii}) = \pi(f_{ii})$, so this is not a very common situation, we have to arrange it somehow.

To do that, we enlarge N_1 and N_2 very slightly using small projections, p_i .

$$1+p \left\{ \left(\begin{array}{c|c} N_i & N_i \\ \hline N_i & N_i \end{array} \right) \right.$$

On this new piece, we will define $\pi(e_{ij}) = \pi(f_{ij}) = 1_p \otimes E_{ij}$. In particular, it's easier to manipulate the representation on this small part.

Faces of finite-dimensional traces

In order to slightly enlarge our representation of $\mathbf{M}_n * \mathbf{M}_n$, we need a very small projection. But what if our original representation is finite-dimensional?

We therefore need to approximate finite-dimensional representations using other *factor* representations of arbitrarily high dimension. We do this using algebraic geometry, and get a nice corollary as a by-product.

Suppose A is a "nice" C^* -algebra generated by x_1, \dots, x_n with relations r_1, \dots, r_p . Then $V_k = \text{Hom}(A, M_k)$, the set of unital $*$ -representations, can be thought of as a (real) algebraic variety. i.e., the algebraic subset of $x \in M_k^n$ such that $r_i(x) = 0$.

Continuing the finite-dimensional proof

We claim that the subset of *surjective* representations is open in this variety. Denote it by V_k^{sur} .

Take $\pi \in V_k^{sur}$. This means, in particular, that there exists k^2 polynomials p_i such that $p_i(\pi(x_j))$ linearly span M_k , or are linearly independent. But this is an open condition, and so the set of representations such that $p_i(\pi(x_j))$ are linearly independent is open in the Zariski topology.

Finally, if V_k is irreducible and V_k^{sur} is non-empty, this also implies that V_k^{sur} is dense in the real topology.

This is indeed the case for algebras like $C^*(F_n)$ and $\mathbf{M}_n * \mathbf{M}_n$. This is because for $C^*(F_n)$ the representation variety is just $U(k)^n$, which is an algebraic connected group and hence irreducible. V_k^{sur} is non-empty since M_k can be generated by 2 unitaries. The same is true for $\mathbf{M}_n * \mathbf{M}_n$ for similar reasons.

A corollary on finite dimensional traces

As a corollary, we get:

Corollary

Take A to be $C^(F_n)$ or $\mathbf{M}_n * \mathbf{M}_n$. Then for every $m \in \mathbb{N}$, the set of extremal traces of dimension at least m is dense in $T_{fin}(A)$. In particular, the face $\overline{T_{fin}(A)}$ is a Poulsen simplex.*

Groups with property (T)

As for other groups, we prove the following:

Theorem (Levit, S', Vigdorovich 23')

Let G be a discrete group with property (T). Then $T(G)$ is a Bauer simplex.

(This follows also from the work of Kennedy and Shamovich on nc-simplices). Our motivation was the following corollary:

Theorem

Let Γ be a higher rank lattice with property (T). Take for example $\Gamma = \mathrm{SL}_3(\mathbb{Z})$. Then for every sequence of distinct extremal traces φ_n we have

$$\varphi_n(g) \rightarrow 0$$

for every $e \neq g \in \Gamma$.

Sketch of Proof

By character rigidity, all extremal traces are finite dimensional except one, δ_e . By the first theorem, this space is closed, hence compact, so every sequence has a converging subsequence.

But it is not hard to show that an infinite sequence of finite dimensional representations can not converge to a fixed finite-dimensional representation, so every such limit must be δ_e .

Application:

Fix your favourite matrix $g \in \mathrm{SL}_3(\mathbb{Z})$ (unless your favourite matrix is the identity). Now pick classical characters φ_p on the finite simple groups $\mathrm{SL}_3(\mathbb{F}_p)$.

Then $\varphi_p(\bar{g}) \rightarrow 0$ as $p \rightarrow \infty$.

Thanks!