Finite free probability and hypergeometric polynomials

Daniel Perales Anaya
Texas A & M University

Probabilistic Operator Algebra Seminar, UC Berkeley

November 20, 2023

Joint work with Andrei Martinez-Finkelshtein and Rafael Morales, arXiv:2309.10970

1. Finite Free Probability
2. Hypergeometric polynomials.
3. Real roots of Hypergeometric polynomials.
4. Ongoing and future work.

Finite Free Probability

Polynomials

$\mathbb{P}_n(S) :=$ monic polynomials of degree n with all its roots contained in the set $S \subset \mathbb{C}$.

Given $p \in \mathbb{P}_n(\mathbb{C})$ we denote:

Roots: $\lambda_1(p), \dots, \lambda_n(p)$.

Normalized k-th elementary symmetric sums of the roots:

$$\tilde{e}_k(p) := \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1}(p) \cdots \lambda_{i_k}(p).$$

$$p(x) = \prod_{k=1}^{n} (x - \lambda_k(p)) = \sum_{k=0}^{n} x^{n-k} (-1)^k \binom{n}{k} \tilde{e}_k(p).$$

The **empirical root distribution (zero counting measure)** of p is

$$\mu_p := \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(p)}. \qquad \text{with moments} \qquad m_k(p) := m_k(\mu_p) = \frac{1}{n} \sum_{i=1}^{n} (\lambda_i(p))^k.$$

Finite free additive convolution

Definition

Given $p, q \in \mathbb{P}_n$, their **finite free additive convolution** is the polynomial $p \boxplus_n q \in \mathbb{P}_n$ with coefficients given by

$$\tilde{e}_k(p \boxplus_n q) = \sum_{i+j=k} \binom{k}{i} \tilde{e}_i(p) \cdot \tilde{e}_j(q), \qquad \text{for } k = 1, 2, \dots, n.$$

Every p can be expressed as some differential operator $P(\frac{\partial}{\partial x})$ applied to x^n:

$$p(x) = P(\tfrac{\partial}{\partial x})x^n.$$

Alternative definition of convolution:

$$p \boxplus_n q(x) = P(\tfrac{\partial}{\partial x})Q(\tfrac{\partial}{\partial x})x^n.$$

$\boxplus_n$ is a bilinear. $(\alpha p + q) \boxplus_n r = \alpha(p \boxplus_n r) + (q \boxplus_n r)$

The identity element is given by x^n.

Finite free multiplicative convolution

Definition

Let $p, q \in \mathbb{P}_n$ their **finite free multiplicative convolution** is the polynomial $p \boxtimes_n q \in \mathbb{P}_n$ with coefficients given by

$$\tilde{e}_k(p \boxtimes_n q) = \tilde{e}_k(p)\tilde{e}_k(q), \qquad \text{for } k = 1, 2, \dots, n.$$

$\boxtimes_n$ is a bilinear.

The identity element is given by $(x-1)^n$.

(Marcus, Spielman, Srivastava '15) In terms of randomly rotated matrices. Let A and B be $n \times n$ selfadjoint matrices with $\det(xI - A) = p(x)$ and $\det(xI - B) = q(x)$. Then

$$[p \boxplus_n q](x) = \mathbb{E}_Q[\det(xI - (A + QBQ^*))] \qquad \text{and}$$
$$[p \boxtimes_n q](x) = \mathbb{E}_Q[\det(xI - AQBQ^*)]$$

where $Q \sim$ Haar measure over orthogonal matrices.
Or $Q \sim$ Haar measure over unitary matrices.
Or $Q \sim$ uniformly distributed over signed permutation matrices.

These operations behave well with respect to real roots

$p, q \in \mathbb{P}_n(\mathbb{R}) \quad \Rightarrow \quad p \boxplus_n q \in \mathbb{P}_n(\mathbb{R}).$ (Walsh '22)

$p, q \in \mathbb{P}_n(\mathbb{R}_{>0}) \quad \Rightarrow \quad p \boxplus_n q \in \mathbb{P}_n(\mathbb{R}_{>0}).$

$p \in \mathbb{P}_n(\mathbb{R}), \quad q \in \mathbb{P}_n(\mathbb{R}_{>0}) \quad \Rightarrow \quad p \boxtimes_n q \in \mathbb{P}_n(\mathbb{R}).$ (Szegö '22)

$p, q \in \mathbb{P}_n(\mathbb{R}_{>0}) \quad \Rightarrow \quad p \boxtimes_n q \in \mathbb{P}_n(\mathbb{R}_{>0}).$

These operations preserve interlacing:
Given $p, \widetilde{p} \in \mathbb{P}_n(\mathbb{R})$, we say that p **interlaces** $\widetilde{p}$ (denoted $p \preccurlyeq \widetilde{p}$) if

$$\lambda_1(p) \leq \lambda_1(\widetilde{p}) \leq \lambda_2(p) \leq \lambda_2(\widetilde{p}) \leq \cdots \leq \lambda_n(p) \leq \lambda_n(\widetilde{p})$$

If $p, \widetilde{p}, q \in \mathbb{P}_n(\mathbb{R})$, then

$$p \preccurlyeq \widetilde{p} \quad \Rightarrow \quad p \boxplus_n q \ \preccurlyeq \ \widetilde{p} \boxplus_n q.$$

If $p, \widetilde{p} \in \mathbb{P}_n(\mathbb{R})$, $q, \widetilde{q} \in \mathbb{P}_n(\mathbb{R}_{>0})$ then

$$p \preccurlyeq \widetilde{p} \quad \Rightarrow \quad p \boxtimes_n q \ \preccurlyeq \ \widetilde{p} \boxtimes_n q.$$

$$q \preccurlyeq \widetilde{q} \quad \Rightarrow \quad p \boxtimes_n q \ \preccurlyeq \ p \boxtimes_n \widetilde{q}.$$

If $p \in \mathbb{P}_n$, the order n finite free cumulants of p, denoted $\kappa_1^{(n)}(p), \kappa_2^{(n)}(p), \ldots, \kappa_d^{(n)}(p)$, are determined by the coefficient-cumulant formula

$$\tilde{e}_k(p) = \frac{1}{n^k} \sum_{\pi \in P(k)} n^{|\pi|} \text{Möb}(0_k, \pi) \kappa_\pi^{(n)}(p), \qquad \text{for } k = 1, 2, \ldots, n.$$

Where Möb is the Möbius function in the lattice of set partitions.

Note: using a moment-coefficient formula (Newton identities) we can compute a moment-cumulant formula.

(Arizmendi, P '16) For any $p, q \in \mathbb{P}_n$ and $r = 1, \ldots, n$ it holds that

$$\kappa_r^{(n)}(p \boxplus_n q) = \kappa_r^{(n)}(p) + \kappa_r^{(n)}(q).$$

Why finite free probability?

Let $\mathfrak{p} = (p_n)_{n=1}^{\infty}$ and $\mathfrak{q} = (q_n)_{n=1}^{\infty}$ be sequences of polynomials with $p_n, q_n \in \mathbb{P}_n(\mathbb{R})$, such that their root distribution converge weakly to compactly supported measures $\nu(\mathfrak{p}), \nu(\mathfrak{q}) \in \mathcal{M}_c(\mathbb{R})$. Namely we have the convergence in moments:

$$\mu_{p_n} \longrightarrow \nu(\mathfrak{p}) \qquad \text{and} \qquad \mu_{q_n} \longrightarrow \nu(\mathfrak{q}).$$

Theorem (Arizmendi, P '16)

Then $\lim \kappa_r^{(n)}(p_n) = \kappa_r(\nu(\mathfrak{p}))$.

Theorem (Marcus '16, Arizmendi, P '16)

Then $\mu_{p_n \boxplus_n q_n} \longrightarrow \nu(\mathfrak{p}) \boxplus \nu(\mathfrak{q})$.

Theorem (Arizmendi,Garza-Vargas, P '21)

Then $\mu_{p_n \boxtimes_n q_n} \longrightarrow \nu(\mathfrak{p}) \boxtimes \nu(\mathfrak{q})$.

Hypergeometric polynomials

Generalized hypergeometric series

Let $a_0 \in \mathbb{R}$, $\boldsymbol{a} = (a_1, \ldots, a_i) \in \mathbb{R}^i$ and $\boldsymbol{b} = (b_1, \ldots, b_j) \in \mathbb{R}^j$ be vectors of parameters. The **generalized hypergeometric series** with the given parameters is

$$ {}_{i+1}F_j\left(\begin{matrix} a_0, \boldsymbol{a} \\ \boldsymbol{b} \end{matrix} ; x \right) = \sum_{k=0}^{\infty} \frac{(a_0)^{\overline{k}} (\boldsymbol{a})^{\overline{k}}}{(\boldsymbol{b})^{\overline{k}}} \frac{x^k}{k!}. $$

where $(\boldsymbol{a})^{\overline{k}} := (a_1)^{\overline{k}} (a_2)^{\overline{k}} \ldots (a_s)^{\overline{k}}$, and $(a)^{\overline{k}} := a(a+1)\ldots(a+k-1)$ denotes the rising factorial.

Examples:

$$ {}_0F_0\left(\begin{matrix} - \\ - \end{matrix} ; x \right) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x. $$

$$ {}_1F_0\left(\begin{matrix} -\alpha \\ - \end{matrix} ; x \right) = \sum_{k=0}^{\infty} \frac{(-\alpha)(-\alpha+1)\ldots(-\alpha+k-1)x^k}{k!} = (1-x)^{\alpha} $$

$$ {}_{i+1}F_j\left(\begin{matrix} -n, \boldsymbol{a} \\ \boldsymbol{b} \end{matrix} ; x \right) = \sum_{k=0}^{n} \frac{(-n)^{\overline{k}} (\boldsymbol{a})^{\overline{k}}}{(\boldsymbol{b})^{\overline{k}}} \frac{x^k}{k!} $$

$a_1, \ldots, a_i \in \mathbb{C} \setminus \{-1, -2, \ldots, -n+1\}, \quad b_1, \ldots b_j \in \mathbb{C} \setminus \{-1, -2, \ldots, -n+1, -n\}.$

A convenient parametrizarion in finite free probability:

Definition (Hypergeometric polynomials)

Given $i, j, n \in \mathbb{N}$, $\boldsymbol{a} = (a_1, \dots, a_i) \in \mathbb{R}^i$ and $\boldsymbol{b} = (b_1, \dots, b_j) \in \mathbb{R}^j$, we denote by $\mathcal{H}_n \left[\begin{smallmatrix} \boldsymbol{b} \\ \boldsymbol{a} \end{smallmatrix} \right] \in \mathbb{P}_n$ the polynomial with coefficients

$$\tilde{e}_k \left(\mathcal{H}_n \left[\begin{smallmatrix} \boldsymbol{b} \\ \boldsymbol{a} \end{smallmatrix} \right] \right) := \frac{(\boldsymbol{b}n)^{\underline{k}}}{(\boldsymbol{a}n)^{\underline{k}}}, \qquad \text{for } k = 1, \dots, n.$$

where $(n)^{\underline{k}} := \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1)$ is the falling factorial.

To avoid indeterminacy, we assume $a_s \notin \left\{ \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n} \right\}$, $s = 1, \dots, i$.

Direct connection with hypergeometric series:

$$\mathcal{H}_n \left[\begin{smallmatrix} \boldsymbol{b} \\ \boldsymbol{a} \end{smallmatrix} \right](x) = \frac{(-1)^n (\boldsymbol{b}n)^{\underline{n}}}{(\boldsymbol{a}n)^{\underline{n}}} \; {}_{i+1}F_j\left(\begin{matrix} -n, \boldsymbol{a}n - n + 1 \\ \boldsymbol{b}n - n + 1 \end{matrix} ; x \right).$$

Theorem (Martinez-Filkenshtein, Morales, P '23+)

Consider tuples $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ *of sizes* $i_1, i_2, i_3, j_1, j_2, j_3$*. Then,*

1. *Reciprocal polynomials:*

$$x^n\,\mathcal{H}_n\left[\begin{smallmatrix}\boldsymbol{b}_1\\ \boldsymbol{a}_1\end{smallmatrix}\right](1/x) = c\,\mathcal{H}_n\left[\begin{smallmatrix}-\boldsymbol{a}_1+1-1/n\\ -\boldsymbol{b}_1+1-1/n\end{smallmatrix}\right]\left((-1)^{i_1+j_1}x\right).$$

2. *The multiplicative convolution is given by:*

$$\mathcal{H}_n\left[\begin{smallmatrix}\boldsymbol{b}_1\\ \boldsymbol{a}_1\end{smallmatrix}\right]\boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix}\boldsymbol{b}_2\\ \boldsymbol{a}_2\end{smallmatrix}\right] = \mathcal{H}_n\left[\begin{smallmatrix}\boldsymbol{b}_1,\ \boldsymbol{b}_2\\ \boldsymbol{a}_1,\ \boldsymbol{a}_2\end{smallmatrix}\right].$$

3. *Assume that the following factorization holds,*

$${}_{j_1}F_{i_1}\left(\begin{matrix}-n\boldsymbol{b}_1\\ -n\boldsymbol{a}_1\end{matrix};x\right)\,{}_{j_2}F_{i_2}\left(\begin{matrix}-n\boldsymbol{b}_2\\ -n\boldsymbol{a}_2\end{matrix};x\right) = {}_{j_3}F_{i_3}\left(\begin{matrix}-n\boldsymbol{b}_3\\ -n\boldsymbol{a}_3\end{matrix};x\right),$$

   and consider the signs $s_r = (-1)^{i_r+j_r+1}$ *for* $r = 1, 2, 3$*. Then the additive convolution is given by*

$$\mathcal{H}_n\left[\begin{smallmatrix}\boldsymbol{b}_1\\ \boldsymbol{a}_1\end{smallmatrix}\right](s_1x)\boxplus_n \mathcal{H}_n\left[\begin{smallmatrix}\boldsymbol{b}_2\\ \boldsymbol{a}_2\end{smallmatrix}\right](s_2x) = \mathcal{H}_n\left[\begin{smallmatrix}\boldsymbol{b}_3\\ \boldsymbol{a}_3\end{smallmatrix}\right](s_3x).$$

Dirac polynomials and LLN

The simplest families of real rooted polynomials are:

Identity for the additive convolution

$$\mathcal{H}_n\left[\begin{smallmatrix} 0 \\ \boldsymbol{a} \end{smallmatrix}\right](x) = x^n.$$

Identity for the multiplicative convolution

$$\mathcal{H}_n\left[\begin{smallmatrix} \boldsymbol{a} \\ \boldsymbol{a} \end{smallmatrix}\right](x) = \mathcal{H}_n\left[\begin{matrix} - \\ - \end{matrix}\right](x) = (x-1)^n.$$

A law of large numbers is valid for the finite free additive convolution [Marcus '21] and the limiting polynomials are precisely

$$p_n(x) = c^n \mathcal{H}_n\left[\begin{matrix} - \\ - \end{matrix}\right](x/c) = (x-c)^n.$$

Notice that $\mu_{p_n} = \delta_c$. So, trivially when $n \to \infty$ the limiting measure is δ_c.

Laguerre polynomials and Poisson limit

Laguerre polynomials $\mathcal{H}_n\left[\begin{smallmatrix} b \\ - \end{smallmatrix}\right]$,

- $\mathcal{H}_n\left[\begin{smallmatrix} b \\ - \end{smallmatrix}\right] \in \mathbb{P}(\mathbb{R}_{>0})$ when $b > 1 - \frac{1}{n}$. And $\mathcal{H}_n\left[\begin{smallmatrix} b \\ - \end{smallmatrix}\right] \preccurlyeq \mathcal{H}_n\left[\begin{smallmatrix} b+\varepsilon \\ - \end{smallmatrix}\right]$ when $0 < \varepsilon < \frac{2}{n}$
- $\mathcal{H}_n\left[\begin{smallmatrix} b \\ - \end{smallmatrix}\right] \in \mathbb{P}(\mathbb{R}_{\geq 0})$ when $b \in \{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\}$, with a multiplicity of $(1-b)n$ at 0.
- $\mathcal{H}_n\left[\begin{smallmatrix} b \\ - \end{smallmatrix}\right] \in \mathbb{P}(\mathbb{R})$ when $b \in (\frac{n-2}{n}, \frac{n-1}{n})$.

$b \in (1-\frac{1}{n}, \infty)$, $b \to b+\varepsilon$		0	λ_1 $\to$	λ_2 $\to$	λ_3 $\to$	$\dots$	λ_n $\to$
$b = 1 - \frac{1}{n}$		$\lambda_1 = 0$	λ_2		λ_3	$\dots$	λ_n
$b \in (1-\frac{2}{n}, 1-\frac{1}{n})$	λ_1	0	λ_2		λ_3	$\dots$	λ_n
$b = 1 - \frac{2}{n}$		$\lambda_1 = 0 = \lambda_2$	λ_3	λ_4		$\dots$	λ_n
$b = 1 - \frac{k}{n}$		$\lambda_1 = 0 = \lambda_k$	λ_{k+1}	λ_{k+2}		$\dots$	λ_n

Asymptotic behaviour: Let

$$\widehat{L}_n^{(b)} := \mathrm{Dil}_{1/n}\mathcal{H}_n\left[\begin{smallmatrix} b \\ - \end{smallmatrix}\right] = \frac{1}{n^n}\mathcal{H}_n\left[\begin{smallmatrix} b \\ - \end{smallmatrix}\right](nx),$$

For $b > 1$, the limiting measure (when $n \to \infty$) is the Marchenko-Pastur law μ_{MP_b}:

$$d\mu_{\mathrm{MP}_b} = \frac{1}{2\pi}\frac{\sqrt{(r_+ - x)(x - r_-)}}{x}dx, \qquad \text{where} \qquad r_\pm = b + 1 \pm 2\sqrt{b}.$$

For $b \in (0,1)$, in the limit we get the Marchenko-Pastur distribution with an additional atom (mass point) at $x = 0$.

Cumulants: $\kappa_r^{(n)}\left(\mathcal{H}_n\left[\begin{smallmatrix} b \\ - \end{smallmatrix}\right]\right) = b$ for all r.

So $\mathcal{H}_n\left[\begin{smallmatrix} a \\ - \end{smallmatrix}\right] \boxplus_n \mathcal{H}_n\left[\begin{smallmatrix} b \\ - \end{smallmatrix}\right] = \mathcal{H}_n\left[\begin{smallmatrix} a+b \\ - \end{smallmatrix}\right]$

Poisson limit: $\mathcal{H}_n\left[\begin{smallmatrix} 1/n \\ - \end{smallmatrix}\right] = x^{n-1}(x-1)$, so

$$\left(\mathcal{H}_n\left[\begin{smallmatrix} 1/n \\ - \end{smallmatrix}\right]\right)^{\boxplus_n k} = \mathcal{H}_n\left[\begin{smallmatrix} k/n \\ - \end{smallmatrix}\right]$$

Bessel polynomials (reciprocal Laguerre polynomials)

Bessel polynomials are the reciprocal of Laguerre:

$$\mathcal{H}_n\left[\begin{smallmatrix} - \\ a \end{smallmatrix}\right] = \qquad c\, x^n\, \mathcal{H}_n\left[\begin{smallmatrix} -a+1-1/n \\ - \end{smallmatrix}\right](-1/x)\,.$$

Real roots:

- $\mathcal{H}_n\left[\begin{smallmatrix} - \\ a \end{smallmatrix}\right] \in \mathbb{P}(\mathbb{R}_{<0})$ when $a < 0$.
- $\mathcal{H}_n\left[\begin{smallmatrix} - \\ a \end{smallmatrix}\right] \in \mathbb{P}(\mathbb{R})$ when $a \in (0, \frac{1}{n})$.

Asymptotic behaviour: Let $\widehat{B}_n^{(a)}(x) := \mathrm{Dil}_n \mathcal{H}_n\left[\begin{smallmatrix} - \\ a \end{smallmatrix}\right] = n^n \mathcal{H}_n\left[\begin{smallmatrix} - \\ a \end{smallmatrix}\right](x/n)$.

For $a < 0$, the limiting measure μ_{RMP_a} is the reversed of a Marchenko-Pastur law of parameter $1 - a$:

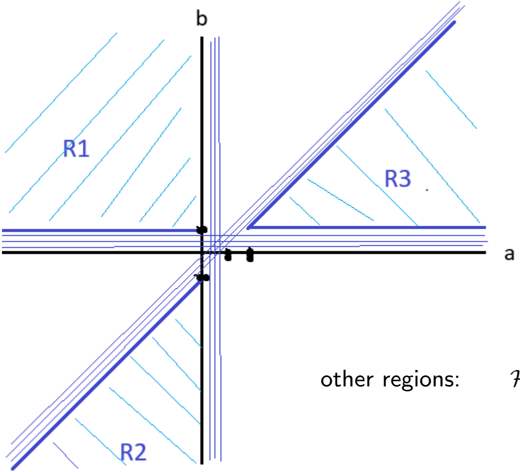
$$d\mu_{\mathrm{RMP}_a} = \frac{-a}{2\pi} \frac{\sqrt{(r_+ - x)(x - r_-)}}{x^2} dx, \qquad \text{where} \qquad r_\pm = \frac{1}{a - 2 \pm 2\sqrt{1-a}}.$$

Curious fact: for $b > 0$

$$\mathcal{H}_n\left[\begin{smallmatrix} -b \\ - \end{smallmatrix}\right] = \left(\mathcal{H}_n\left[\begin{smallmatrix} b \\ - \end{smallmatrix}\right]\right)^{\boxplus_n - 1} = \left(\mathcal{H}_n\left[\begin{smallmatrix} - \\ -b \end{smallmatrix}\right]\right)^{\boxtimes_n - 1}$$

Jacobi polynomials $\mathcal{H}_n\begin{bmatrix} b \\ a \end{bmatrix}$

R1. $\mathcal{H}_n\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] \in \mathbb{P}(\mathbb{R}_{<0})$ when $b > 1$ and $a < 0$.
R2. $\mathcal{H}_n\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] \in \mathbb{P}(\mathbb{R}_{>0})$ when $a < 0$ and $b < a - 1$.
R3. $\mathcal{H}_n\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] \in \mathbb{P}([0,1])$ when $b > 1$ and $a > b + 1$.



other regions: $\mathcal{H}_n\begin{bmatrix} k/n \\ 1 \end{bmatrix}(x) = (x-1)^k x^{n-k}$

The asymptotic zero distribution $\mu_{b,a} := \nu(\mathcal{H}_n\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right])$ depends on the region:

R1. When $b > 1$ and $a < 0$,

$$d\mu_{b,a} = \frac{-a}{4\pi}\frac{x-1}{x}\sqrt{(r_+ - x)(x - r_-)}dx, \qquad r_\pm = -\left(\frac{\sqrt{1-a} \mp \sqrt{b(b-a)}}{\sqrt{(1-a)(b-a)} \pm \sqrt{b}}\right)^2.$$

R2. When $a < 0$ and $b < a - 1$.

$$d\mu_{b,a} = \frac{-ax}{4\pi}\frac{\sqrt{(r_+ - x)(x - r_-)}}{x-1}dx, \qquad \text{with} \qquad r_\pm = \left(\frac{b-1}{\sqrt{(a-1)b} \mp \sqrt{a-b}}\right)^2.$$

R3. When $b > 1$ and $a > b + 1$,

$$d\mu_{b,a} = \frac{a}{4\pi}\frac{\sqrt{(r_+ - x)(x - r_-)}}{x(1-x)}dx, \qquad \text{with} \qquad r_\pm = \left(\frac{\sqrt{a-b} \pm \sqrt{(a-1)b}}{a}\right)^2.$$

The last distribution $\mu_{b,a}$ was studied in the realm of free probability in [Yoshida, '20].
For $c, d > 1$, the free beta distribution is given by $f\beta(c, d) = \mu_{c,c+d}$.
Notice the identity:

$$\mathcal{H}_n\left[\begin{smallmatrix} c \\ c+d \end{smallmatrix}\right] \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} c+d \\ - \end{smallmatrix}\right] = \mathcal{H}_n\left[\begin{smallmatrix} c \\ - \end{smallmatrix}\right],$$

Real roots of hypergeometric polynomials

Application: Construct several hypergeometric polynomials that are real-rooted.

Idea: Use simple hypergeometric polynomial (Laguerre, Bessel, Jacobi) as building blocks, and use finite free convolutions.

As a byproduct we get their asymptotic distribution.

$$p_n := \mathcal{H}_n\left[\begin{smallmatrix} b_1, & b_2, & \dots, & b_j \\ a_1, & a_2, & \dots, & a_i \end{smallmatrix}\right].$$

Theorem

If $b_1, \dots, b_j > 1$ and $a_1, \dots, a_i < 0$, then

$$p_n = \mathcal{H}_n\left[\begin{smallmatrix} - \\ a_1 \end{smallmatrix}\right] \boxtimes_n \cdots \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} - \\ a_i \end{smallmatrix}\right] \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} b_1 \\ - \end{smallmatrix}\right] \boxtimes_n \cdots \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} b_j \\ - \end{smallmatrix}\right] \in \mathbb{P}_n(\pm\mathbb{R}_{>0}).$$

Moreover, the root distribution of $\mathfrak{p} = (p_n)_{n\geq 1}$ converges to

$$\nu(\mathfrak{p}) = \mu_{RMPa_1} \boxtimes \cdots \boxtimes \mu_{RMPa_i} \boxtimes \mu_{MPb_1} \boxtimes \cdots \boxtimes \mu_{MPb_j}.$$

Theorem

If $j \geq i$, $b_1, \dots, b_j > 0$, and $a_1, \dots, a_i \in \mathbb{R}$ such that $a_s \geq b_s + 1$ for $s = 1, \dots, i$, then

$$p_n = \mathcal{H}_n\left[\begin{smallmatrix} b_1 \\ a_1 \end{smallmatrix}\right] \boxtimes_n \cdots \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} b_i \\ a_i \end{smallmatrix}\right] \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} b_{i+1} \\ - \end{smallmatrix}\right] \boxtimes_n \cdots \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} b_j \\ - \end{smallmatrix}\right] \in \mathbb{P}_n(\pm\mathbb{R}_{>0}),$$

$$\nu(\mathfrak{p}) = f\beta(b_1, a_1 - b_1) \boxtimes \cdots \boxtimes f\beta(b_i, a_i - b_i) \boxtimes \mu_{MPb_{i+1}} \boxtimes \cdots \boxtimes \mu_{MPb_j}.$$

- For $a_1, a_2 > 1$, $b > a_1 + 1$, $b > a_2 + 1$, and $a_1 + a_2 - b > 1$:

$$\mathcal{H}_n\left[\begin{smallmatrix} a_1+a_2-b \\ - \end{smallmatrix}\right] \boxplus_n \left(\mathcal{H}_n\left[\begin{smallmatrix} b-a_1 \\ b \end{smallmatrix}\right] \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} b-a_2 \\ - \end{smallmatrix}\right]\right) = (\mathcal{H}_n\left[\begin{smallmatrix} a_1 \\ b \end{smallmatrix}\right] \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} a_2 \\ - \end{smallmatrix}\right]).$$

In the limit:

$$\mu_{\mathrm{MP}a_1+a_2-b} \boxplus (f\beta(b-a_1, a_1) \boxtimes \mu_{\mathrm{MP}b-a_2}) = f\beta(a_1, b-a_1) \boxtimes \mu_{\mathrm{MP}a_2}.$$

- For $a, b > 1$, $a > b + 1$:

$$\left(\mathcal{H}_n\left[\begin{smallmatrix} a \\ a+b-\frac{1}{2n} \end{smallmatrix}\right] \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} b \\ - \end{smallmatrix}\right]\right)^{(\boxplus_n)2} = \mathcal{H}_n\left[\begin{smallmatrix} 2b \\ a+b-\frac{1}{2n} \end{smallmatrix}\right] \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} 2a \\ 2a+2b \end{smallmatrix}\right] \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} a+b \\ - \end{smallmatrix}\right].$$

In the limit:

$$(f\beta(a,b) \boxtimes \mu_{\mathrm{MP}b})^{\boxplus 2} = f\beta(2b, a-b) \boxtimes f\beta(2a, 2b) \boxtimes \mu_{\mathrm{MP}a+b}.$$

- For $a < 0$ and $b < a - 1$:

$$\mathcal{H}_n\left[\begin{smallmatrix} - \\ 2a \end{smallmatrix}\right] \boxplus_n \mathcal{H}_n\left[\begin{smallmatrix} - \\ 2b \end{smallmatrix}\right] = \mathcal{H}_n\left[\begin{smallmatrix} a+b \\ 2a \end{smallmatrix}\right] \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} a+b-\frac{1}{2n} \\ 2a+2b-\frac{1}{n} \end{smallmatrix}\right] \boxtimes_n \mathcal{H}_n\left[\begin{smallmatrix} - \\ 2b \end{smallmatrix}\right].$$

In the limit:

$$\mu_{\mathrm{RMP}a} \boxplus \mu_{\mathrm{RMP}b} = \mu_{a+b,2a} \boxtimes \mu_{a+b,2a+2b} \boxtimes \mu_{\mathrm{RMP}2b}.$$

Real zeros of ${}_2\mathcal{F}_2\left(\begin{matrix}-n, & a \\ b_1, & b_2\end{matrix}; x\right)$

a	b₁	b₂	Roots in
$\mathbb{R}_{<-n+1}$	$(-\mathbb{Z}_n) \cup \mathbb{R}_{>0}$	$\mathbb{R}_{>0}$	$\mathbb{R}_{\leq 0}$
$\{b_1 + k\} \cup \mathbb{R}_{>b_1+n-2}$	$(-\mathbb{Z}_n) \cup \mathbb{R}_{>0}$	$\mathbb{R}_{>0}$	$\mathbb{R}_{\geq 0}$
$\mathbb{R}_{<-n+1}$	$\mathbb{R}_{<a-n+2} \cup \{a-1, a-2, \dots\}$	$\mathbb{R}_{>0}$	$\mathbb{R}_{>0}$
$(-n+1, -n+2)$	$(-\mathbb{Z}_n) \cup \mathbb{R}_{>-1}$	$\mathbb{R}_{>0}$	$\mathbb{R}$
$(-n+1, -n+2)$	$\mathbb{R}_{<a-n+2} \cup \{a-1, a-2, \dots\}$	$\mathbb{R}_{>0}$	$\mathbb{R}$
$\mathbb{R}_{<-n+1} \cup \mathbb{R}_{>b_1+n-2}$	$(-1, 0)$	$\mathbb{R}_{>0}$	$\mathbb{R}$
$k + 1/2$	$2 - b_2 > 0$ or $1 - b_2 > 0$	$\mathbb{R}_{>0}$	$\mathbb{R}_{>0}$
$b_1 + k - 1/2$	$(b_2 + t + 1)/2$	$\mathbb{R}_{>0}$	$\mathbb{R}_{>0}$
$(b_1 + 1)/2 + k$	$2(b_2 - 1 + t)$	$\mathbb{R}_{>0}$	$\mathbb{R}_{>0}$
$b_1/2 + k$	$2(b_2 + t) - 1$	$\mathbb{R}_{>0}$	$\mathbb{R}_{>0}$
$b_2 - 1/2$	$2b_2 - 2$	$(0, 1)$	$\mathbb{R}$
$b_2 - 1/2$	$2b_2 - 1$	$(-1, 0)$	$\mathbb{R}$
$b_2 + k - 1/2$	$2b_2 - 2$	$(1/2, 1)$	$\mathbb{R}$
$k + 1/2$	$1 - b_2$ or $2 - b_2$	$(-1, 0)$	$\mathbb{R}$

In the table above $k \in \mathbb{Z}_n$ while $t \in \mathbb{Z}_n \cup \mathbb{R}_{>n-2}$.
$a \notin (-\mathbb{Z}_n)$, and the polynomial is of degree exactly n.
Moreover, a zero at $x = 0$ appears only when either $b_i \in (-\mathbb{Z}_n)$.

Real zeros of ${}_3\mathcal{F}_2\left(\begin{matrix}-n,\ a_1,\ a_2\\ b_1,\ b_2\end{matrix};x\right)$

$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{b_1}$	$\mathbf{b_2}$	**Roots in**
$\mathbb{R}_{<-n+1}$	$\mathbb{R}_{>\min\{b_1,b_2\}+n-2}$	$\mathbb{R}_{>0}$	$\mathbb{R}_{>0}$	$\mathbb{R}_{<0}$
$\mathbb{R}_{>b_1+n-2}$	$\mathbb{R}_{>b_2+n-2}$	$\mathbb{R}_{>0}$	$\mathbb{R}_{>0}$	$\mathbb{R}_{>0}$
$\mathbb{R}_{<-n+1}$	$\mathbb{R}_{<-n+1}$	$\mathbb{R}_{>0}$	$\mathbb{R}_{>0}$	$\mathbb{R}_{>0}$
$\mathbb{R}_{<-n+1}$	$\mathbb{R}_{<-n+1}$	$\mathbb{R}_{>0}$	$\mathbb{R}_{<\min\{a_1,a_2\}-n+2}$	$\mathbb{R}_{<0}$
$\mathbb{R}_{<-n+1}$	$\mathbb{R}_{<-n+1}$	$\mathbb{R}_{<a_1-n+2}$	$\mathbb{R}_{<a_2-n+2}$	$\mathbb{R}_{>0}$
$\mathbb{R}_{<-n+1}$	$\mathbb{R}_{>b_2+n-2}$	$\mathbb{R}_{<a_1-n+2}$	$\mathbb{R}_{>0}$	$\mathbb{R}_{>0}$

Future work

Ongoing and future research

- (Ongoing project) Convolution of polynomials of the form $\mathcal{H}_n\left[\begin{smallmatrix}\boldsymbol{b}\\ \boldsymbol{a}\end{smallmatrix}\right](x^2)$.
- (Ongoing project) Detailed study of some specific interesting hypergeometric polynomials, in connection with Multiple orthogonal families of polynomials.
- Complete characterization of which

$$ {}_2F_2\left(\begin{matrix} -n, a_1 \\ b_1, b_2 \end{matrix}; x\right) $$

polynomials are real-rooted.
- Complete characterization of which

$$ {}_3F_2\left(\begin{matrix} -n, a_1, a_2 \\ b_1, b_2 \end{matrix}; x\right) $$

polynomials are real-rooted.

Thanks!