Free integral calculus

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Probabilistic Operator Algebra Seminar, UC Berkeley

4th December 2023

Linearization

Why linearization shows up?

Consider $\psi = (1 - z(XY + YX))^{-1}$. We have $\mathbb{E}_X(\psi) = \beta^b_Y(\psi) + (\beta^b_Y \otimes \mathbb{E}_X)[\overrightarrow{\delta}_X(\psi)]$.

We have $\overrightarrow{\delta}_X(XY + YX) = 1 \otimes XY + Y \otimes X$,hence $\overrightarrow{\delta}_X(\psi) = z\psi \otimes XY\psi + z\psi Y \otimes X\psi$.

Two functionals: Boolean cumulants with products as entries

Two functionals

Blocked cumulant

Recall that on $(\mathbb{C}\langle X, Y\rangle, \mu)$ we defined *blocked Boolean cumulant* β^b_Y linear functional by prescribing its values on monomials, we define

$$\beta^b_Y(Y^{k_0}X^{k_1}Y^{k_2}\cdots Y^{k_{2n}}) = \beta_{2n+1}(Y^{k_0}, X^{k_1}, Y^{k_2}, \dots, Y^{k_{2n}})$$

Two functionals

Fully factored cumulant

On $(\mathbb{C}\langle X, Y\rangle, \mu)$ we define the *fully factored Boolean cumulant* β^{δ} by prescribing its values on monomials, for $X_{i_l} \in \{X, Y\}$ we define

$$\beta^{\delta}_{Y}(1) = 1,$$
$$\beta^{\delta}_{Y}(X_{i_1} X_{i_2} \cdots X_{i_k}) = \beta_k(X_{i_1}, X_{i_2}, \ldots, X_{i_k}).$$

if $X_{i_1} = X_{i_k} = Y$, and $\beta^{\delta}_{Y}(P) = 0$ for other monomials.

Boolean cumulants of products

Proposition

Let $a_1, a_2, \ldots, a_n \in \mathcal{A}$ *be random variables then*

$$\beta_{m+1}(a_1 a_2 \cdots a_{d_1}, a_{d_1+1} a_{d_1+2} \cdots a_{d_2}, \ldots, a_{d_m+1} a_{d_m+2} \cdots a_n) = \sum_{\substack{\pi \in \mathrm{Int}(n) \\ \pi \vee \rho = 1_n}} \beta_\pi(a_1, a_2, \ldots, a_n),$$

where $\rho = \{\{1, 2, \ldots, d_1\}, \{d_1 + 1, d_1 + 2, \ldots, d_2\}, \ldots, \{d_m + 1, \ldots, n\}\} \in \mathrm{Int}(n)$.

Boolean cumulants with products as entries

Corollary

Let $a_1, a_2, \ldots, a_n \in \mathcal{A}$ *consider partition* $\rho = \{\{1, \ldots, d_1\}, \{d_1 + 1, \ldots, d_2\}, \ldots, \{d_m + 1, \ldots, n\}\} \in \mathrm{Int}(n)$. *We write* $\rho = \{B_1, B_2, \ldots, B_{m+1}\}$, *where blocks are ordered in natural order. For* $j \in \{1, \ldots, n\}$ *denote by* $\rho(j)$ *the number of block containing* j, *i.e. we have* $\rho(j) = k$ *if* $j \in B_k$, *then*

$$\begin{aligned}
&\beta_{m+1}(a_1 a_2 \cdots a_{d_1}, a_{d_1+1} a_{d_1+2} \cdots a_{d_2}, \ldots, a_{d_m+1} a_{d_m+2} \cdots a_n) \\
&\quad = \sum_{j \in \{1,\ldots,n\} \setminus \{d_1, d_2, \ldots, d_m\}} \beta_j(a_1, a_2, \ldots, a_j) \beta_{m-\rho(j)+1)}(a_{j+1} a_{j+2} \cdots a_{d_{\rho(j)}}, \ldots, a_{d_m+1} \cdots a_n).
\end{aligned}$$

Product as entries via derivatives

Products via derivatives

For any $P \in \mathbb{C}\langle X, Y\rangle$ we have

$$\begin{aligned}\beta_X^b(P) &= \epsilon(P) + (\beta_X^\delta \otimes \beta_X^b)(\overleftarrow{\delta}_X\, P)\\ &= \epsilon(P) + (\beta_X^b \otimes \beta_X^\delta)(\overrightarrow{\delta}_X\, P).\end{aligned}$$

Two functionals: Boolean cumulants with free entries

VNRP

Theorem (Fevrier, Mastnak, Nica, Sz. and Jekel, Liu)

Subalgebras $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_s \subseteq \mathcal{M}$ *of a ncps* $(\mathcal{M}, \varphi)$ *are free if and only if for any colouring* $c : \{1, \dots, n\} \to \{1, \dots, s\}$ *we have*

$$\beta_n(a_1, a_2, \dots, a_n) = \sum_{\pi \in NC^{irr}(n) \text{ with VNRP}} \beta_\pi(a_1, a_2, \dots, a_n)$$

whenever $a_i \in \mathcal{A}_{c(i)}$. *Here a partition* $\pi \in NC^{irr}(n)$ *is said to have VNRP if* $\pi \leq \ker c$ *and every inner block is covered by a block of different colour, i.e.,* c *induces a proper coloring on the nesting tree of* π.

VNRP for alternating entries

Proposition

Let $\{a_1, a_2, \ldots, a_n\}$ *and* $\{b_1, b_2, \ldots, b_{n-1}\}$ *be free,* $n \geq 1$*. Then*

$$\beta_{2n-1}(a_1, b_1, \ldots, a_n, b_{n-1}, a_n)$$
$$= \sum_{k=2}^{n} \sum_{1=j_1<j_2<\cdots<j_k=n} \beta_k(a_{j_1}, a_{j_2}, \ldots, a_{j_k}) \prod_{\ell=1}^{k-1} \beta_{2(j_{\ell+1}-j_\ell)-1}(b_{j_l}, a_{j_\ell+1}, \ldots, a_{j_{\ell+1}-1}, b_{j_{\ell+1}-1}).$$

VNRP for alternating entries

Lemma

Suppose $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$ *are free and let* $a_1, a_2, \ldots, a_n \in \mathcal{A}$ *and* $b_1, b_2, \ldots, b_{n-1} \in \mathcal{B}$. *Assume further that for each* $j = 1, 2, \ldots, n-1$ *we have* $b_i = c_1^{(i)} \cdots c_{j_i}^{(i)}$ *with* $c_1^{(i)}, \ldots, c_{j_i}^{(i)} \in \mathcal{B}$, *then we have*

$$\beta_{2n-1}(a_1, b_1, a_2, \ldots, b_{n-1}, a_n) = \beta_{n+j_1+\ldots+j_{n-1}}(a_1, c_1^{(1)}, \ldots, c_{j_1}^{(1)}, a_2, \ldots, c_1^{(n-1)}, \ldots, c_{j_{n-1}}^{(n-1)}, a_n).$$

VNRP via derivatives

VNRP via derivatives

For any $P \in \mathbb{C}\langle X, Y\rangle$ we have

$$\beta^{\delta}_{X}(P) = \epsilon(P) + \sum_{k=1}^{\infty} \beta_k(X) \left[\epsilon \otimes \left(\beta^{b}_{Y}\right)^{\otimes(k-1)} \otimes \epsilon\right] \left(\partial^{k}_{X} P\right)$$

Example additive convolution

$\Psi = (1 - z(X+Y))^{-1} = \sum_{n=0}^{\infty} (z(X+Y))^n$.

First formula

$$\beta_X^b(P) = \epsilon(P) + (\beta_X^b \otimes \beta_X^\delta)(\overrightarrow{\delta}_X P).$$

Taking derivatives we obtain

$$\overrightarrow{\delta}_X (\Psi) = 1 + z\Psi \otimes X\Psi, \qquad \overrightarrow{\delta}_Y (\Psi) = 1 + z\Psi \otimes Y\Psi.$$

Thus

$$\beta_X^b(\Psi) = 1 + z\beta_X^b(\Psi)\beta_X^\delta(X\Psi), \qquad \beta_Y^b(\Psi) = 1 + z\beta_Y^b(\Psi)\beta_Y^\delta(Y\Psi).$$

Hence we obtain

$$\beta_X^b(\Psi) = \left(1 - z\beta_X^\delta(X\Psi)\right)^{-1}, \qquad \beta_Y^b(\Psi) = \left(1 - z\beta_Y^\delta(Y\Psi)\right)^{-1}.$$

Example additive convolution

Second formula

$$\beta_X^\delta(P) = \epsilon(P) + \sum_{n=1}^{\infty} \beta_n(X) \left[\epsilon \otimes \left(\beta_Y^b\right)^{\otimes(n-1)} \otimes \epsilon\right] (\partial_X^n P)$$

$$\partial_X^n(X\Psi) = z^{n-1} 1 \otimes \Psi^{\otimes n} + z^n X\Psi \otimes \Psi^{\otimes n}, \qquad \partial_Y^n(Y\Psi) = z^{n-1} 1 \otimes \Psi^{\otimes n} + z^n Y\Psi \otimes \Psi^{\otimes n}.$$

Thus

$$\beta_X^\delta(X\Psi) = \sum_{n=1}^{\infty} \beta_n(X) \beta_Y^b(\Psi)^{n-1} z^{n-1} = \widetilde{\eta}_X(z\beta_Y^b(\Psi)),$$

$$\beta_Y^\delta(Y\Psi) = \sum_{n=1}^{\infty} \beta_n(Y) \beta_X^b(\Psi)^{n-1} z^{n-1} = \widetilde{\eta}_Y(z\beta_X^b(\Psi)).$$

Finally we obtain the following system of equations

$$\beta_X^\delta(X\Psi) = \widetilde{\eta}_X\left(z\left(1 - z\beta_Y^\delta(Y\Psi)\right)^{-1}\right), \quad \beta_Y^\delta(Y\Psi) = \widetilde{\eta}_Y\left(z\left(1 - z\beta_X^\delta(X\Psi)\right)^{-1}\right).$$

Statement on matrices

Proposition

Let $\mathcal{M} = M_N(\mathbb{C}\langle X, Y\rangle)$ *and* X, Y *free with respect to* $\mu : \mathcal{M} \to \mathbb{C}$. *Then*

$$\mathbb{E}_X^{(N)}[M] = {\beta_Y^b}^{(N)}(M) + (\beta_Y^b \otimes \mathbb{E}_X)^{(N)}[\vec{\delta}_X^{(N)}(M)].$$

Statement on matrices

Proposition

Let $M \in M_N(\mathbb{C}\langle X, Y\rangle)$*, then we have*

$$\begin{aligned}\beta_X^{b(N)}(M) &= \epsilon^{(N)}(M) + (\beta_X^{\delta} \otimes \beta_X^{b})^{(N)}(\overleftarrow{\delta}_X^{(N)} M)\\ &= \epsilon^{(N)}(M) + (\beta_X^{b} \otimes \beta_X^{\delta})^{(N)}(\overrightarrow{\delta}_X^{(N)} M)\end{aligned}$$

Proposition

Let $M \in M_N(\mathbb{C}\langle X, Y\rangle)$*, then*

$$\beta_X^{\delta(N)}(M) = \epsilon^{(N)}(M) + \sum_{k=1}^{\infty} \beta_k(X) \left[\epsilon \otimes \left(\beta_Y^{b}\right)^{\otimes(k-1)} \otimes \epsilon\right]^{(N)} \left(\partial_X^{k(N)}(M)\right).$$

Examples

Anti–commutator

Consider $P(X,Y) = XY + YX$ then one has

$$\mathbb{E}_X\left[(1 - z^2(XY+YX))^{-1}\right] = \left(1 - f_{Y,43}z - z^2X\left(f_{Y,33} + f_{Y,44}\right) - f_{Y,34}X^2z^3\right)^{-1},$$
$$\mathbb{E}_Y\left[(1 - z^2(XY+YX))^{-1}\right] = \left(1 - f_{X,12}z - z^2Y\left(f_{X,11} + f_{X,22}\right) - f_{X,21}Y^2z^3\right)^{-1}.$$

We find $u, v \in \mathbb{R}^4$ and $C_X, C_Y \in \mathbb{R}^{4\times 4}$ such that
$(1 - z^2(XY+YX))^{-1} = u^t(1 - z(C_X X + C_Y Y))^{-1}v = u^t\Psi(z)v.$
Then $\mathbb{E}_X\left[(1 - z^2(XY+YX))^{-1}\right] = u^t\mathbb{E}_X^{(N)}(\Psi(z))v.$

This result is obtained with the linearization involving the matrices

$$C_X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad C_Y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Anti–commutator

We consider

$$\tilde{F}_X = \beta_X^{\delta\,(N)}(X\Psi(z))$$
$$\tilde{F}_Y = \beta_Y^{\delta\,(N)}(Y\Psi(z))$$

Using kernels of matrices C_X and C_Y we can reduce the size of matrices $\tilde{F}_X$ and $\tilde{F}_Y$.

$$F_X = \begin{bmatrix} f_{X,11} & f_{X,12} & 0 & 0 \\ f_{X,21} & f_{X,22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad F_Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & f_{Y,33} & f_{Y,34} \\ 0 & 0 & f_{Y,43} & f_{Y,44}, \end{bmatrix}$$

and show that they satisfy

$$\begin{cases} F_X &= \widetilde{\eta}_X\left(zQ_X(1 - zC_Y F_Y)^{-1} C_X\right), \\ F_Y &= \widetilde{\eta}_Y\left(zQ_Y(1 - zC_X F_X)^{-1} C_Y\right), \end{cases}$$

Anti–commutator and commutator

Nica and Speicher proved that for symmetric distributions anti–commutator and commutator in free variables have the same distribution.

Proposition

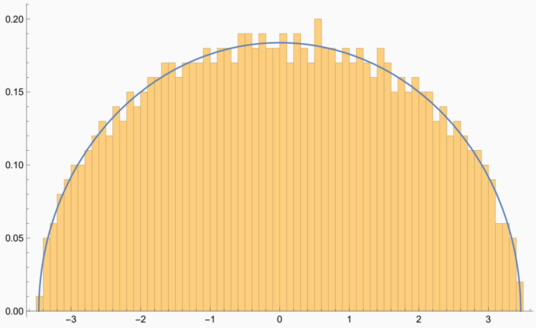
Assume that X, Y are free and symmetric, then

$$\mathbb{E}_X[(1 - z^2 i(XY - YX))^{-1}] = \mathbb{E}_X[(1 - z^2(XY + YX))^{-1}]$$

Free convolution without freeness

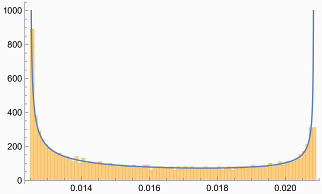
Proposition

Let X, Y be free, X semicircle of variance 1 and Y symmetric Bernoulli, the element $X + i(XY - YX)$ has semicircle distribution with variance 3. Moreover elements X and $i(XY - YX)$ are not free, while $i(XY - YX)$ has semicircle distribution with variance 2.



Further examples in the paper

- $R = X(1-X-Y)^{-1}X$, we find conditional expectation of $\psi = (1-zR)^{-1}$ and distribution of R, when X, Y are symmetric Bernoulli elements.

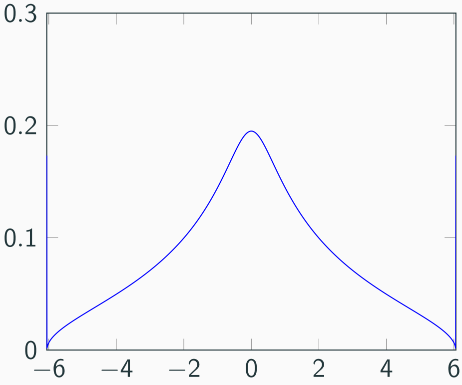


- $P = XZYZX$, we find conditional expectation of $(1-z^5XZYZX)^{-1}$ on X, Y, taking all variables to be Bernoulli elements we have explicit formulas.

Further examples

$P = XYZ + XZY + YXZ + YZX + ZXY + ZYX$, where all variables are semicircle, the Cauchy transform satisfies

$$2G(z)^4 z^2 + 8G(z)^3 z + 8G(z)^2 - 3G(z)z^5 + 3z^4 = 0$$



Thank you!